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DIFFERENTIAL INEQUALITY TECHNIQUES AND SINGULAR PERTURBATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. The use of differential inequality techniques in the study of singular perturbation problems for ordinary and partial differential equations has a short but interesting history. In this paper we delineate briefly several avenues of investigation, starting from the original work and following its influence up to the present.

2. The work of Nagumo. In the late 1930's the Japanese mathematician M. Nagumo wrote two beautiful papers on differential inequalities, one concerned with two-point boundary value problems and the other specifically with a singularly perturbed initial value problem. These papers comprise the opening chapter of our story, and so let us spend a little time describing their contents.

The first paper of Nagumo [29] (cf. also [21]) concerns the existence of solutions of the Dirichlet problem

$$(2.1)\qquad \begin{aligned} &y'' = f(t, y, y'),\ a < t < b,\\ &y(a) = A,\ y(b) = B, \end{aligned}$$

where f is a continuous function on $[a, b] \times \mathbf{R}^2$. Under the assumptions that f grows at most quadratically with respect to y' (that is, $f(t, y, z) = O(|z|^2)$ as $|z| \to \infty$ for (t, y) in bounded subsets of $[a, b] \times \mathbf{R}$) and that there exists a $C^{(2)}$-bounding pair of functions $\{\alpha, \beta\}$ for the problem (2.1) (that is, functions α and β of class $C^{(2)}[a, b]$ satisfying $\alpha \leqq \beta$, $\alpha(a) \leqq A \leqq \beta(a)$, $\alpha(b) \leqq B \leqq \beta(b)$, and for t in (a, b), $\alpha'' \geqq f(t, \alpha, \alpha')$, $\beta'' \leqq f(t, \beta, \beta')$), Nagumo showed the existence of a $C^{(2)}$-solution $y = y(t)$of (2.1) satisfying $\alpha(t) \leqq y(t) \leqq \beta(t)$ in $[a, b]$. Thus he was not only able to prove that a solution exists, but also to give an estimate for this solu-

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tion in terms of the bounding functions α and β. This latter facet of Nagumo's result provides the crucial connection with later work on singularly perturbed boundary value problems of the form

$$\varepsilon y'' = f(t, y, y'), \ a < t < b,$$
$$y(a, \varepsilon) = A, \ y(b, \varepsilon) = B,$$

for small, positive values of ε; cf. §3 below.

Later writers have extended this theory to problems like (2.1) with Neumann and Robin boundary data, and to second-order systems with fairly general types of boundary conditions. We mention only the papers [**9**], [**26**], [**22**] which the interested reader can consult for details and additional references.

The second paper of Nagumo [**30**] was published in 1939 and it concerns the singularly perturbed initial value problem

$$\begin{aligned} &\varepsilon y'' = f(t, y, y'), \ 0 < t \leqq T < \infty, \\ &y(0, \varepsilon) = y_0, \ y'(0, \varepsilon) = y_1, \end{aligned} \tag{2.2}$$

where f is a continuous function on $[0, T] \times \mathbf{R}^2$ and ε is a small, positive parameter. (A specific problem of this form was suggested to Nagumo by a chemist.) In order to get some idea of how the solution of (2.2) behaves as $\varepsilon \to 0^+$, let us consider the simple example

$$\begin{aligned} &\varepsilon y'' = -ky', \ 0 < t \leqq T, \\ &y(0, \varepsilon) = y_0, \ y'(0, \varepsilon) = y_1 \neq 0, \end{aligned} \tag{2.3}$$

for k a positive constant. (The results that follow are not valid if $k \leqq 0$.) The exact solution $y = y(t, \varepsilon)$ of (2.3) satisfies in $[0, T]$

$$y(t, \varepsilon) = y_0 + O(\varepsilon k^{-1}|y_1|\exp[-kt\varepsilon^{-1}])$$

and

$$y'(t, \varepsilon) = O(|y_1|\exp[-kt\varepsilon^{-1}]),$$

that is,

$$\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = y_0 \text{ for } 0 \leqq t \leqq T$$

and

$$\lim_{\varepsilon \to 0^+} y'(t, \varepsilon) = 0 \text{ for } 0 < \delta \leqq t \leqq T$$

