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ANALOGOUS FUNCTION THEORIES FOR THE HEAT, WAVE, AND LAPLACE EQUATIONS

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ABSTRACT. Transmutation operators are used to establish analogous function theories for standard and radial versions of the heat, wave, and Laplace equation. Under these transformations correspondences are established relating fundamental solutions, polynomial solutions, associated functions, generating functions, Fourier transform criteria, and expansion theorems. In some cases, the transmutation operators must be interpreted in the generalized sense as acting on distributions.

1. Introduction. In 1966, D.V. Widder [11] pointed out numerous analogies between classical function theory and representation theory for solutions of the heat equation. He did this by comparing a table of properties for representations of analytic functions to a corresponding table of properties for representations of heat functions. This work did not directly connect the results in the two tables by means of constructive isomorphisms. One of the purposes of this paper is to indicate how the results in related partial differential equations can be used to accomplish this.

In a series of papers, [4–9] the authors have shown how various elliptic, parabolic, and hyperbolic partial differential equations can be related through transmutation operators. In particular, [8] we have shown that various polynomial solutions of elliptic and hyperbolic equations can be obtained from the heat polynomials and radial heat polynomials, and how these in turn can be used to represent solutions of problems involving these equations. In many cases, the transmutation operators do not exist in the classical sense and therefore must be interpreted in a generalized sense as acting on distributions. Once this is done, it is possible to show that classical analytic function theory can be obtained directly from the representation theory for heat functions. In this case, representation of heat functions in a strip in terms of heat polynomials [10] corresponds to

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representation of analytic functions inside a circle. Associated heat functions (Appell transforms of heat polynomials) are transformed by the same transmutations into reciprocal solutions of Laplace's equation. In this case, expansion of heat functions in a half-plane in terms of associated heat functions corresponds to representation of analytic functions outside a circle using reciprocal powers of a complex variable.

In the case of the wave equation, the heat polynomials transform into basic polynomial solutions which can be used for expansions of solutions of the wave equation, which converge inside squares bounded by characteristic lines. The associated heat functions transform into distributions which can be used to expand generalized solutions of the wave equation.

The analogies between these various function theories reveal many other striking similarities. In fact, under the relevant transmutation operators, fundamental solutions (Green's functions) correspond to fundamental solutions, generating functions correspond to generating functions, and many basic properties of solutions also correspond. There is also a similar Fourier transform criterion for expansion in terms of associated functions in each case.

Clearly, these same ideas can be used to develop analogies between solutions of other partial differential equations. The same transmutation operators mentioned above carry solutions of the radial heat equation into solutions of the radial Laplace equation and the radial wave equation. In this case, one starts with the radial heat polynomials [2] and associated functions and transforms them into corresponding solutions of the Laplace and wave equations, for which corresponding expansion theorems are developed. These results will be mentioned in this paper. It is well known [12] that there are multinomial versions of the heat polynomials and associated heat functions. These can be used to develop analogous theories for the higher dimensional Laplace and wave equations. Other transmutation operators are available to set the correspondences between the heat equation and the equation of generalized axially symmetric potential theory (GASPT) and the Euler-Poisson-Darboux equation (EPD), [4, 7]. These equations will be treated in a separate paper.

The plan of this paper is as follows: A summary of all the transformations needed will be given in section two. Section three will be devoted to a review of all the relevant material for the standard heat equation. Section four will give the analogous function theory for Laplace's equation and related analytic functions. The function theory for the wave equation will be given in section five, and the analogous results for the radial versions of the wave and Laplace equations will be sketched in section six.

2. Transformations. In this section, we summarize the properties of the basic transformations needed later. We begin with the Appell transform [1]. If $u(x, t)$ is a solution of the heat equation,

$$u_t(x, t) = u_{xx}(x, t), \tag{2.1}$$

then the Appell transform of $u(x, t)$, defined by

$$A\{u(x, t)\} = k(x, t)u(x/t, -1/t), \tag{2.2}$$

is also a solution of (2.1), where

$$k(x, t) = (4\pi t)^{-1/2}e^{-x^2/4t} \tag{2.3}$$

is the fundamental solution of (2.1).

Similarly, if $u(r, t)$ is a solution of the radial heat equation,

$$u_t(r, t) = u_{rr}(r, t) + (\mu - 1)\, r^{-1}\, u_r(r, t) \tag{2.4}$$

for $r > 0$, then the generalized Appell transform, defined by

$$A_\mu\{u(r, t)\} = k_\mu(r, t)u(r/t, -1/t), \tag{2.5}$$

also satisfies (2.4), where

$$k_\mu(r, t) = (4\pi t)^{-\mu/2}e^{-r^2/4t} \tag{2.6}$$

is the fundamental source solution of (2.4).

Let $h(x, t)$ be a solution of (2.1) for $-\infty < x < \infty$, $t > 0$, with $h(x, 0) = \phi(x)$. Then under appropriate assumptions on $\phi(x)$, according to [9],

$$\begin{aligned} u(x, y) &= \frac{y}{\sqrt{4\pi}} \int_0^\infty s^{-3/2}e^{-y^2/4s}\, h(x, s)\, ds \\ &= \frac{y}{\sqrt{\pi}} \mathscr{L}_\sigma\{\sigma^{-1/2}\, h(x, 1/4\sigma)\}_{\sigma\to y^2} \\ &= T_1 h(x, t) \end{aligned} \tag{2.7}$$

is a solution of the Dirichlet problem:

$$u_{yy}(x, y) + u_{xx}(x, y) = 0, \; -\infty < x < \infty, \; y > 0, \;\; u(x, 0) = \phi(x). \tag{2.8}$$

In the definition of T_1, the notation $\mathscr{L}_\sigma\{\cdots\}_{\sigma\to y^2}$ refers to Laplace transformation with respect to σ, with the variable of the transform y^2. If the first integral does not exist, as for example, if $h(x, t)$ has polynomial growth in t, we shall take the transform in the generalized sense as defined in Zemanian [13]. All of the functions we wish to transform in this paper have Laplace transforms either in the conventional or generalized sense. Similarly, if $h(x, t)$ is defined as above,

$$
\begin{aligned}
v(x, y) &= -\frac{1}{\pi}\int_0^\infty s^{-1/2}\, e^{-y^2/4s}\, h(x, s)\, ds \\
&= -\frac{1}{\sqrt{4\pi}}\, \mathscr{L}_\sigma\{\sigma^{-3/2} h(x, 1/4\sigma)\}_{\sigma\to y^2} \qquad (2.9)\\
&= T_2 h(x, t)
\end{aligned}
$$

is a solution of the Neumann problem:

$$
v_{yy}(x, y) + v_{xx}(x, y) = 0,\ -\infty < x < \infty,\ y > 0,\ v_y(x, 0) = \phi(x). \qquad (2.10)
$$

Again referring to the same $h(x, t)$ above, we have according to **[5, 6]**,

$$
h(x, t) = \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-s^2/4t} w(x, s)\, ds\,,
$$

where $w(x, t)$ is a solution of the Cauchy problem:

$$
\begin{aligned}
&w_{tt}(x, t) = w_{xx}(x, t),\ -\infty < x < \infty,\ -\infty < t < \infty\,, \\
&\qquad w(x, 0) = \phi(x),\ w_t(x, 0) = 0\,.
\end{aligned} \qquad (2.11)
$$

Inverting this transform, we have

$$
\begin{aligned}
w(x, t) &= t\sqrt{\pi}\, \mathscr{L}_\sigma^{-1}\{\sigma^{-1/2}\, h(x, 1/4\sigma)\}_{\sigma\to t^2} \\
&= T_3 h(x, t)
\end{aligned} \qquad (2.12)
$$

in which $\mathscr{L}_\sigma^{-1}\{\cdots\}_{\sigma\to t^2}$ denotes the inverse Laplace transform with t^2 the variable of inversion.

Similarly, we have

$$
\begin{aligned}
h(x, t) &= \frac{1}{\sqrt{4\pi t^3}}\int_0^\infty s e^{-s^2/4t}\, \tilde{w}(x, s)\, ds \\
\tilde{w}(x, t) &= \frac{1}{2}\sqrt{\pi}\, \mathscr{L}_\sigma^{-1}\{\sigma^{-3/2}\, h(x, 1/4\sigma)\}_{\sigma\to t^2} \qquad (2.13)\\
&= T_4 h(x, t)
\end{aligned}
$$

where $\tilde{w}(x, t)$ is a solution of the Cauchy problem:

$$
\begin{aligned}
&\tilde{w}_{tt}(x, t) = \tilde{w}_{xx}(x, t),\ -\infty < x < \infty,\ -\infty < t < \infty, \\
&\qquad \tilde{w}(x, 0) = 0,\ \tilde{w}_t(x, 0) = \phi(x)\,.
\end{aligned} \qquad (2.14)
$$

