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PIECEWISE LINEAR APPROXIMATE FIBRATIONS

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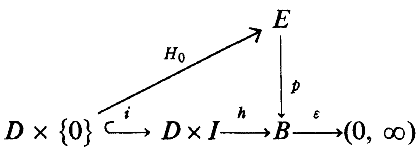
ABSTRACT. A pl map is an approximate fibration if and only if all the maps of any iterated mapping cylinder decomposition are homotopy equivalences. This leads to a classifying space.

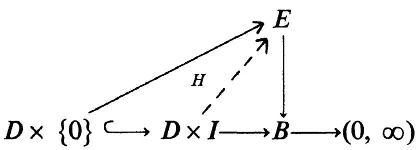
In this paper we provide a local characterization for piecewise linear maps which are also approximate fibrations. We use the approach taken by A. E. Hatcher in [5] to apply his higher simple homotopy theory to the classification of PL fibrations. This leads to a classifying space for PL approximate fibrations. All of the proofs are elementary but, hopefully, the results have sufficient intrinsic interest to justify themselves.

I. Local characterization. We begin with a few basic definitions and assumptions. All metric spaces will be equipped with a fixed metric, denoted by $d(a, b)$. In the case of the unit interval, $I = [0, 1]$, the metric is the usual absolute value of $a - b$.

DEFINITION I.1. (See [1].) A proper surjection p: $E \to B$ of metric spaces is an approximate fibration if, for every metric space D, every lifting problem



has an approximate solution



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That is to say, given maps H_0, h, and ε with the indicated domains and ranges such that $hi = pH_0$, there is a homotopy $H: D \times I \to E$ such that

1) $Hi = H_0$ and

2) $d(pH(d, t), h(d,t)) < \varepsilon h(d,t)$ for every $(d, t) \in D \times I$. (This will be denoted briefly as $d(pH, h) < \varepsilon$.)

Note that if B (and hence E) is compact, we may take ε to be constant. Theorem 2.6 of [2] permits us to replace "any metric space" with "any disc" in the choice of D. This theorem, together with the simplicial approximation theorem, shows that a PL map which is an approximate fibration must be an "approximate fibration in the PL category", and conversely. That is, we may require D to be a PL space and h, H_0 and H to be PL maps as suits our needs. This, of course, is subject to the obvious restriction that if H is to be PL, H_0 must be PL also.

An apparently weaker condition, that of being completely movable, has been shown by Coram and Duvall [2] to be equivalent provided that E and B are locally compact, separable, metric ANRs. An apparently stronger condition, local homotopy product structure, has been shown by the author [4] to be equivalent provided, in addition, the fiber has the shape of a compact ANR. We shall need the definition of complete movability.

Definition I.2. [See [2].) A proper map $p: E \to B$ is a completely movable map provided that for each $b \in B$ and each neighborhood U of the fiber F_b there is a neighborhood V of F_b in U such that if F_c is any fiber in V and W is any neighborhood of F_c in V, then there is a homotopy $H: V \times I \to U$ such that $H(x, 0) = x$ and $H(x, 1) \in W$ for each x in V and $H(x, t) = x$ for each x in F_c and each t in I.

Proposition I.3. *Let $p: E \to B$ be a proper surjection of locally compact separable metric* ANR*s. Then p is an approximate fibration if and only if p is completely movable.*

Proof. This is Proposition 3.6 of [2].

We shall also need a construction described by Hatcher [5].

Definition I.4. Let

$$L_0 \xrightarrow{f_1} L_1 \xrightarrow{f_2} L_2 \cdots \xrightarrow{f_k} L_k$$

be a chain of PL maps. The iterated mapping cylinder, imc, $M(f_1, \ldots, f_k)$ is defined inductively as follows:

1. $M(f_1)$ is the mapping cylinder of f_1, and
2. $M(f_1, \ldots, f_k)$ is the mapping cylinder of the composition

$$M(f_1, \ldots, f_{k-1}) \xrightarrow{p} L_{k-1} \xrightarrow{f_k} L_k$$

where P is the projection of the previously defined (iterated) mapping cylinder to its image. Thus $M(f_1, f_2)$ is the mapping cylinder of

$$M(f_1) \longrightarrow L_1 \xrightarrow{f_2} L_2$$

and so on.