$(0 < \delta < T)$. Thus the solution of (2.3) converges uniformly to the constant value y_0 as $\varepsilon \to 0^+$ in $[0, T]$, while its derivative converges uniformly to zero as $\varepsilon \to 0^+$ in any proper closed subinterval of $(0, T]$. The nature of this nonuniformity is made even clearer when we note that

$$y_1 = \lim_{\varepsilon\to 0^+}\lim_{t\to 0^+} y'(t, \varepsilon) \neq \lim_{t\to 0^+}\lim_{\varepsilon\to 0^+} y'(t, \varepsilon) = 0.$$

The limiting value y_0 is of course the solution of the "reduced" initial value problem $0 = -ku'$, $u(0) = y_0$, obtained from (2.3) by formally setting $\varepsilon = 0$ and dropping the second initial condition.

Based on the behavior of the solution of the simple linear problem (2.3) we expect that the solution of the general problem (2.2) should behave in a similar manner provided the function f has certain properties. In **[30]** Nagumo was able to prove such a result by assuming the following: (1) the reduced problem $0 = f(t, u, u')$, $0 < t \leqq T$, $u(0) = y_0$, has a solution $u = u(t)$ of class $C^{(2)}[0, T]$;
(2) the function u is "stable" in the sense that there exists a positive constant k such that

$$f_{y'}(t, u(t), u'(t)) \leqq -k < 0 \text{ in } [0, T]; \text{ and}$$

(3) The function u is "stable in the boundary layer (region of nonuniform behavior)" at $t = 0$ in the sense that if $u'(0) \neq y_1$, then

$$f_{y'}(0, y_0, \lambda) \leqq -k < 0,$$

for all values of λ between $u'(0)$ and y_1.

The precise result of Nagumo is that under these assumptions there exists an $\varepsilon_0 > 0$ such that for each ε in $(0, \varepsilon_0]$ the problem (2.2) has a $C^{(2)}$-solution $y = y(t, \varepsilon)$ satisfying in $[0, T]$

$$(2.4)\qquad \begin{aligned} y(t, \varepsilon) &= u(t) + O(\varepsilon k^{-1}|y_1 - u'(0)|\exp[-kt\varepsilon^{-1}]) + O(\varepsilon),\\ y'(t, \varepsilon) &= u'(t) + O(|y_1 - u'(0)|\exp[-kt\varepsilon^{-1}]) + O(\varepsilon). \end{aligned}$$

Nagumo proved this theorem by means of a differential inequality lemma on first-order systems, which we now describe.

Consider then the initial value problem

$$(2.5)\qquad \begin{aligned} \boldsymbol{x}' &= \boldsymbol{g}(t, \boldsymbol{x}), \; 0 < t \leqq T,\\ \boldsymbol{x}(0) &= \boldsymbol{\xi}, \end{aligned}$$