3. Heat equation. In this section, we shall recall some of the results of Rosenbloom and Widder **[10]** pertaining to representation of solutions of the heat question. A solution $h(x, t)$ of (2.1) satisfying $h(x, 0) = x^n$, n a nonnegative integer, is called a *heat polynomial* of degree n. We denote these polynomials by $h_n(x, t)$, $n = 0, 1, 2, \ldots$. The heat polynomials can be defined in various ways:

$$
\begin{aligned}
h_n(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} k(x-\xi, t)\xi^n \, d\xi \\
&= k(x, t) * x^n \\
&= n! \sum_{j=0}^{[n/2]} x^{n-2j} t^j/(n-2j)!\, j!\,,
\end{aligned} \tag{3.1}
$$

or by the generating function

$$\sum_{n=0}^{\infty} h_n(x, t)\, a^n/n! = e^{ax+a^2t}\,. \tag{3.2}$$

We also note the recurrence relation

$$\partial h_n/\partial x = n h_{n-1}\,. \tag{3.3}$$

An *associated heat function*, $H_n(x, t)$, is a solution of (2.1) given by the Appell transform of $H_n(x, t)$.

$$H_n(x, t) = A\{h_n(x, t)\} = (-2)^n \partial^n k(x, t)/\partial x^n\,. \tag{3.4}$$

$H_n(x, t)$ can also be defined by the generating function

$$\sum_{n=0}^{\infty} H_n(x, t) a^n/n! = k(x-2a, t)\,. \tag{3.5}$$

There is also a bilateral generating function connecting the heat polynomials and the associated functions

$$\sum_{n=0}^{\infty} h_n(x, t) H_n(y, s)/2^n n! = k(x-y, t+s), \quad |t| < s\,. \tag{3.6}$$

DEFINITION 3.1. An entire function $\phi(x) = \sum_{n=0}^{\infty} a_n x^n$ is of growth (ρ, τ), if and only if

$$\limsup_{n\to\infty} (n/e\rho)|a_n|^{\rho/n} \leqq \tau\,. \tag{3.7}$$

We denote this class of growth (ρ, τ) by $\mathfrak{A}(\rho, \tau)$ and note that $\mathfrak{A}(\rho', \tau) \subset \mathfrak{A}(\rho, \tau)$ if $\rho' < \rho$.

We are now in a position to state some of the Rosenbloom-Widder results which will be used later. (see also **[12]**).

THEOREM 3.1. *Let $h(x, t)$ be a solution of* (2.1) *corresponding to $h(x, 0) = \phi(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathfrak{A}(2, \tau)$. If* (3.7) *holds with equality then the series*

$$\sum_{n=0}^{\infty} a_n h_n(x, t) \tag{3.8}$$

converges absolutely to $h(x, t)$ in the time strip $|t| < 1/4\tau$, but does not converge everywhere in any including strip. Conversely, if the series (3.8) *converges for* $|t| < 1/4\tau$, *then the function* $h(x, t)$ *defined by* (3.8) *satisfies* (2.1) *and* $h(x, 0)$ *is an entire function in* $\mathfrak{A}(2, \tau)$.

THEOREM 3.2. *If* $\limsup_{n\to\infty} (2n/e)|b_n|^{2/n} = \sigma$, *then the series*

$$\sum_{n=0}^{\infty} b_n H_n(x, t) \tag{3.9}$$

converges absolutely for $t > \sigma$ *to a solution of* (2.1), *but does not converge everywhere for* $t > \sigma - \varepsilon$, $\varepsilon > 0$.

THEOREM 3.3. *The series* (3.9) *converges for* $t > \sigma \geq 0$ *if and only if*

$$\sum_{n=0}^{\infty} b_n H_n(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixs - ts^2} \psi(s)\, ds \tag{3.10}$$

where $\psi(x) \in \mathfrak{A}(2, \sigma)$ *and* $b_n = \psi^{(n)}(0)/[n!\,(-2i)^n]$.

Depending on the entireness properties of $\phi(x)$, a solution of (2.1) corresponding to $h(x, 0) = \phi(x)$ may have representations in terms of only one of the set $\{h_n(x, t)\}$ and $\{H_n(x, t)\}$ or in terms of both sets. However, both representations cannot exist in the same time strip. In order to obtain some criterion for which representation holds, it is useful to obtain a relationship between $\phi(x)$ and the function $\psi(x)$ in the integral representation (3.10). Using the Poisson integral representation along with (3.10). it follows that $\phi(x)$ and $\psi(x)$ define the same solution $h(x, t)$ of (2.1) if

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t}\, \phi(\xi) d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixs - ts^2}\, \psi(s)\, ds\,. \tag{3.11}$$

If we assume, for the moment, that $\phi(x) \in L^1(-\infty, \infty)$, it is not difficult to show that

$$\psi(x) = \int_{-\infty}^{\infty} \phi(\xi) e^{-ix\xi} d\xi\,, \tag{3.12}$$

i.e., that ψ is a Fourier transform of ϕ. (See [3] for analogous results for the radial heat equation). Similarly,

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixs} \psi(-s)\, ds\,. \tag{3.13}$$

To the authors' knowledge, the class of entire functions which have Fourier transforms which are also entire has not been characterized. However, the following results provide information about certain subclasses of entire functions.

THEOREM 3.4. *Let* $\phi(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathfrak{A}(\rho, \sigma)$, $\rho < 2$, *and let* $h(x, t)$ *be a solution of* (2.1) *corresponding to* $h(x, 0) = \phi(x)$. *Then the series* (3.8) *converges for* $0 \leqq t < \infty$.

PROOF. Since $\phi(x) \in \mathfrak{A}(\rho, \sigma)$, it follows by Definition 3.1 that for every $\varepsilon > 0$, there is a positive N such that $n \geqq N$ implies $|a_n| < [(\sigma + \varepsilon)e\rho/n]^{n/\rho}$. From Theorem 3.4 of [10], we have for $-\infty < x < \infty$, $0 \leqq t < \infty$, $0 < \delta < \infty$, $n = 1, 2, 3, \ldots$,

$$|h_n(x, t)| \leqq (1 + t/\delta)^{1/2}[2n(t + \delta)/e]^{n/2}e^{x^2/4\delta}.$$

Using these estimates, we have

$$\sum_{n=N}^{\infty} |a_n|\,|h_n(x, t)| \leqq (1 + t/\delta)^{1/2}e^{x^2/4\delta} \sum_{n=N}^{\infty} \left\{\left[\frac{(\sigma + \varepsilon)e\rho}{n}\right]^{1/\rho} \sqrt{\frac{2n(t + \delta)}{e}}\right\}^n.$$

An application of the ratio test to the last series shows that the dominating series converges for $0 \leqq t < \infty$.

COROLLARY 3.1. *If $\phi(x) \in \mathfrak{A}(\rho, \sigma)$, $\rho < 2$, then the solution $h(x, t)$ of (2.1) corresponding to $h(x, 0) = \phi(x)$ cannot be represented in terms of the associated functions $H_n(x, t)$.*

If we consider the last result with Theorem 3.3, it follows that, if $\phi(x) \in \mathfrak{A}(\rho, \sigma)$, $\rho < 2$, there cannot exist a function $\psi(x) \in \mathfrak{A}(\rho', \tau)$, $\rho' \leqq 2$, $\tau > 0$, corresponding to $\phi(x)$. We need the following definition.

DEFINITION 3.2. An entire function $\phi(x)$ is said to be of strict class (ρ, σ) if $\phi(x) \in \mathfrak{A}(\rho, \sigma)$ but $\phi(x) \notin \mathfrak{A}(\rho', \sigma)$, $\rho' < \rho$. We denote this class by $\mathfrak{A}^s(\rho, \sigma)$.

It follows that $\phi(x)$ can give rise to an entire function $\psi(x)$ of appropriate class only if $\phi(x) \in \mathfrak{A}^s(2, \sigma)$ for some $\sigma > 0$. Similarly, $\psi(x)$ can give rise to an appropriate $\phi(x)$ only if $\psi(x) \in \mathfrak{A}^s(2, \sigma)$ for some $\sigma > 0$. Of course, the corresponding $\phi(x)$, if it exists, must be in $\mathfrak{A}^s(2, \tau)$ for some $\tau > 0$, An example of a pair of such functions is $\phi(x) = e^{-ax^2}$, $a > 0$, $\psi(x) = \sqrt{\pi/a}\, e^{-x^2/4a}$. The first gives rise to an expansion (3.8) for $|t| < 1/4a$, while the second an expansion (3.9) for $t > 1/4a$.

4. Laplace's equation. In developing the analogy between the Rosenbloom-Widder theory and analytic function theory, we first note that the fundamental solutions for the Dirichlet and Neumann problems correspond to $k(x, t)$ under the transformations T_1 and T_2:

$$K(x, y) = T_1k(x, t) = (1/\pi)(y/(x^2 + y^2)), \tag{4.1}$$

$$L(x, y) = T_2k(x, t) = (1/2\pi)\log(x^2 + y^2). \tag{4.2}$$

The first exists in the ordinary sense, while the second is the transform of a distribution.