I.5. Note that there is a canonical projection of $M(f_1, \ldots, f_k)$ to Δ^k, which can be obtained by iterating the canonical projection of $M(f_1)$ to $I = \Delta^1$. This projection is easier to visualize if one views Δ^k as the iterated mapping cylinder of $* \to * \cdots \to *$, where $*$ denotes any singleton space. The projection is then obtained by ignoring the L_i and projecting onto the I "factors" of the iterated mapping cylinder.

DEFINITION I.6. An iterated mapping cylinder decomposition, imcd, of $p\colon E \to B$ is a triangulation T of B such that, for each simplex σ of T, $p^{-1}(\sigma)$ is given as $M(f_1^\sigma, f_2^\sigma, \ldots, f_k^\sigma)$ where $k = \dim \sigma$ and $p\colon p^{-1}(\sigma) \to \sigma$ is the canonical projection. Moreover, these identifications are to be compatible with passage to faces of σ.

Hatcher shows [5] that any PL map admits an iterated mapping cylinder decomposition. Briefly, one first chooses triangulations of E and B such that p is simplicial and then chooses barycentric subdivisions of these triangulations such that p is still simplicial. To define the f_γ^j, one looks at a simplex $\gamma = \langle \hat{\sigma}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_k \rangle$ spanned by barycenters $\hat{\sigma}_j$ of simplices σ_j such that σ_{j+1} is a face of σ_j for each j. If $\hat{\tau}_j$ is a barycenter such that $p(\hat{\tau}_j) = \hat{\sigma}_j$, then τ_j maps to σ_j and $\tau_j \cap p^{-1}(\partial\sigma_j)$ is a face τ_{j+1} of τ_j, which maps to σ_{j+1}. The maps f_j^σ are obtained by sending each $\hat{\tau}_j$ to $\hat{\tau}_{j+1}$ and extending linearly. (See [5, p. 105]).

Hatcher shows that, for PL fibrations, the f_σ^j are always PL contractible maps. We show that for PL approximate fibrations they are always homotopy equivalences, but first we need a lemma which permits passage to subcomplexes.

LEMMA I.7. *If $p\colon E \to B$ is a* PL *approximate fibration, and C is a finite subcomplex of B, then $p|\colon p^{-1}(C) \to C$ is also an approximate fibration.*

PROOF. The approximate homotopy lifting property is almost hereditary in that approximate liftings of homotopies in C take values in neighborhoods of $p^{-1}(C)$. If ε' is chosen carefully, the lifting can be pushed into $p^{-1}(C)$ (using a regular neighborhood) in such a way that the total error at C is less than ε.

We note in passing that the lemma is also true for an arbitrary subcomplex C. We shall only use the lemma in the case $C = \Delta^1$.

PROPOSITION I.8. *If $p\colon E \to B$ is a* PL *approximate fibration, then the* *maps in any iterated mapping cylinder decomposition of p are homotopy equivalences.*

PROOF. Let f be such a map and restrict attention to $p\colon p^{-1}(I) \to I$ where $p^{-1}(I) = M_f$. According to Lemma I.7, p is still an approximate fibration. For nomenclature, let $I = [0, 1]$, $L_0 = p^{-1}(0)$, $L_1 = p^{-1}(1)$ and $f\colon L_0 \to L_1$. We apply the approximate homotopy lifting property to the following lifting problem:

$$\begin{array}{ccccc} & & & & M_f \\ & \nearrow^{1} & & & \downarrow p \\ M_f \times \{0\} & \hookrightarrow & M_f \times I & \xrightarrow{h} & [0, 1] \quad \varepsilon = 1/2 \end{array}$$

where $h(x, s, t) = \min(s, 1 - t)$ if $(x, s) \in L_0 \times I$ and $h(y, t) = 1 - t$ if $y \in L_1$.

Thus we obtain $H\colon M_f \times I \to M_f$ which deforms M_f into $L_0 \times [0, 1/2]$, keeping $L_0 \times \{0\}$ in $L_0 \times [0, 1/2]$. This can easily be improved to a strong deformation retraction of M_f to L_0. Thus f is a homotopy equivalence.

A converse holds.