where $\boldsymbol{x}, \boldsymbol{g}$ and $\boldsymbol{\xi}$ are in $\mathbf{R}^n$, and $\boldsymbol{g}$ is a continuous function on $[0, T] \times \mathbf{R}^n$. If there exists a $C^{(1)}$-bounding pair of functions $(\boldsymbol{w}, \boldsymbol{W})$ for the problem (2.5) (that is, functions $\boldsymbol{w} = (w_1, \ldots, w_n)$ and $\boldsymbol{W} = (W_1, \ldots, W_n)$ of class $C^{(1)}[0, T]$ satisfying $w_i \leqq W_i$, $w_i(0) \leqq \xi_i \leqq W_i(0)$, and for t in $(0, T]$, $w_i' \leqq g_i(t, \hat{\boldsymbol{w}}_i)$, $W_i' \geqq g_i(t, \hat{\boldsymbol{W}}_i)$ for $i = 1, \ldots, n$, where $\hat{\boldsymbol{w}}_i = (x_1, \ldots, x_{i-1}, w_i, x_{i+1}, \ldots, x_n)$ and $\hat{\boldsymbol{W}}_i = (x_1, \ldots, x_{i-1}, W_i, x_{i+1}, \ldots, x_n)$ for all x_j in $[w_j, W_j]$, $j \neq i$), Nagumo showed the existence of a $C^{(1)}$-solution $\boldsymbol{x} = \boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))$ of (2.5) satisfying $w_i(t) \leqq x_i(t) \leqq W_i(t)$ in $[0, T]$ for $i = 1, \ldots, n$. Let us hasten to point out that the differential inequalities for the bounding functions $\hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{W}}$ place no restriction on the monotonicity properties of the function $\boldsymbol{g}$. However, if $\boldsymbol{g}$ is quasi-monotone nondecreasing with respect to $\boldsymbol{x}$ for each fixed t in $(0, T]$ (that is, if for $i = 1, \ldots, n$, $g_i(t, \boldsymbol{x}) \geqq g_i(t, \boldsymbol{y})$ for any vectors $\boldsymbol{x}, \boldsymbol{y}$ such that $x_i = y_i$ and $x_j \geqq y_j$, $j \neq i$), then the differential inequalities for w_i and W_i simplify to $w_i' \leqq g_i(t, \boldsymbol{w})$ and $W_i' \geqq g_i(t, \boldsymbol{W})$; cf. [3].

Using this result Nagumo was able to study the perturbed problem (2.2) by converting it to the first-order system

$$\begin{aligned} &x_1' = x_2,\ x_1(0) = y_0, \\ &\varepsilon x_2' = f(t, x_1, x_2),\ x_2(0) = y_1, \end{aligned} \tag{2.6}$$

and then constructing a $C^{(1)}$-bounding pair $(\boldsymbol{w}, \boldsymbol{W})$ in order to obtain the estimates in (2.4). This construction is based on the expectation that the actual solution of (2.6) for small $\varepsilon > 0$ follows the solution u of the corresponding reduced problem except in a vanishingly small neighborhood of $t = 0$. Near $t = 0$ the functions u and u' must be supplemented by the "boundary layer corrector" terms containing the rapidly changing exponential function (cf. (2.4)). Thus, in one stroke, Nagumo established the existence of a solution of (2.2) and gave an estimate for it which improves as ε decreases to zero.

This basic result has been improved to include a sharper estimate for the admissible "boundary layer jump" $|y_1 - u'(0)|$ (cf. Assumption (3) above) in **[15]**. It was also extended to system analogs of the problem (2.2) in **[27]** and **[16]**, which the reader can consult for further discussion and additional references.

In the remainder of this paper we shall examine how these results of Nagumo have been used by later writers to examine singularly perturbed boundary value problems.

3. The work of Brish. In 1954 the Soviet mathematician N. I. Brish published a short Doklady note **[2]** (cf. also **[10]**) in which he successfully applied Nagumo's result on the boundary value problem (2.1) to the singularly perturbed Dirichlet problems

$$\begin{aligned} &\varepsilon y'' = F(t, y, y'),\ a < t < b, \\ &y(a, \varepsilon) = A,\ y(b, \varepsilon) = B, \end{aligned} \tag{3.1}$$

where ε is a small, positive parameter, and $F(t, y, y') = h(t, y)$ or $F(t, y, y') = f(t, y, y')$ with $f_{y'} \not\equiv 0$ are continuous functions on their appropriate domains. He also applied Nagumo's result on the perturbed initial value problem (2.2) to the Robin problem

$$\begin{aligned} &\varepsilon y'' = f(t, y, y'),\ a < t < b, \\ &p_1 y(a, \varepsilon) - p_2 y'(a, \varepsilon) = A,\ q_1 y(b, \varepsilon) + q_2 y'(b, \varepsilon) = B, \end{aligned} \tag{3.2}$$

where $p_i, q_i \geqq 0$, $i = 1, 2$, and $p_1^2 + p_2^2 > 0$, $q_1^2 + q_2^2 > 0$. Let us take a moment to discuss his results.