Taking transforms of the heat polynomials in the generalized sense, we have

$$
\begin{aligned}
u_n(x, y) &= T_1 h_n(x, t) \\
&= \frac{y}{\sqrt{\pi}} \mathscr{L}_\sigma \left\{ n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!\,k!} \frac{\sigma^{-1/2}}{(4\sigma)^k} \right\}_{\sigma \to y^2} \\
&= n! \frac{y}{\sqrt{\pi}} \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!\,4^k k!} \mathscr{L}_\sigma \left\{ \sigma^{-1/2-k} \right\}_{\sigma \to y^2} \\
&= n! \frac{y}{\sqrt{\pi}} \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!\,4^k k!} \frac{\Gamma(1/2-k)}{y^{1-2k}} \\
&= n! \sum_{k=0}^{[n/2]} (-1)^k x^{n-2k} y^{2k} / (n-2k)!(2k)! \\
&= \mathrm{Re}(z^n), \quad n = 0, 1, 2, \ldots .
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
v_n(x, y) &= T_2 h_n(x, t) \\
&= \frac{-n!}{\sqrt{4\pi}} \mathscr{L}_\sigma \left\{ \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!k!} \frac{\sigma^{-3/2}}{(4\sigma)^k} \right\}_{\sigma \to y^2} \\
&= \frac{-n!}{\sqrt{4\pi}} \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!\,4^k k!} \frac{\Gamma(-1/2-k)}{y^{-1-2k}} \\
&= n! \sum_{k=0}^{[n/2]} (-1)^k x^{n-2k} y^{2k+1} / (n-2k)!(2k+1)! \\
&= \mathrm{Im}(z^{n+1})/(n+1), \quad n = 0, 1, 2, \ldots .
\end{aligned} \tag{4.4}
$$

The following recurrence relations follow immediately from the corresponding one for the heat polynomials

$$\partial u_n(x, y)/\partial x = n u_{n-1}(x, y), \quad \partial v_n(x, y)/\partial x = n v_{n-1}(x, y) . \tag{4.5}$$

For the expansion theorems we shall need the obvious bounds

$$|u_n(x, y)| \leqq |z|^n, \quad |v_n(x, y)| \leqq |z|^{n+1}/(n+1) . \tag{4.6}$$

Next we show that the generating functions correspond under the transformations T_1 and T_2. For the set $\{u_n(x, y)\}$, we have

$$G(x, y, a) = T_1 e^{ax+a^2t} = \frac{ye^{ax}}{\sqrt{\pi}} \mathscr{L}_\sigma \{\sigma^{-1/2} e^{a^2/4\sigma}\}_{\sigma \to y^2} . \tag{4.7}$$

Let $a = \xi + i\eta$, then

$$G(x, y, a) = \frac{ye^{ax}}{\sqrt{\pi}} \mathscr{L}_\sigma \{\sigma^{1/2} e^{(\xi^2-\eta^2+2i\xi\eta)/4a}\}_{\sigma \to y^2}$$

which is analytic for $\mathrm{Re}(a^2) < 0$, i.e., $\xi^2 < \eta^2$. If we let $\xi = 0$

$$
\begin{aligned}
G(x, y, i\eta) &= \frac{ye^{ix\eta}}{\sqrt{\pi}} \mathscr{L}_\sigma \{\sigma^{-1/2} e^{-\eta^2/4\sigma}\}_{\sigma \to y^2} \\
&= e^{ix\eta} e^{-\eta y} = e^{(x+iy)i\eta} .
\end{aligned}
$$

By analytic continuation, we have $G(x, y, a) = e^{(x+iy)a}$. Actually, what we want is Re $G(x, y, a)$ when a is real. Therefore, our generating function is

$$(e^{(x+iy)a} + e^{(x-iy)a})/2 = e^{ax} \cos ay\,. \tag{4.8}$$

By a similar argument we can show that the generating function for the set $\{v_n(x, y)\}$ is

$$\sum_{n=0}^{\infty} v_n(x, y)a^n/n! = (e^{ax} \sin ay)/a\,. \tag{4.9}$$

For the associated functions, we make use of the identity $H_n(x, t) = (-2)^n \partial^n k(x, t)/\partial x^n$. Hence,

$$\begin{aligned} U_n(x, y) &= T_1 H_n(x, t) \\ &= \frac{y}{\sqrt{\pi}} \mathscr{L}_\sigma\{\sigma^{-1/2}(-2)^n \partial^n k(x, 1/4\sigma)/\partial x^n\}_{\sigma \to y^2} \\ &= (-2)^n \frac{\partial^n}{\partial x^n} \mathscr{L}_\sigma\{\sigma^{-1/2} k(x, 1/4\sigma)\}_{\sigma \to y^2} \\ &= (-2)^n \frac{\partial^n}{\partial x^n} K(x, y) = -\frac{(-2)^n}{\pi} \frac{\partial^n}{\partial x^n} \operatorname{Im}(1/z) \\ &= -\frac{2^n n!}{\pi} \operatorname{Im}(1/z^{n+1}) = -\frac{2^n n!}{\pi} \operatorname{Im}(\bar{z}^{n+1}/|z|^{2n+2}) \\ &= \frac{2^n n!}{\pi} \operatorname{Im}(z^{n+1}/|z|^{2n+2}), \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{4.10}$$

For the other set of associated functions corresponding to the Neumann problem, we first note from (4.2) that $T_2 H_0 = (2\pi)^{-1} \log (x^2 + y^2)$. Since we do not want this singular function to enter the expansion theorems, we begin the transformation of the associated heat functions starting with $n = 1$. For $n \geqq 1$, $H_n(x, t) = (-2)^{n-1}(\partial^{n-1}/\partial x^{n-1})g(x, t)$ where $g(x, t) = (x/t)k(x, t)$. Then

$$\begin{aligned} V_n(x, y) &= T_2 H_n(x, t) \\ &= \frac{-(-2)^{n-1}}{\sqrt{4\pi}} \frac{\partial^{n-1}}{\partial x^{n-1}} \mathscr{L}_\sigma\{4x\sigma^{-1/2} k(x, 1/4\sigma)\}_{\sigma \to y^2} \\ &= \frac{(-2)^n}{\pi} \frac{\partial^{n-1}}{\partial x^{n-1}} x \mathscr{L}_\sigma\{e^{-x^2\sigma}\}_{\sigma \to y^2} \\ &= \frac{(-2)^n}{\pi} \frac{\partial^{n-1}}{\partial x^{n-1}} \left(\frac{x}{x^2 + y^2}\right) \\ &= \frac{(-2)^n}{\pi} \frac{\partial^{n-1}}{\partial x^{n-1}} \operatorname{Re}(1/z) = \frac{-2^n}{\pi} (n-1)! \operatorname{Re}(1/z^n) \\ &= \frac{-2^n}{\pi} (n-1)! \operatorname{Re}(z^n/|z|^{2n}), \quad n = 1, 2, 3, \ldots . \end{aligned} \tag{4.11}$$

For the expansion theorems we will need the obvious bounds:

(4.12) $|U_n(x, y)| \leqq (2^n n!/\pi)|z|^{-(n+1)}$, $|V_n(x, y)| \leqq (2^n(n-1)!/\pi)|z|^{-n}$.

For the generating functions, we make use of equation (3.5) and the fact that the fundamental solutions correspond under the transformations T_1 and T_2.

$$\sum_{n=0}^{\infty} U_n(x, y)a^n/n! = (1/\pi)y/((x-a)^2 + y^2). \tag{4.13}$$

$$\sum_{n=1}^{\infty} V_n(x, y)a^n/n! = (1/2\pi)\log[((x-2a)^2 + y^2)/(x^2 + y^2)]. \tag{4.14}$$

These series converge for $|2a| < |z|$.

The result analogous to (3.6) is

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x, y)U_n(\xi, \eta)/2^n n! \\ = (1/2)[K(x-\xi, \eta+y) + K(x-\xi, \eta-y)],\end{aligned} \tag{4.15}$$

which converges for $|z| < |\zeta|$, $\zeta = \xi + i\eta$. This is proved as follows

$$\sum_{n=0}^{\infty} u_n(x, y)U_n(\xi, \eta)/2^n n! = \frac{1}{\rho\pi}\sum_{n=0}^{\infty}(r/\rho)^n \cos n\theta \sin(n+1)\phi,$$

where $z = r(\cos\theta + i\sin\theta)$, $\zeta = \rho(\cos\phi + i\sin\phi)$. The series converges for $r < \rho$. Furthermore,

$$\begin{aligned}&\sum_{n=0}^{\infty} u_n(x, y)U_n(\xi, \eta)/2^n n! \\ &\quad = \frac{1}{\rho\pi}\sum_{n=0}^{\infty}(r/\rho)^n \cos n\theta[\sin n\phi\cos\phi + \cos n\phi\sin\phi] \\ &\quad = \frac{1}{2\rho\pi}\sum_{n=0}^{\infty}\cos\phi[(r/\rho)^n \sin n(\theta+\phi) + (r/\rho)^n \sin n(\phi-\theta)] \\ &\quad\quad + \frac{1}{2\rho\pi}\sum_{n=0}^{\infty}\sin\phi[(r/\rho)^n \cos n(\theta-\phi) + (r/\rho)^n \cos n(\phi-\theta)].\end{aligned}$$

The terms in brackets can be expressed as real or imaginary parts of power series. After some elementary manipulations the result (4.15) follows.

Next we turn to some expansion theorems.