PROPOSITION I.9. *Let $p\colon E \to B$ be a* PL *map. If the maps in some iterated mapping cylinder decomposition for p are all homotopy equivalences, then p is an approximate fibration.*

PROOF. We will show that p is completely movable. From the proof of Proposition I.8 we see that the proposition holds for the special case of $p\colon M_f \to I$, where f is a homotopy equivalence from L_0 to L_1. For if in Definition I.2 we take V to be the inverse image of a small open set, we have that V is either the cartesian product of L_0 with an interval or a copy of the open mapping cylinder of f. In either case, since f is a homotopy equivalence, V will strong deformation retract to any F_c in V.

For the general case, we reduce to the special case as follows. Let b and U be given and choose $V = p^{-1}(V')$ where V' is a neighborbood of b chosen so that

1) V' intersects each simplex in either an empty or a convex set,

2) $p^{-1}(V') \subseteq U$ (p is proper), and

3) $V' \subseteq \bigcup\{\mathring{\sigma}\colon b \in \sigma\} \cup \{b\}$. Here σ represents a simplex of the iterated mapping cylinder decomposition of p. For instance, take V' to be the open star of b in some small triangulation.

Now, let $c \in V'$ and $p^{-1}(c) \subseteq W \subseteq V$ be given. We must show that $p^{-1}(V')$ will deform through U into W fixing $p^{-1}(c)$. In fact, we show $p^{-1}(V')$ will strong deformation retract to $p^{-1}(c)$. This is done in separate stages.

Stage 1. Let σ be the simplex of lowest dimension containing both b and c. By downward induction, we may strong deformation retract V to $p^{-1}(\sigma)$, using the mapping cylinders. Thus, without loss of generality, we may assume that B is a single simplex σ and E is a single iterated mapping cylinder.

Stage 2. Join b and c by an interval I. Note that, by convexity, $I \subseteq V'$. There are four possible cases.

Case a. b and c are both in $\mathring{\sigma}$. In this case, $V' \subseteq \mathring{\sigma}$, $p^{-1}(I) = L_0 \times I$ and $V = p^{-1}(V') = L_0 \times V'$. In this case we can cross a strong deformation retraction of V' to c with the identity on L_0 to complete the strong deformation retraction of V to $p^{-1}(c)$.

Case b. $b \in \sigma$ and $c \in \partial\sigma$. This case does not occur, by construction of V (condition 3).

Case c. $b \in \partial\sigma$ and $c \in \partial\sigma$. In this case, by construction of V again, b and c must be in a common face τ of σ. This violates minimality of σ and so this case does not occur.

Case d. $b \in \partial\sigma$ and $c \in \mathring{\sigma}$. In this case, $p^{-1}(I)$ is a copy of a mapping cylinder from $L_0 = p^{-1}(c)$ to $L_1 = p^{-1}(b)$. Now, $V' \cap \partial\sigma$ is contained in a single face τ of $\partial\sigma$ and so, as in Case a, $V' \cap \partial\sigma$ will strong deformation retract to b. This will extend to a strong deformation retraction of V' to I which is covered by a strong deformation retraction of V to $p^{-1}(I)$. Now, since $p^{-1}(I) = M_f$ and f is a homotopy equivalence, the special case completes Case *d*. Thus, in any case, V will strong deformation retract to $p^{-1}(c)$. This completes the proof of proposition I.9.

COROLLARY I.10. *Let $f: E \to B$ be a* PL *map. There is a triangulation of B such that f is an approximate fibration if and only if $f|f^{-1}(B^1): f^{-1}(B^1) \to B^1$ is an approximate fibration.*

II. The classifying space. In this part we provide some of the basic structure and definitions for working with approximate fibrations in the PL category. In this category, the situation is considerably improved over the more customary setting in the category of ANRs. For instance, pullbacks of PL approximate fibrations by PL maps are PL approximate fibrations.

To establish notation, if C is a topological category, the associated semi-simplicial set is denoted NC and the geometric realization of NC is denoted BC. (See [13].) The associated homotopy category is denoted $\bar{C}$, that is, the objects of $\bar{C}$ are those of C, but the morphisms are homotopy classes of those of C. We shall work with the category H of finite simplicial complexes and pl maps, with a full subcategory $H(K)$ with only one object K, and with a full subcategory H_K whose objects are all homotopy equivalent to K.