THEOREM 4.1. *Let* a_n *and* c_n *be real*, $n = 0, 1, 2, \ldots$, *and* $\limsup_{n\to\infty}|a_n|^{1/n} = \limsup_{n\to\infty}|c_n|^{1/n} = 1/\sigma$. *Then the series*

$$u(x, y) = \sum_{n=0}^{\infty} a_n u_n(x, y) = \mathrm{Re}\left(\sum_{n=0}^{\infty} a_n z^n\right) \tag{4.16}$$

(4.17) $$v(x, y) = \sum_{n=0}^{\infty} c_n v_n(x, y) = \mathrm{Im}\Big(\sum_{n=0}^{\infty} c_n z^n/(n+1)\Big)$$

converge to solutions of Laplace's equation for $|z| < \sigma$, *but do not converge everywhere in any including disk. Furthermore,* $u(x, 0) = \phi(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $v_y(x, 0) = \psi(x) = \sum_{n=0}^{\infty} c_n x^n$ *are analytic for* $|x| < \sigma$. *Conversely, if* (4.16) *and* (4.17) *converge for* $|z| < \sigma$, $\phi(x)$ *and* $\psi(x)$ *are analytic for* $|x| < \sigma$ *and* $\limsup_{n\to\infty} |a_n|^{1/n} \leqq 1/\sigma$, $\limsup_{n\to\infty} |c_n|^{1/n} \leqq 1/\sigma$.

This theorem is the counterpart of Theorem 3.1. The proof should be obvious from analytic function theory. Function theory in the complex plane can be obtained from these series. Let $c_{n-1} = na_n$, $n = 1, 2, 3, \ldots$, a_n real. Then if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $\mathrm{Re} f(z) = \sum_{n=0}^{\infty} a_n u_n(x, y)$, and $\mathrm{Im} f(z) = \sum_{n=1}^{\infty} (c_{n-1}/n) \mathrm{Im}(z^n) = \sum_{n=0}^{\infty} c_n v_n(x, y)$, and $f(z)$ is analytic for $|z| < \sigma$. An obvious modification will take care of the case where a_n is complex.

THEOREM 4.2. *If* $\limsup_{n\to\infty} 2n|b_n|^{1/n}/e = \sigma$, *then the series*

(4.18) $$U(x, y) = \sum_{n=0}^{\infty} b_n U_n(x, y), \quad V(x, y) = \sum_{n=1}^{\infty} b_n V_n(x, y)$$

converge absolutely for $|z| > \sigma$ *to solutions of Laplace's equation, but do not converge everywhere for* $|z| > \sigma - \varepsilon$, $\varepsilon > 0$.

PROOF. Using Stirling's formula for $n!$, it is easy to show that $\limsup_{n\to\infty} (2^n n! |b_n|)^{1/n} = \sigma$. Then using the estimates (4.12) one can show that the series (4.18) converge for $|z| > \sigma$. The first series diverges for $x = 0$, $|z| < \sigma$, while the second series diverges for $y = 0$, $|z| < \sigma$.

Let b_n be real and $\limsup_{n\to\infty} 2n|b_n|^{1/n}/e = \sigma$, then

$$V(x, y) = (1/2) \sum_{n=0}^{\infty} b_n V_{n+1}(x, y) = -\frac{1}{\pi} \sum_{n=0}^{\infty} 2^n n! b_n \mathrm{Re}(1/z^{n+1})$$

$$U(x, y) = \sum_{n=0}^{\infty} b_n U_n(x, y) = -\frac{1}{\pi} \sum_{n=0}^{\infty} 2^n n! b_n \mathrm{Im}(1/z^{n+1})$$

$$f(z) = V(x, y) + iU(x, y) = -\frac{1}{\pi} \sum_{n=0}^{\infty} 2^n n! b_n z^{-(n+1)}.$$

Obviously the radius of convergence is σ and the function $f(z)$ is analytic for $|z| > \sigma$, so the stated convergence criteria for the series (4.18) corresponds to analyticity in the complex plane outside a disk of radius σ.

There is also a theorem analogous to Theorem 3.3.

THEOREM 4.3. *The series* $\sum_{n=0}^{\infty} b_n U_n(x, y)$ *converges for* $y > \sigma \geqq 0$, *if and only if*

$$\sum_{n=0}^{\infty} b_n U_n(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixs}e^{-|s|y}\psi(s)\, ds\,, \tag{4.19}$$

where $\psi(z) \in \mathfrak{A}(1, \sigma)$ *and* $b_n = \psi^{(n)}(0)/[n!(-2i)^n]$.

PROOF. If the series converges, $\limsup_{n\to\infty} 2n|b_n|^{1/n}/e \leqq \sigma$, which implies that $\psi(z) \in \mathfrak{A}(1, \sigma)$. Conversely, since

$$K(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixs}e^{-|s|y} ds, \quad y > 0\,,$$

$$U_n(x, y) = (-2)^n \frac{\partial^n K(x, y)}{\partial x^n} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-2is)^n e^{ixs}e^{-|s|y}\, ds\,.$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} b_n U_n(x, y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}(-2is)^n b_n e^{ixs}e^{-|s|y}\, ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixs}e^{-|s|y}\psi(s)\, ds\end{aligned}$$

provided the term-by-term integration is valid. It will be if $\psi(s) \in \mathfrak{A}(1, \sigma)$ and $y > \sigma$.

Finally, we wish to investigate the question of when a series of heat polynomials or associated heat functions can be transformed by T_1 or T_2 term-by-term to obtain corresponding solutions of Laplace's equation.

THEOREM 4.4. *Suppose the series* (3.8) *converges for* $|t| < 1/4\tau$ *to a solution of* (2.1) *corresponding to the initial data function* $\phi(x) = \sum_{n=0}^{\infty} a_n x^n$. *Then the series*

$$u(x, y) = \sum_{n=0}^{\infty} a_n T_1 h_n(x, t) \tag{4.20}$$

$$v(x, y) = \sum_{n=0}^{\infty} a_n T_2 h_n(x, t) \tag{4.21}$$

converge to solutions of Laplace's equation for $|z| < \infty$, *with* $u(x, 0) = \phi(x)$ *and* $v_y(x, 0) = \phi(x)$.

PROOF. By Theorem 3.1, $\phi(x) \in \mathfrak{A}(2, \tau)$ and (3.7) holds. Therefore, $\phi(x)$ is entire and (4.16) and (4.17) converge for $|z| < \infty$ to harmonic functions satisfying the boundary conditions.

THEOREM 4.5. *Suppose the series* (3.9) *converges for* $0 < t < \infty$, *where* b_n *corresponds to an entire function* $\psi(x) \in \mathfrak{A}(1, \sigma)$ *as in Theorem* 3.3. *Then the series* (4.18) *converge to solutions of Laplace's equation for* $|z| > \sigma$.

PROOF. By Definition 3.1, $\limsup_{n\to\infty} n|a_n|^{1/n}/e \leqq \sigma$, where $a_n =$

$(-2i)^n b_n$. Then using Stirling's formula, it is easy to show that $\limsup_{n\to\infty}(|a_n|n!)^{1/n} \leqq \sigma$. The results follow from Theorem 4.2.

THEOREM 4.6. *Suppose the series* (3.9) *converges for* $0 < t < \infty$, *where* b_n *corresponds to an entire function* $\phi(x) \in \mathfrak{A}(\rho, \sigma)$, $\rho < 1$, *as in Theorem* 3.3. *Then the series* (4.18) *converge to solutions of Laplace's equation for* $|z| > 0$.

PROOF. By Definition 3.1, $\limsup_{n\to\infty} n|a_n|^{\rho/n}/\rho e \leqq \sigma$, where $a_n = (-2i)^n b_n$. Given $\varepsilon > 0$, there is an N, such that for $n \geqq N$, $n2^\rho|b_n|^{\rho/n}/\rho e \leqq \sigma + \varepsilon$ or $|b_n| \leqq 2^{-n}[\rho e(\sigma + \varepsilon)/n]^{n/\rho}$. Therefore,

$$\left|\sum_{n=N}^{\infty} b_n U_n(x, y)\right| \leqq \sum_{n=N}^{\infty} (2^n n!/\pi)|b_n||z|^{-(n+1)}$$

$$\leqq \frac{1}{\pi|z|}\sum_{n=N}^{\infty} [\rho e(\sigma + \varepsilon)/n]^{n/\rho} n!/|z|^n .$$

Applying the ratio test to this last series, we can show that $\sum_{n=0}^{\infty} b_n U_n(x, y)$ converges for $|z| > 0$. A similar argument applies to the other series in (4.18).