An imcd T of a pl approximate fibration $p: E \to B$ determines a map $C_p: B \to BH$ as follows. For any simplex σ of T consider $p^{-1}(\sigma) = m(f_1, \ldots, f_k)$. Define $C_p(\sigma)$ to be the image in BH of the simplex

$$L_0 \xrightarrow{f_1} L_1 \cdots \xrightarrow{f_k} L_k$$

of NH. For each triangulation T of B the assignment $p \to C_p$ is clearly a bijection between the set of imc decompositions of pl approximate fibrations over T and semisimplicial maps of B to NH. We shall also refer to the map $B \to NH$ as C_p. Indeed, all of our results could have been done using NH, but we prefer to use the concrete world of spaces and maps when possible. We first show that the homotopy class $[C_p]$ in $[B, BH]$ is independent of the choices of triangulation T and the maps of the decomposition $f_1 \ldots f_k$.

LEMMA II.1. *Let $p: E \to B$ have* imcds *for the triangulation T of B whose maps are denoted f_i^σ and g_i^σ respectively for each simplex σ of T. The corresponding classifying maps C_p and C'_p are homotopic.*

PROOF. Note that, for each i and σ, $m(f_i^\sigma) \cong p^{-1}(\sigma_i) \cong m(g_i^\sigma)$ where σ_i is the edge of σ covered by $p: m(f_i^\sigma) \to \sigma$. We thus have

$$L_{i-1} \hookrightarrow p^{-1}(\sigma_i) \xrightarrow{r_f} L_i$$

and

$$L_{i-1} \hookrightarrow p^{-1}(\sigma_i) \xrightarrow{r_g} L_i$$

where r_f and r_g are mapping cylinder contractions and the compositions are f_i^σ and g_i^σ, respectively. Since each of r_f and r_g is a homotopy equivalence, we have that f and g are homotopic. (In fact, more is true, but this is all we need.)

Let h_i^σ be a homotopy from f_i^σ to g_i^σ for each σ and i. Now define k_i^σ: $L_{i-1} \times I \to L_i \times I$ by $k_i(\ell, t) = (h_i^\sigma(\ell, t), t)$.

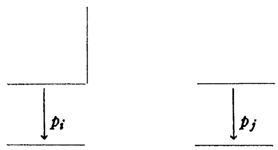
Now, for each σ, note that $\bar{E}^\sigma = m(k_1^\sigma, \ldots, k_i^\sigma)$ maps to $\sigma \times I$ in an obvious way. In fact, since each k_i^σ restricts to a homotopy equivalence of $L_{i-1} \times \{t\}$ to $L_i \times \{t\}$ at each stage, $\bar{E}^\sigma$ is a parameterized family of pl approximate fibrations. It is easily shown by methods similar to those of Proposition I.9 that $\bar{E}^\sigma \to \sigma \times I$ is a pl approximate fibration and that the union of these projections over all σ in T is a pl approximate fibration E over $B \times I$. We can easily choose a triangulation of $B \times I$ extending T on each end so that the projection is simplicial and admits an imcd extending the given ones on each end. The map $C\bar{p}$ which classifies $\bar{p}$: $\bar{E} \to B \times I$ is thus a homotopy of C_p to $C'_{\bar{p}}$.

LEMMA II.2. *Let $p: E \to B$ be a* PL *approximate fibration with a given* imcd *for a triangulation T of B. If T' is any subdivision of T, then the classifying maps C_p for T and C_p' for T' are homotopic.*