THEOREM 4.7. *Suppose that the series* (3.9) *converges for* $0 < t < \infty$, *where* b_n *is real and corresponds to an entire function* $\phi(x) \in \mathfrak{A}(\rho, \sigma)$, $\rho < 1$, *as in Theorem* 3.3. *Then the function*

$$f(z) = \sum_{n=0}^{\infty} c_n z^{-(n+1)} = -\frac{1}{\pi}\sum_{n=0}^{\infty} 2^n n! b_n z^{-(n+1)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} b_n V_{n+1}(x, y) + i\sum_{n=0}^{\infty} b_n U_n(x, y)$$

is an entire function of $1/z$ *of growth* $(\rho/(1 - \rho), (1 - \rho)\ (\sigma\rho^\rho)^{1/(1-\rho)})$

PROOF. As in Theorem 4.6, for $n \geq N$, $|b_n| \leqq 2^{-n}[\rho e(\sigma + \varepsilon)/n]^{n/\rho}$. Using Stirling's formula,

$$|c_n| \sim \sqrt{2/\pi}\, n^{n+1/2} e^{-n}[\rho e(\sigma + \varepsilon)/n]^{n/\rho} .$$

Now consider $(n/e\rho')|c_n|^{\rho'/n}$ with $\rho' = \rho/(1 - \rho)$. Then

$$|c_n|^{\rho'/n} \leqq (2/\pi)^{\rho'/2n} n^{\rho' + \rho/2n} e^{-\rho'}[\rho e(\sigma + \varepsilon)/n]^{\rho'/\rho},$$

$$(n/e\rho')|c_n|^{\rho'/n} \leqq (2/\pi)^{\rho'/2n}\left[\frac{e^{\rho'/\rho - \rho' - 1}\rho^{\rho'/\rho} n^{\rho'/2n}(\sigma + \varepsilon)^{\rho'/\rho}}{\rho' n^{\rho'/\rho - \rho' - 1}}\right].$$

However, $\rho'/\rho - \rho' - 1 = 0$. Hence,

$$(n/e\rho')|c_n|^{\rho'/n} \leqq (2/\pi)^{\rho'/2n}[\rho^{\rho'/\rho} n^{\rho'/2n}(\sigma + \varepsilon)^{\rho'/\rho}/\rho'$$

for $n \geq N$. Since ε is arbitrary,

$$\limsup_{n\to\infty} (n/e\rho')|c_n|^{\rho'/n} = (\sigma\rho)^{\rho'/\rho}/\rho' = (1-\rho)(\sigma\rho^\rho)^{1/(1-\rho)}.$$

Therefore, $f(1/z) \in \mathfrak{A}(\rho/(1-\rho), (1-\rho)(\sigma\rho^\rho)^{1/(1-\rho)})$.

5. Wave equation. In this section, we develop the analogous results for the one-dimensional wave equation

$$w_{tt}(x, t) = w_{xx}(x, t) \tag{5.1}$$

using the transmutation operators T_3(2.12) and T_4(2.13). When we apply these operators to the heat polynomials, the transforms exist in the conventional sense. However, when applied to the associated heat functions, they must be considered in the generalized sense and the results are distributions. We begin by noting that the fundamental solutions correspond under the transformations T_3 and T_4, i.e.,

$$\begin{aligned} D(x, t) &= T_3k(x, t) = t\sqrt{\pi}\mathscr{L}_\sigma^{-1}\{\sigma^{-1/2}k(x, 1/4\sigma)\}_{\sigma\to t^2} \\ &= t\mathscr{L}_\sigma^{-1}\{e^{-x^2\sigma}\}_{\sigma\to t^2} = (1/2)[\delta(x+t) + \delta(x-t)], \end{aligned} \tag{5.2}$$

where δ is the Dirac distribution. In the other case,

$$\begin{aligned} E(x, t) &= T_4k(x, t) = \frac{\sqrt{\pi}}{2}\mathscr{L}_\sigma^{-1}\{\sigma^{-3/2}k(x, 1/4\sigma)\}_{\sigma\to t^2} \\ &= \frac{1}{2}\mathscr{L}_\sigma^{-1}\{\sigma^{-1}e^{-x^2\sigma}\}_{\sigma\to t^2} = (1/2)[\theta(x+t) - \theta(x-t)], \end{aligned} \tag{5.3}$$

where θ is the Heaviside distribution.

When we apply the transformations T_3 and T_4 to the heat polynomials, we have the following sets of polynomials: for $n = 0, 1, 2, \ldots$.

$$\begin{aligned} w_n(x, t) &= t\sqrt{\pi}\mathscr{L}_\sigma^{-1}\Big\{n!\sum_{k=0}^{[n/2]} x^{n-2k}/2^{2k}k!(n-2k)!\sigma^{k+1/2}\Big\}_{\sigma\to t^2} \\ &= n!\sum_{k=0}^{[n/2]} x^{n-2k}t^{2k}/(n-2k)!(2k)! = ((x+t)^n + (x-t)^n)/2 \end{aligned} \tag{5.4}$$

$$\begin{aligned} \tilde{w}_n(x, t) &= \frac{1}{2}\sqrt{\pi}\mathscr{L}_\sigma^{-1}\Big\{n!\sum_{k=0}^{[n/2]} x^{n-2k}/2^{2k}k!(n-2k)!\sigma^{k+3/2}\Big\}_{\sigma\to t^2} \\ &= n!\sum_{k=0}^{[n/2]} x^{n-2k}t^{2k+1}/(n-2k)!(2k+1)! \\ &= ((x+t)^{n+1} - (x-t)^{n+1})/2(n+1). \end{aligned} \tag{5.5}$$

The following recurrence relations follow directly from (3.3):

$$\partial w_n/\partial x = nw_{n-1}, \quad \partial\tilde{w}_n/\partial x = n\tilde{w}_{n-1}. \tag{5.6}$$

For the associated functions, we have from (3.4)

$$\begin{aligned}W_n(x,t) &= T_3H_n(x,t) = (-2)^n(\partial^n/\partial x^n)D(x,t)\\ &= (-2)^n(\delta^{(n)}(x+t)+\delta^{(n)}(x-t))/2,\ n=0,1,2,\dots .\end{aligned} \tag{5.7}$$

$$\begin{aligned}\tilde{W}_n(x,t) &= T_4H_n(x,t) = (-2)^n(\partial^n/\partial x^n)E(x,t)\\ &= (-2)^n(\delta^{(n-1)}(x+t)-\delta^{(n-1)}(x-t))/2,\ n=1,2,3,\dots,\\ \tilde{W}_0(x,t) &= E(x,t)\,.\end{aligned} \tag{5.8}$$

The generating functions correspond under the transformations T_3 and T_4 as well. For the polynomials (5.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} w_n(x,t)a^n/n! &= T_3e^{ax+a^2t}\\ &= t\sqrt{\pi}\,\mathscr{L}_\sigma^{-1}\{\sigma^{-1/2}e^{ax}e^{a^2/4\sigma}\}_{\sigma\to t^2}\\ &= e^{ax}\cosh at\,.\end{aligned} \tag{5.9}$$

For the polynomials (5.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \tilde{w}_n(x,t)a^n/n! &= T_4e^{ax+a^2t}\\ &= \frac{1}{2}\sqrt{\pi}\,\mathscr{L}_\sigma^{-1}\{\sigma^{-3/2}e^{ax}e^{a^2/4\sigma}\}_{\sigma\to t^2}\\ &= (e^{ax}/a)\sinh at\,.\end{aligned} \tag{5.10}$$

For the associated functions, we make use of the fact that the Dirac distribution is a tempered distribution and therefore an ultradistribution [13]. Hence, it has a Taylor series expansion, and

$$\sum_{n=0}^{\infty} W_n(x,t)a^n/n! = T_3k(x-2a,t) = D(x-2a,t) \tag{5.11}$$

$$\sum_{n=0}^{\infty} \tilde{W}_n(x,t)a^n/n! = T_4k(x-2a,t) = E(x-2a,t)\,. \tag{5.12}$$

Corresponding to (3.6), we have the bilateral generating function

$$\begin{aligned}\sum_{n=0}^{\infty} w_n(x,t)W_n(y,s)/2^nn! &= (1/2)[D(x-y,s+t)\\ &\quad + D(x-y,s-t)]\,.\end{aligned} \tag{5.13}$$

Next we consider some expansion theorems.

THEOREM 5.1. *Let* $\limsup_{n\to\infty}|a_n|^{1/n} = 1/\sigma$. *Then the series*

$$w(x,t) = \sum_{n=0}^{\infty} a_nw_n(x,t) \tag{5.14}$$

converges absolutely for $|x| + |t| < \sigma$ *and does not converge everywhere* *for* $|x| + |t| < \sigma + \varepsilon$, $\varepsilon > 0$. *Furthermore*, $w(x, t)$ *is a solution of* (5.1) *with* $w(x, 0) = \phi(x) = \sum_{n=0}^{\infty} a_n x^n$, *analytic for* $|x| < \sigma$, $w_t(x, 0) = 0$. *Conversely, if the series* (5.14) *converges for* $|x| + |t| < \sigma$, *then the function* $w(x, t)$ *satisfies* (5.1) *and* $w(x, 0)$ *is analytic for* $|x| < \sigma$.

PROOF. A simple geometric argument shows that the region described by $|x| + |t| < \sigma$ is the region bounded by the lines $x + t = \sigma$, $x - t = \sigma$, $t - x = \sigma$, and $-t - x = \sigma$, and in this region $|x + t| < \sigma$ and $|x - t| < \sigma$. Since $\limsup_{n\to\infty} |a_n|^{1/n} = 1/\sigma$, for every $\varepsilon > 0$ there is an N such that $n \geqq N$ implies $|a_n|^{1/n} \leqq 1/\sigma + \varepsilon$. Let $|x + t| \leqq \rho < \sigma$ and $|x - t| \leqq \rho < \sigma$, then

$$\sum_{n=N}^{\infty} \left| a_n((x + t)^n + (x - t)^n)/2 \right| \leqq \sum_{n=N}^{\infty} (1 + \sigma\varepsilon)^n (\rho/\sigma)^n .$$

Since ε is arbitrary and $\rho < \sigma$, the comparison series converges, showing that the series (5.14) converges absolutely for $|x| + |t| < \sigma$ and uniformly in $|x| + |t| \leqq \rho < \sigma$. A similar argument holds for the second partial derivatives with respect to x and t. Therefore, the series (5.14) can be differentiated twice with respect to x and t, showing that equation (5.1) and the initial conditions are satisfied. At $t = 0$, the series $\sum_{n=0}^{\infty} a_n x^n$ diverges if $|x| > \sigma$. Conversely, if (5.14) converges for $|x| + |t| < \sigma$, then $\sum_{n=0}^{\infty} a_n x^n = w(x, 0)$ converges for $|x| < \sigma$ where $w(x, 0)$ is analytic.

THEOREM 5.2. *Let* $\limsup_{n\to\infty} |a_n|^{1/n} = 1/\sigma$. *Then the series*

$$\tilde{w}(x, t) = \sum_{n=0}^{\infty} a_n \tilde{w}_n(x, t) \tag{5.15}$$

converges absolutely for $|x| + |t| < \sigma$ *and does not converge everywhere for* $|x| + |t| < \sigma + \varepsilon$, $\varepsilon > 0$. *Furthermore*, $\tilde{w}(x, t)$ *is a solution of* (5.1) *with* $\tilde{w}(x, 0) = 0$, $\tilde{w}_t(x, 0) = \phi(x) = \sum_{n=0}^{\infty} a_n x^n$, *analytic for* $|x| < \sigma$. *Conversely, if the series* (5.14) *converges for* $|x| + |t| < \sigma$, *then* $\tilde{w}(x, t)$ *satisfies* (5.1) *and* $w_t(x, 0)$ *is analytic for* $|x| < \sigma$.

PROOF. The proof is very similar to the proof of Theorem 5.1.

THEOREM 5.3. *Suppose the series* (3.8) *converges for* $|t| < 1/4\tau$ *to a solution of* (2.1) *corresponding to the initial data function* $\phi(x) = \sum_{n=0}^{\infty} a_n x^n$. *Then the series*

$$w(x, t) = \sum_{n=0}^{\infty} a_n T_3 h_n(x, t) \tag{5.16}$$

$$\tilde{w}(x, t) = \sum_{n=0}^{\infty} a_n T_4 h_n(x, t) \tag{5.17}$$

converge to solutions of (5.1) *for* $|x| + |t| < \infty$, *with* $w(x, 0) = \phi(x)$, $w_t(x, 0) = 0$, $\tilde{w}(x, 0) = 0$, $\tilde{w}_t(x, 0) = \phi(x)$.

PROOF. By Theorem 3.1, $\phi(x) \in \mathfrak{A}(2, \tau)$ and (3.7) holds. Therefore, $\phi(x)$ is entire and (5.14) and (5.15) converge for $|x| + |t| < \infty$ to solutions of the wave equation (5.1) satisfying the stated initial conditions.

THEOREM 5.4. *If* $\limsup_{n\to\infty}(n/e\rho)|b_n|^{\rho/n} \leqq \tau$, $\rho > 0$, *then*

$$\sum_{n=0}^{\infty} b_n W_n(x, t), \quad \sum_{n=0}^{\infty} b_n \tilde{W}_n(x, t) \tag{5.18}$$

converge for all t, in the topology of ultradistributions [13], *to generalized solutions of* (5.1).

PROOF. We give the proof for the first series. The argument for the second will be similar. For any fixed t_0

$$\sum_{n=0}^{\infty} b_n W_n(x, t_0) = \frac{1}{2}\sum_{n=0}^{\infty} b_n(-2)^n[\delta^{(n)}(x + t_0) + \delta^{(n)}(x - t_0)]\,. \tag{5.19}$$

We shall show that this series converges in the space of ultradistributions Z'. Let $\psi(x)$ be any testing function in Z. Then $\psi \in \mathfrak{A}(1, \sigma)$ for some $\sigma > 0$, and for every $\varepsilon > 0$ there is an N such that $n \geqq N$ implies

$$|\psi^{(n)}(\pm t_0)/n!| \leqq [e(\sigma + \varepsilon)/n]^n, \quad |b_n| \leqq [\rho e(\tau + \varepsilon)/n]^{n/\rho}\,.$$

Let $m > n \geqq N$, and $\langle f(x), \psi(x)\rangle$ denote the continuous linear functional for $f \in Z'$. Then

$$\begin{aligned}
&\left\langle \frac{1}{2}\sum_{k=n}^{m} b_k w_k(x, t_0),\ \psi(x)\right\rangle \\
&\quad = \left\langle \frac{1}{2}\sum_{k=n}^{m} b_k(-2)^k[\delta^{(k)}(x + t_0) + \delta^{(k)}(x - t_0)],\ \psi(x)\right\rangle \\
&\quad = \frac{1}{2}\sum_{k=n}^{m} b_k 2^k[\psi^{(k)}(-t_0) + \psi^{(k)}(t_0)]\,, \\
&\frac{1}{2}\sum_{k=n}^{m}\left| b_k 2^k[\psi^{(k)}(-t_0) + \psi^{(k)}(t_0)]\right| \\
&\quad \leqq \sum_{k=n}^{m} 2^k k![\rho e(\tau + \varepsilon)/k]^{k/\rho}[e(\sigma + \varepsilon)/k]^k\,.
\end{aligned}$$

A ratio test now shows that the last sum is a Cauchy sequence as $n, m \to \infty$. Therefore, since Z' is closed under convergence the series (5.19) converges for each t_0.

THEOREM 5.5. *The series* $\sum_{n=0}^{\infty} b_n W_n(x, t)$ *converges in* Z', *if and only if*

$$\sum_{n=0}^{\infty} b_n W_n(x, t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \psi(s)[e^{-is(x+t)} + e^{-is(x-t)}]\, ds \tag{5.20}$$

where $\psi(s) = \sum_{n=0}^{\infty}(2is)^n b_n$ *is a distribution in* $\mathscr{D}'$.

PROOF. Since $D(x, t) = (1/2)[\delta(x + t) + \delta(x - t)] \in Z'$, it is a Fourier transform, i.e.,

$$D(x, t) = (1/2)[\delta(x, + t) + \delta(x - t)] = \mathscr{F}\{(1/2\pi)\cos ts\}$$
$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} (e^{its} + e^{-its})e^{-ixs}\, ds$$
$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} [e^{-is(x+t)} + e^{-is(x-t)}]\, ds$$

where the integral is taken in the generalized sense. Now

$$W_n(x, t) = (-2)^n \frac{\partial^n}{\partial x^n} D(x, t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} (2is)^n[e^{-is(x+t)} + e^{-is(x-t)}]\, ds$$

and

$$\sum_{n=0}^{\infty} b_n W_n(x, t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} (2is)^n b_n[e^{-is(x+t)} + e^{-is(x-t)}]\, ds$$
$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \phi(s)[e^{-is(x+t)} + e^{-is(x-t)}]\, ds$$

provided the term-by-term integration is justified. It is by the continuity of the Fourier transform in Z'.

6. Radial equations. In this section, we shall sketch the analogous results for the radial versions of the heat, Laplace, and wave equations. The radial heat equation is equation (2.4). The radial heat polynomials are given by

$$R_n^\mu(r, t) = n!(4t)^n L_n^{(\alpha)}(-r^2/4t) \tag{6.1}$$

$n = 0, 1, 2, \ldots, \alpha = \mu/2 - 1$, and $L_n^{(\alpha)}$ denotes the generalized Laguerre polynomial of degree n and index α. $R_n^\mu(r, t)$ satisfies (2.4) and $R_n^\mu(r, 0) = r^{2n}$. The associated radial heat functions are defined as generalized Appell transforms of radial heat polynomials,

$$\tilde{R}_\mu^n(r, t) = A_\mu\{R_n^\mu(r, t)\} = t^{-2n}k_\mu(r, t)R_n^\mu(r, -t) \tag{6.2}$$

$t > 0, n = 0, 1, 2, \ldots$. They satisfy equation (2.4) for $0 < t < \infty$. There are generating functions for the radial heat polynomials and associated functions as follows:

$$\sum_{n=0}^{\infty} R_n^\mu(r, t)a^n/n! = e^{ar^2/(1-4at)}/(1 - 4at)^{\mu/2},\ 1 - 4at > 0 \tag{6.3}$$

$$\sum_{n=0}^{\infty} \tilde{R}_n^\mu(r, t)a^n/n! = k_\mu(r, t + 4a),\ t + 4a \neq 0. \tag{6.4}$$

From [2], we state the following expansion theorems for the radial heat equation.

THEOREM 6.1. *Let* $\limsup_{n\to\infty} 4n|a_n|^{1/n}/e = 1/\sigma < \infty$. *Then the series*

$$R(r, t) = \sum_{n=0}^{\infty} a_n R_n^{\mu}(r, t) \tag{6.5}$$

converges absolutely in the time strip $|t| < \sigma$ *and not everywhere in any including stip. Furthermore,* $R(r, t)$ *is a solution of* (2.4) *with* $R(r, 0) = \sum_{n=0}^{\infty} a_n r^{2n}$. *Conversely, if* (6.5) *converges for* $|t| < \sigma$, *it satisfies* (2.4) *and* $R(r, 0) = \phi(r^2) = \sum_{n=0}^{\infty} a_n r^{2n}$ *and* $\phi(z) \in \mathfrak{A}(1, 1/4\sigma)$.

THEOREM 6.2. *If* $\limsup_{n\to\infty} 4n|b_n|^{1/n}/e = \sigma < \infty$, *then the series*

$$\tilde{R}(r, t) = \sum_{n=0}^{\infty} b_n \tilde{R}_n^{\mu}(r, t) \tag{6.6}$$

converges absolutely to a solution of (2.4) *for* $t > \sigma$ *and not everywhere for* $t > \sigma - \varepsilon$, $\varepsilon > 0$.