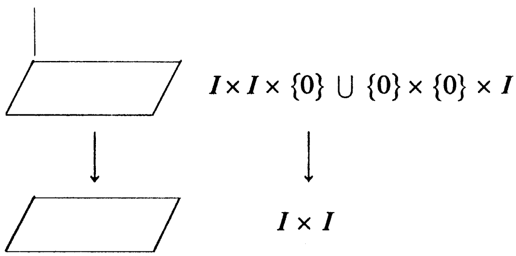
PROOF. Consider the PL approximate fibration $p \times 1: E \times I \to B \times I$. Note that $C = \{\sigma \times \{0\}, \sigma \times \{1\}; \sigma \in T\}$ is a cell decomposition of $B \times \partial I$. By 1.4 of **[6]**, choose a simiplicial subdivision of $B \times I$ with no new vertices. Now, consider the subcomplex $B \times \{0, 1\}$ which is now triangulated by T on each end. We subdivide $B \times \{0, 1\}$ by retaining T on $B \times \{0\}$ and using T' on $B \times \{1\}$. This subdivision of $B \times \{0, 1\}$ extends to a subdivision of $B \times I$. Performing a relative subdivision of $(B \times I, B \times \{0, 1\})$ if necessary, we obtain an imcd of $B \times I$ which agrees on each end with that determined by T and T'. The classifying map of this imcd is a homotopy from C_p to C_p'. This completes the lemma.

COROLLARY II.3. *A* PL *approximate fibration $p: E \to B$ uniquely determines a homotopy class $[C_p]$ in $[B, BH]$.*

The converse to this corollary is false as can be seen from the following simple example. Let $i: \{0\} \to [0, 1]$ be the inclusion and let $j: \{0\} \to \{0\}$ be the identity. Both maps are homotopy equivalences and so $p_i: m(i) \to I$ and $p_j: m(j) \to I$ are approximate fibrations.



They are quite distinct since they have distinct total spaces; however, their classifying maps are homotopic via the classifying map for the following PL approximate fibration:



This example and the proof of the lemma suggest the following definition.

DEFINITION II.4. Two PL approximate fibrations $p: E \to B$ and $p': E' \to B$ are *A*-equivalent if there is a PL approximate fibration $p: \bar{E} \to B \times I$ such that

1) $p^{-1}(B \times \{0\})$ is PL homeomorphic to E,

2) $p^{-1}(B \times \{1\})$ is PL homeomorphic to E', and

3) Under these identifications, $P = p$ over $B \times \{0\}$ and $P = p'$ over $B \times \{1\}$.

Note that A-equivalence is an equivalence relation on the class of PL approximate fibrations. To verify transitivity one will have to show that $P_1 \cup P_2 \colon \bar{E}_1 \cup \bar{E}_2 \to (B \times [0, 1]) \cup (B \times [1, 2])$ is a PL approximate fibration. This is most easily done using the local characterization of Part I. The reflexive and symmetric properties are trivially satisfied.

Note that the proofs of the two preceeding lemmas actually prove that changing a triangulation of B within its equivalence class or changing the maps f_i^q within their homotopy classes yields approximate fibrations which are A-equivalent.

Definition II.5. If $c \colon B \to N\bar{H}$ is a simplicial map (for a triangulation, T of B), the induced PL approximate fibration $c^* \colon E_c \to B$ is constructed using c as a blueprint as follows. For each 1-simplex τ of T, choose a homotopy equivalence f_τ out of $c(\tau)$ which is a homotopy class of homotopy equivalences. This induces a lifting also denoted $c \colon B \to NH$. E_c is the union of iterated mapping cylinders so that $c^{*-1}(\sigma) = m(c(\sigma))$. These mapping cylinders intersect according to the incidence relations of T. That is, if σ_1 and σ_2 share a common face τ, then $m(c(\tau)) \subseteqq m(c(\tau_1))$ and $m(c(\tau)) \subseteqq m(c(\sigma_2))$ are identified. Since c is well defined, the projections of $m(c(\sigma_1))$ and $m(c(\sigma_2))$ agree on $m(c(\tau))$. Note that by construction c^* has an imcd using homotopy equivalences and that $C_{c^*} = c$. Note also that altering our selection of these homotopy equivalences within homotopy class will only alter E_c within its A-equivalence class.

Lemma II.6. *If $c \colon B \to N\bar{H}$ and $d \colon B \to N\bar{H}$ are homotopic (via h), then c^* and d^* are A-equivalent.*

Proof. Via h^*.

Lemma II.7. *If $p \colon E \to B$ and $p' \colon E' \to B$ are A-equivalent* PL *approximate fibrations (via P), then C_p and $C_{p'}$ are homotopic.*

Proof. Via C_P.

If we let $[AF(B)]$ be the set of A-equivalence classes of pl approximate fibrations over B, we obtain maps $\Theta \colon [AF(B)] \to [B, NH]$ and $\Psi \colon [B, N\bar{H}] \to [AF(B)]$ by setting $\Theta(p) = C_p$ and $\Psi(c) = c^*$.

The two preceding lemmas show that Θ and Ψ are well-defined. By construction, we have the following proposition.

Proposition II.8. *The following diagram commutes for each* pl *space B.*

$$\begin{array}{ccc} [AF(B)] & \longrightarrow & [B, NH] \\ & \nwarrow & \downarrow \\ & & [B, N\bar{H}] \end{array}$$

COROLLARY II.9. *Each of the above maps is a bijection.*

COROLLARY II.10. *NH and $N\bar{H}$ are (weakly) homotopy equivalent semi-simplicial sets.*

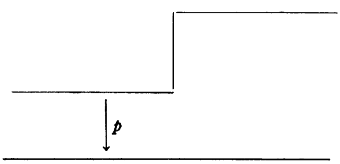
COROLLARY II.11. *There is a bijection between $[B, NH_K]$ and the set of A-equivalence classes of* pl *approximate fibrations overB with fibre homotopy equivalent to K, $[AF_K(B)]$.*

PL fibrations and quasi-fibrations. According to A. E. Hatcher [5], the space S_K of PL contractible maps of spaces simply equivalent to K classifies PL fibrations in just the same way as we have done for PL approximate fibrations. We thus obtain a commuting diagram.

$$\begin{array}{ccc} [F_K(B)] & \longrightarrow & [B, S_K] \\ \downarrow & & \downarrow \\ [AF_K(B)] & \longrightarrow & [B, NH_K] \end{array}$$

Here, $[F_K(B)]$ is the set of fiber homotopy equivalence classes of PL fibrations over B with homotopy fibre K. Hatcher uses quasi-fibrations (QF) (see [3]) to obtain his exact sequence by mapping S_K to BG_K. It would be of interest to compare NH of $N\bar{H}$ with BG_K. The following examples may provide some insight.

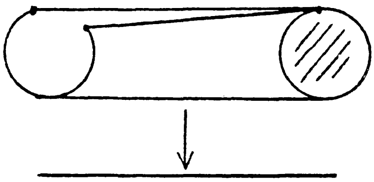
EXAMPLE II.12. Let E be the union of M_i and M_j, where $i: \{0\} \to [0, 1]$ and $j: \{1\} \to [0, 1]$ are inclusions. Let $E = M_i \cup M_j$ where the intersection is along $[0, 1]$. Note that both i and j are elementary expansions and so they are homotopy equivalences (in fact, simple equivalences). The union of the mapping cylinder projections thus provides a PL approximate fibration $p: E \to I = [0, 1]$.



We note that P is not a quasi-fibration. Let $h: \{*\} \times I \to I$ be given by $H((*, t)) = t$. Note that h has unique liftings when restricted to $\{*\} \times [0, 1/2]$ and $\{*\} \times [1/2, 1]$, but these do not agree at $(*, 1/2)$. Thus there is no lifting. Since p is $1 - 1$ over 0, any quasi-lifting (permitting an initial vertical homotopy) would also be a lifting. Thus there is no quasi-lifting.

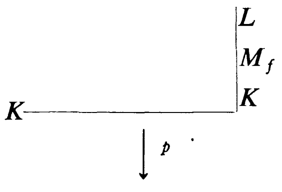
The following example, due to John Walsh, shows that simple equivalences may not yield quasi-fibrations even for a single mapping cylinder projection.

EXAMPLE II.13. Let f be the exponential map wrapping the unit interval once around the boundary of the unit disk D. Note that f is a (simple) homotopy equivalence since both I and D are contractible. Thus p: $M_f \to I$ is an approximate fibration.



One can also show that p is not a quasi-fibration by considering a homotopy of the projection of the $I \vee I$ which occurs as the mapping cylinder restricted to $\{0, 1\}$ at the top of the diagram. The projection maps each copy of I to I by the identity. We follow this by the homotopy (downstairs) $h(s, t) = \min(s, 1 - t)$. It is easily seen that any vertical homotopy of the initial inclusion must fix the wedge point, by continuity, so that any quasi-lifting would imply a lifting. Again, there is clearly no lifting.