The radial Laplace equation for $u(r, y)$ is

$$u_{yy} + u_{rr} + (\mu - 1)(1/r)u_r = 0\,, \tag{6.7}$$

$r > 0$, $\mu > 1$. We can obtain polynomial solutions of (6.7) by applying the transformation T_1 to the radial heat polynomials:

$$\begin{aligned} u_n^{\mu}(r, y) &= T_1 R_n^{\mu}(r, t) = \frac{n!y}{\sqrt{\pi}} \mathscr{L}_{\sigma}\{\sigma^{-n-1/2} L_n^{(\alpha)}(-r^2\sigma)\}_{\sigma\to y^2} \\ &= \frac{(-1)^n n!\Gamma(1/2)}{\Gamma(n + 1/2)} (y^2 + r^2)^n P_n^{(\mu/2-1,-1/2)}\left(\frac{y^2 - r^2}{y^2 + r^2}\right), \end{aligned} \tag{6.8}$$

where the transform is interpreted in the generalized sense. Here $P_n^{(\alpha,\beta)}(z)$ is the Jacobi polynomial of degree n. The polynomials (6.8) satisfy the boundary conditions,

$$u_n^{\mu}(r, 0) = r^{2n}, \quad \partial u_n^{\mu}(r, 0)/\partial y = 0\,. \tag{6.9}$$

Another set of polynomial solutions of (6.7) can be obtained by transforming with T_2:

$$\begin{aligned} v_n^{\mu}(r, y) &= T_2 R_n^{\mu}(r, t) = -(n!/\sqrt{4\pi})\mathscr{L}_{\sigma}\{\sigma^{-n-3/2} L_n^{(\alpha)}(-r^2\sigma)\}_{\sigma\to y^2} \\ &= \frac{(-1)^n n!\Gamma(3/2)}{\Gamma(n + 3/2)} y(y^2 + r^2)^n P_n^{(\mu/2-1,1/2)}\left(\frac{y^2 - r^2}{y^2 + r^2}\right), \end{aligned} \tag{6.10}$$

where the transform again is in the generalized sense. The polynomials (6.10) satisfy the boundary conditions,

$$v_n^{\mu}(r, 0) = 0, \quad (\partial v_n^{\mu}/\partial y)(r, 0) = r^{2n}\,. \tag{6.11}$$

Associated functions can be obtained by transforming the associated heat functions:

$$
\begin{aligned}
U_n^\mu(r, y) &= T_1\tilde{R}_n^\mu(r, t) \\
&= \frac{(-16)^n n! y}{\pi^{\mu/2+1/2}} \mathscr{L}_\sigma\{\sigma^{n+\mu/2-1/2}e^{-r^2\sigma}L_n^{(\alpha)}(r^2\sigma)\}_{\sigma\to y^2} \qquad (6.12)\\
&= \frac{(-16)^n n!}{\pi^{\mu/2+1/2}} \frac{y\Gamma(n+\mu/2+1/2)}{(y^2+r^2)^{n+\mu/2+1/2}} P_n^{(\mu/2-1,1/2)}\left(\frac{y^2-r^2}{y^2+r^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
V_n^\mu(r, y) &= T_2\tilde{R}_n^\mu(r, t) \\
&= \frac{-(-16)^n n!}{2\pi^{\mu/2+1/2}} \mathscr{L}_\sigma\{\sigma^{n+\mu/2-3/2}e^{-r^2\sigma}L_n^{(\alpha)}(r^2\sigma)\}_{\sigma\to y^2} \qquad (6.13)\\
&= \frac{-(-16)^n n!}{2\pi^{\mu/2+1/2}} \frac{\Gamma(n+\mu/2-1/2)}{(y^2+r^2)^{n+\mu/2-1/2}} P_n^{(\mu/2-1,-1/2)}\left(\frac{y^2-r^2}{y^2+r^2}\right).
\end{aligned}
$$

There are various generating functions for the Jacobi polynomials, but the one which serves our purposes best is

$$
\begin{aligned}
&\sum_{n=0}^{\infty} ((1+\alpha+\beta)_n/(1+\alpha)_n)P_n^{(\alpha,\beta)}(x)a^n \\
&\quad = (1-a)^{-\alpha-\beta-1}{}_2F_1((\alpha+\beta+1)/2,\ (\alpha+\beta+2)/2; \qquad (6.14)\\
&\qquad 1+\alpha;\ 2(x-1)a/(1-a)^2),
\end{aligned}
$$

where $(c)_n$ is the factorial function

$$(c)_n = c(c+1)(c+2)\cdots(c+n-1) = \Gamma(c+n)/\Gamma(c)$$

and ${}_2F_1$ is a hypergeometric function. Using (6.14) it can easily be shown that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} \frac{\Gamma((\mu-1)/2+n)}{\Gamma(\mu/2+n)} a^n u_n^\mu(r, y) \qquad (6.15)\\
&\quad = \frac{\Gamma(1/2)\Gamma((\mu-1)/2)}{\Gamma(\mu/2)} \frac{{}_2F_1((\mu-1)/4, (\mu+1)/4; \mu/2; 4ar^2/R^2)}{R^{\mu/2-1/2}}.
\end{aligned}
$$

where $R = 1 + ay^2 + ar^2$. Similarly,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\Gamma(n+3/2)}{n!} \frac{\Gamma((\mu+1)/2+n)}{\Gamma(\mu/2+n)} a^n v_n^\mu(r, y) \qquad (6.16)\\
&\quad = \frac{y\Gamma(3/2)\Gamma((\mu+1)/2)}{\Gamma(\mu/2)} \frac{{}_2F_1((\mu+1)/4, (\mu+2)/4; \mu/2; 4ar^2/R^2)}{R^{\mu/2+1/2}}.
\end{aligned}
$$

For the associated functions, we have the generating functions

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{a^n}{n!} U_n^\mu(r, y)/\Gamma(\mu/2+n) \qquad (6.17)\\
&\quad = \frac{y\Gamma((\mu+1)/2)}{\Gamma(\mu/2)\pi^{\mu/2+1/2}} \frac{{}_2F_1((\mu+1)/4, (\mu+2)/4; \mu/2; 64ar^2/\rho^2)}{\rho^{\mu/2+1/2}},
\end{aligned}
$$

where $\rho = 16a + y^2 + r^2$, and

$$(6.18)\quad \begin{aligned}&\sum_{n=0}^{\infty} \frac{a^n}{n!} V_n^{\mu}(r, y))/\Gamma(\mu/2 + n)\\ &\quad = \frac{-\Gamma((\mu - 1)/2)}{2\Gamma(\mu/2)\pi^{\mu/2+1/2}} \frac{{}_2F_1((\mu - 1)/4, (\mu + 1)/4; \mu/2; 64ar^2/\rho^2)}{\rho^{\mu/2-1/2}}.\end{aligned}$$

Next we state, without proofs, some expansion theorems involving the polynomials (6.8) and (6.10), and the associated functions (6.12) and (6.13).

THEOREM 6.3. *Let* $\limsup_{n\to\infty}|a_n|^{1/n} = 1/\sigma < \infty$. *Then the series*

$$(6.19)\qquad u(r, y) = \sum_{n=0}^{\infty} a_n u_n^{\mu}(r, y)$$

converges absolutely for $r^2 + y^2 < \sigma^2$ *and diverges for* $r^2 + y^2 > \sigma^2$. *Furthermore,* $u(r, y)$ *is a solution of* (6.7) *with* $u(r, 0) = \sum_{n=0}^{\infty} a_n r^{2n}$, $u_y(r, 0) = 0$. *Conversely, if* (6.19) *converges for* $y^2 + r^2 < \sigma^2$, *then* $\limsup_{n\to\infty}|a_n|^{1/n} \leqq 1/\sigma$.

THEOREM 6.4. *Let* $\limsup_{n\to\infty}|a_n|^{1/n} = 1/\sigma < \infty$. *Then the series*

$$(6.20)\qquad v(r, y) = \sum_{n=0}^{\infty} a_n v_n^{\mu}(r, y)$$

converges absolutely for $r^2 + y^2 < \sigma^2$ *and diverges for* $r^2 + y^2 > \sigma^2$ *when* $y \neq 0$. *Furthermore,* $v(r, y)$ *is a solution of* (6.7) *with* $v(r, 0) = 0$, $v_y(r, 0) = \sum_{n=0}^{\infty} a_n r^{2n}$. *Conversely, if* (6.20) *converges for* $y^2 + r^2 < \sigma^2$, *then* $\limsup_{n\to\infty}|a_n|^{1/n} \leqq 1/\sigma$.

THEOREM 6.5. *Let* $\limsup_{n\to\infty} 4n|b_n|^{1/2n}/e = \sigma < \infty$. *Then the series*

$$(6.21)\qquad U(r, y) = \sum_{n=0}^{\infty} b_n U_n^{\mu}(r, y)$$

converges to a solution of (6.7) *for* $y^2 + r^2 > \sigma^2$ *and diverges for* $y^2 + r^2 < \sigma^2$ *when* $y \neq 0$.