EXAMPLE II.14. Let f: $K \to L$ be any pl homotopy equivalence and let j: $K \to M_f$ be inclusion. Note that j is contractible over its image and that M_f will strong deformation retract to the image of j. Consider p: $M_f \to I$ given by $p(k, t) = t$ for $(k, t) \in K \times I \subseteqq M_j$ and $p(m) = 1$ for $m \in M_f \subseteqq M_j$.



Note that M_j will fibre deformation retract to $K \times I$, a fibration. Thus M_j is fibre homotopy equivalent to a fibration and so it is a quasi-fibration [3]. Note that f and j have the same Whitehead torsion.

The question of quasi-fibrations now raises some interesting problems. Clearly $F \subseteqq QF \subseteqq AF$. One would expect a classifying space somewhere between H_K and S_K to identify the maps in the imcd of a quasi-fibration. Our second example shows that a space containing even the simple equivalences is too large. Our first example is, over each half of its range, a quasi-fibration. Thus one would expect to find, in addition to some possible restriction on the maps, a restriction on the intersection of adjacent iterated mapping cylinders.

Our third example shows that there are quasi-fibrations whose imcd maps are not simple. Joint work (unpublished) with John Walsh generalizes Example 3 to identify a class of maps which can be used in imcd to yield a projection which, over each simplex, is a quasi-fibration. Unfortunately it is not known that an arbitrary pl quasi-fibration yields maps in this class. Our first example is the limit of PL fibrations. (One can simply tilt E a little before projecting.) It is also easily seen that it is $\mathcal{A}$-equivalent to a fibration (hence to a quasi-fibration).

REMARK II.15. I propose the following questions.

1. Is there a quasi-fibration which is not a limit of fibrations?
2. Is there a PL approximate fibration which is not $\mathcal{A}$-equivalent to a quasi-fibration?
3. What class of homotopy equivalences is used in the iterated mapping cylinder decomposition of quasi-fibrations?
4. What conditions are imposed on the intersections?

Frank Quinn has shown that, provided $Wh(\pi_1(K) \oplus Z^j) = 0$ for all j, an approximate fibration of high dimensional generalized manifolds may be approximated by a block bundle projection. See **[12]** for a precise statement of results. A negative answer to questions 1 or 2 would provide interesting insight at a fairly accessible level to this very deep work of Quinn.

In view of this apparent scarcity of information about the local topological structure of quasi-fibrations, the following result is perhaps of interest, since $NH(K)$ classifies pl quasi-fibrations up to fibre homotopy equivalence **[14]**.

PROPOSITION II.16. *The inclusion functor I: $H(K) \to H_K$ induces a (weak) homotopy equivalence $NH(K) \to NH_K$.*

PROOF. The following diagram of categories and functors commutes.

$$\begin{array}{ccc} H_K & \xrightarrow{R} & \bar{H}_K \\ {\scriptstyle I}\uparrow & & \uparrow{\scriptstyle I} \\ H(K) & \xrightarrow{S} & \bar{H}(K) \end{array}$$

Thus, the corresponding diagram of semi-simplicial sets and maps commutes.

$$\begin{array}{ccc} NH_K & \xrightarrow{NR} & N\bar{H}_K \\ {\scriptstyle NI}\uparrow & & \uparrow{\scriptstyle N\bar{I}} \\ NH(K) & \xrightarrow{NS} & N\bar{H}(K) \end{array}$$

Now, we have shown that NR is a weak homotopy equivalence. $\bar{I}$ is the inclusion of a monoid $\bar{H}(K)$ into a category $\bar{H}_K$ in which every object is isomorphic to every other, thus $\bar{I}$ is a natural equivalence of categories [**9**, p. 52 and p. 61]. Thus $N\bar{I}$ is a weak homotopy equivalence.

To show that NS is a weak homotopy equivalence, we use the same method as for NR. That is to say, we must find something to classify. To do this let $AF(B, K)$ be the pl approximate fibrations over B with fibre K. Let two such maps be B-equivalent if they are A-equivalent by $\bar{E}$ such that $\bar{E}$ is in $AF(B \times I, K)$. All of the results of the previous section have direct analogs in this setting. In particular, NS is a weak homotopy equivalence. This completes the proposition.