THEOREM 6.6. *Let* $\limsup_{n\to\infty} 4n|b_n|^{1/2n}/e = \sigma < \infty$. *Then the series*

$$(6.22)\qquad V(r, y) = \sum_{n=0}^{\infty} b_n V_n^{\mu}(r, y)$$

converges to a solution of (6.7) *for* $y^2 + r^2 > \sigma^2$ *and diverges for* $y^2 + r^2 < \sigma^2$.

The radial wave equation for $w(r, t)$ is

$$(6.23)\qquad w_{tt} = w_{rr} + \frac{\mu - 1}{r} w_r ,$$

$r > 0$, $\mu > 1$. We can obtain polynomial solutions of (6.23) by applying the transforms T_3 to the radial heat polynomials:

$$\begin{aligned} w_n^\mu(r, t) &= T_3 R_n^\mu(r, t) = n!\, t\sqrt{\pi}\, \mathscr{L}_\sigma^{-1}\{\sigma^{-n-1/2} L_n^{(\alpha)}(-r^2\sigma)\}_{\sigma\to t^2} \\ &= \frac{n!\Gamma(1/2)}{\Gamma(n+1/2)}(t^2 - r^2)^n P_n^{(\mu/2-1,-1/2)}\left(\frac{t^2+r^2}{t^2-r^2}\right). \end{aligned} \tag{6.24}$$

These polynomials satsify the initial conditions

$$w_n^\mu(r, 0) = r^{2n}, \quad \partial w_n^\mu(r, 0)/\partial t = 0. \tag{6.25}$$

Another set of polynomial solutions of (6.23) can be obtained by transforming with T_4:

$$\begin{aligned} \tilde{w}_n^\mu(r, t) &= T_4 R_n^\mu(r, t) = n!\, \Gamma(3/2) \mathscr{L}_\sigma^{-1}\{\sigma^{-n-3/2} L_n^{(\alpha)}(-r^2\sigma)\}_{\sigma\to t^2} \\ &= \frac{n!\,\Gamma(3/2)}{\Gamma(n+3/2)}\, t(t^2 - r^2)^n P_n^{(\mu/2-1,1/2)}\left(\frac{t^2+r^2}{t^2-r^2}\right). \end{aligned} \tag{6.26}$$

These polynomials satisfy the initial conditions

$$\tilde{w}_n^\mu(r, 0) = 0, \quad \partial w_n(r, 0)/\partial t = r^{2n}. \tag{6.27}$$

Using the generating relation (6.14), we can obtain the generating functions:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} \frac{\Gamma(n+(\mu-1)/2)}{\Gamma(n+\mu/2)} a^n w_n^\mu(r, t) \\ &= \frac{\Gamma(1/2)\Gamma((\mu-1)/2)}{\Gamma(\mu/2)} \frac{{}_2F_1((\mu-1)/4, (\mu+1)/4; \mu/2; 4ar^2/R^2)}{R^{\mu/2-1/2}}, \end{aligned} \tag{6.28}$$

where $R = 1 - at^2 + ar^2$, and

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\Gamma(n+3/2)}{n!} \frac{\Gamma(n+(\mu+1)/2)}{\Gamma(n+\mu/2)} a^n \tilde{w}_n^\mu(r, t) \\ &= \frac{t\Gamma(3/2)\Gamma((\mu+1)/2}{\Gamma(\mu/2)} \frac{{}_2F_1((\mu+1)/4, (\mu+2)/4, \mu/2, 4ar^2/R^2)}{R^{\mu/2+1/2}}. \end{aligned} \tag{6.29}$$

We have the following expansion theorems involving the polynomials (6.24) and (6.26).

THEOREM 6.7. *Let* $\limsup_{n\to\infty}|a_n|^{1/n} = 1/\sigma < \infty$. *Then the series*

$$w(r, t) = \sum_{n=0}^{\infty} a_n w_n^\mu(r, t) \tag{6.30}$$

converges absolutely for $|r| + |t| < \sigma$ *and diverges for* $|r| + |t| > \sigma$. *Further more,* $w(r, t)$ *is a solution of* (6.23) *with* $w(r, 0) = \sum_{n=0}^\infty a_n r^{2n}$, $w_t(r, 0) = 0$. *Conversely, if* (6.30) *converges for* $|r| + |t| < \sigma$, *then* $\limsup_{n\to\infty}|a_n|^{1/n} \leqq 1/\sigma$.

THEOREM 6.8. *Let* $\limsup_{n\to\infty}|a_n|^{1/n} = 1/\sigma < \infty$. *Then the series*

$$\tilde{w}(r, t) = \sum_{n=0}^{\infty} a_n \tilde{w}_n^{\mu}(r, t) \tag{6.31}$$

converges absolutely for $|r| + |t| < \sigma$ *and diverges for* $|r| + |t| > \sigma$ *when* $t \neq 0$. *Furthermore,* $\tilde{w}(r, t)$ *is a solution of* (6.23) *with* $\tilde{w}(r, 0) = 0$, $\tilde{w}_t(r, 0) = \sum_{n=0}^{\infty} a_n r^{2n}$. *Conversely, if* (6.31) *converges for* $|r| + |t| < \sigma$ *then* $\limsup_{n\to\infty}|a_n|^{1/n} \leqq 1/\sigma$.

The associated functions are obtained by transforming the associated radial heat functions with the transformations T_3 and T_4;

$$\begin{aligned} W_n^{\mu}(r, t) &= T_3 \tilde{R}_n^{\mu}(r, t) \\ &= \frac{tn!(-16)^n}{\pi^{\mu/2-1/2}} \mathscr{L}_\sigma^{-1}\{\sigma^{n+\mu/2-1/2} e^{-r^2\sigma} L_n^{(\alpha)}(r^2\sigma)\}_{\sigma\to t^2} \\ &= \frac{tn!(-16)^n}{\pi^{\mu/2-1/2}} \cdot \frac{1}{\Gamma(1/2 - \mu/2 - n)} \cdot \frac{P_n^{(\mu/2-1,\,1/2)}\left(\dfrac{t^2 + r^2}{t^2 - r^2}\right)}{(t^2 - r^2)^{n+\mu/2+1/2}}, \end{aligned} \tag{6.32}$$

$t > r$, μ not an odd integer. Here the transform is taken is the generalized sense and the functions are treated as pseudo functions at $t = r$.

$$\begin{aligned} \tilde{W}_n^{\mu}(r, t) &= T_4 \tilde{R}_n^{\mu}(r, t) \\ &= \frac{n!(-16)^n}{2\pi^{\mu/2-1/2}} \mathscr{L}_\sigma^{-1}\{\sigma^{n+\mu/2-3/2} e^{-r^2\sigma} L_n^{(\alpha)}(r^2\sigma)\}_{\sigma\to t^2} \\ &= \frac{n!(-16)^n}{2\pi^{\mu/2-1/2}} \cdot \frac{1}{\Gamma(3/2 - \mu/2 - n)} \cdot \frac{P_n^{(\mu/2-1,\,-1/2)}\left(\dfrac{t^2 + r^2}{t^2 - r^2}\right)}{(t^2 - r^2)^{n+\mu/2-1/2}} \end{aligned} \tag{6.33}$$

$t > r$, μ not an odd integer. The transform is again taken in the generalized sense and the functions are treated as pseudo functions at $t = r$.

In the case where μ is an odd integer, the associated functions involve delta functions and derivatives of delta functions. We present, as illustration, the case $\mu = 3$.

$$W_n^3(r, t) = \frac{tn!(-16)^n}{\pi} \mathscr{L}_\sigma^{-1}\{\sigma^{n+1} e^{-r^2\sigma} L_n^{(1/2)}(r^2\sigma)\}_{\sigma\to t^2}. \tag{6.34}$$

If we use the fact that certain heat polynomials can be expressed in terms of Laguerre polynomials as follows: $h_{2n+1}(x, t) = n!(4t)^n L_n^{(1/2)}(-x^2/4t)$, and the result in equation (5.7), we obtain the transform

$$\begin{aligned} t\sqrt{\pi}\, &\mathscr{L}_\sigma^{-1}\{\sigma^{-1/2} H_{2n+1}(x, 1/4\sigma)\}_{\sigma\to t^2} \\ &= (-1)^n 4^{2n+1} n!\; tx \mathscr{L}_\sigma^{-1}\{\sigma^{n+1} e^{-x^2\sigma} L_n^{(1/2)}(x^2\sigma)\}_{\sigma\to t^2} \\ &= (-2)^{2n+1} \frac{\delta^{(2n+1)}(x + t) + \delta^{(2n+1)}(x - t)}{2}. \end{aligned}$$

Using this in (6.34), we have

$$W_n^3(r,t) = \frac{(-2)^{2n+1}}{4\pi r}\,\frac{\delta^{(2n+1)}(r+t)+\delta^{(2n+1)}(r-t)}{2}\,. \tag{6.35}$$

Similarly, we obtain

$$\tilde{W}_n^3(r,t) = \frac{(-2)^{2n+1}}{4\pi r}\,\frac{\delta^{(2n)}(r+t)-\delta^{(2n)}(r-t)}{2}\,. \tag{6.36}$$

If μ is an odd integer greater than three, the situation is much more complicated. In this case, we can reduce it to consideration of $\mu = 3$ using the relation

$$L_n^{(\alpha+1)}(x) = \sum_{k=0}^{n} L_k^{(\alpha)}(x)\,.$$

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