Corollary II.7. *Each* pl *approximate fibration is A-equivalent to one with fixed fibre.*

Corollary II.18. *There are bijections between the sets of*
1) *A-equivalence classes of* pl *approximate fibrations,*
2) *B-equivalence classes of* pl *approximate fibrations with fibre K, and*
3) *fibre homotopy equivalence classes of* pl *quasi-fibrations.*

Corollary II.19. *There are* pl *approximate fibrations* (*over spheres*) *which are not A-equivalent to* pl *fibrations* [**5**].

Corollary II.20. *The classifying map* $S_K \to BG_K$ *in Hatcher's sequence is induced from the inclusion functor of the category of* pl *contractible maps into the category of* pl *homotopy equivalences.*

Pullbacks.

Definition II.21. Given a diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

we can extend it to a pullback diagram

$$\begin{array}{ccc} f^*(E) & \xrightarrow{f^*} & E \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

as follows. Define $f^*(E) = \{(b', e) \in B' \times E: f(b') = p(e)\}$. The maps p^* and f^* are simply restrictions of the projections of $B' \times E$. p^* is called the pullback of p by f.

The pullback of an approximate fibration is not, generally, an approximate fibration. However, we do have the following result.

PROPOSITION II.22. *If $p: E \to B$ is a PL approximate fibration and if $f: B' \to B$ is a PL map, then $p^*: f^*(E) \to B'$ is a PL approximate fibration.*

PROOF: Triangulate E, B, and B' so that all maps are simplicial and p has an iterated mapping cylinder decomposition. We shall identify the spaces E, B, and B' with these simplicial complexes and we shall show that p^* admits an imcd with homotopy equivalences. To this end, let σ be a simplex of B' and suppose $f(\sigma) = \tau$. Order the vertices of τ consistently with the chain of maps

$$L_0 \xrightarrow{f_1^\tau} L_1 \longrightarrow \cdots \xrightarrow{f_k^\tau} L_k$$

which describes $p^{-1}(\tau)$. Thus $\tau = \langle \tau_1, \ldots, \tau_m \rangle$ and $\tau_j = p(L_j)$. Now, order the vertices of σ consistently with the vertices of τ: $\sigma = \langle \sigma_1, \ldots, \sigma_n \rangle$. We do this for each simplex σ of B' respecting incidence relations. That is, the ordering induced on a common face by two simplices is the same. This is forced by the corresponding property in B except for vertices which have the same image under f. These may be ordered arbitrarily with the provision that the same ordering is retained when the same vertices are reconsidered as vertices of another simplex.

We are now ready to identify $p^{*-1}(\sigma)$ as an imc. First, observe that $p^{*-1}(\sigma_j) = p^{-1}(f(\sigma_j)) = p^{-1}(\tau_k) = L_k$ for some k. We define the maps g_j: $p^*(\sigma_{j-1}) \to p^{*-1}(\sigma_j)$ as follows. If $f(\sigma_{j-1}) = f(\sigma_j) = \tau_k$, then $g_j = 1$: $L_k \to L_k$. If $f(\sigma_{j-1}) = \tau_{k-1}$ and $f(\sigma_j) = \tau_k$, then $g_j = f_k$: $L_{k-1} \to L_k$. By construction of the pullback $p^{*-1}(\sigma) = m(g_1, \ldots, g_n)$. This decomposition is consistent with restriction to faces because this was true of the decomposition of p and since we have ordered the vertices of B' consistently with those of B. Thus we obtain an imcd for p^*. Note that all maps are either the identity or maps appearing in the imcd of p. In either case, they are homotopy equivalences. Thus, p^* is an approximate fibration. This concludes Proposition II.22.

COROLLARY II.23. *Classifying maps are natural. That is, if p^* is the pullback by f of a PL approximate fibration p, then $[C_p \circ f] = [C_{p^*}]$.*

COROLLARY II.24. *If f, g: $B' \to B$ are homotopic and p: $E \to B$ is a PL approximate fibration, then the pullbacks by f and g are A-equivalent.*

Larry Husch has pointed out that the theorem on pullbacks is already known; since for PL maps the notions of shape fibration ([**10**] and [**11**]) and approximate fibration coincide and shape fibrations admit pullbacks ([**7**] and [**8**]).

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