In the case of (3.1) with $F(t, y, y') = h(t, y)$ we obtain an insight into the behavior of a solution as $\varepsilon \to 0^+$ by considering a simple example like

$$\begin{aligned} &\varepsilon y'' = my, 0 < t < 1, \\ &y(0, \varepsilon) = A \neq 0, y(1, \varepsilon) = B \neq 0, \end{aligned} \tag{3.3}$$

for m a positive constant. The exact solution of (3.3) (to asymptotically negligible terms) is

$$y(t, \varepsilon) = O(|A|\exp[-(m\varepsilon^{-1})^{1/2}(t - a)]) + O(|B|\exp[-(m\varepsilon^{-1})^{1/2}(b - t)]),$$

this is,

$$\lim_{\varepsilon\to 0^+} y(t, \varepsilon) = 0 \text{ for } 0 < \delta \leqq t \leqq 1 - \delta < 1, (0 < \delta < 1), \tag{3.4}$$

and so the solution has boundary layers at both $t = 0$ and $t = 1$. We note that the limit $u \equiv 0$ is the solution of the reduced ($\varepsilon = 0$) equation $mu = 0$, and that the limiting relation (3.4) does not obtain if $m \leqq 0$. For the general problem (3.1) with $F = h$ Brish assumed the following:

(1) the reduced equation $h(t, u) = 0$, $a < t < b$, has a solution $u = u(t)$ of class $C^{(2)}[a, b]$;

(2) the function u is "stable" in the sense that there exists a positive constant m such that

$$h_y(t, u(t)) \geqq m > 0 \text{ in } [a, b]; \text{ and}$$

(3) the function u is "stable in the boundary layers" at $t = a$ and $t = b$ in the sense that if $u(a) \neq A$, then

$$h_y(a, \xi) \geqq m > 0$$

for all values of ξ between $u(a)$ and A, while if $u(b) \neq B$, then

$$h_y(b, \eta) \geqq m > 0$$

for all values of η between $u(b)$ and B.

Under these assumptions he was able to show that there exists an $\varepsilon_0 > 0$ such that for each ε in $(0, \varepsilon_0]$ the problem (3.1) with $F = h$ has a $C^{(2)}$-solution $y = y(t, \varepsilon)$ satisfying in $[a, b]$

$$y(t, \varepsilon) = u(t) + O(v_L(t, \varepsilon)) + O(v_R(t, \varepsilon)) + O(\varepsilon),$$

for

$$v_L(t, \varepsilon) = |A - u(a)|\exp[-(m\varepsilon^{-1})^{1/2}(t - a)]$$

and

$$v_R(t, \varepsilon) = |B - u(b)|\exp[-(m\varepsilon^{-1})^{1/2}(b - t)].$$

Brish proved this result by setting

$$\alpha = \alpha(t, \varepsilon) = u(t) - v_L(t, \varepsilon) - v_R(t, \varepsilon) - \varepsilon\gamma m^{-1}$$

and

$$\beta = \beta(t, \varepsilon) = u(t) + v_L(t, \varepsilon) + v_R(t, \varepsilon) + \varepsilon\gamma m^{-1},$$

with $\gamma = \|u''\|_\infty$, and verifying that these functions satisfy the required differential inequalities.

As regards the problem (3.1) with $F(t, y, y') = f(t, y, y')$, a simple example again reveals what to expect of a solution as $\varepsilon \to 0^+$. The problem is

$$(3.5) \qquad \begin{aligned} &\varepsilon y'' = -ky', 0 < t < 1, \\ &y(0, \varepsilon) = A, y(1, \varepsilon) = B, A \neq B, \end{aligned}$$

for k a positive constant, and its exact solution (to asymptotically negligible terms) is

$$y(t, \varepsilon) = B + O(|B - A|\exp[-kt\varepsilon^{-1}]).$$

Thus, we see that

$$\lim_{\varepsilon\to 0^+} y(t, \varepsilon) = B \text{ for } 0 < \delta \leqq t \leqq 1, (0 < \delta < 1),$$

that is, y has a boundary layer at $t = 0$ and $|y'(0, \varepsilon)| = O(\varepsilon^{-1}) \to \infty$ as $\varepsilon \to 0^+$. (If k is a negative constant, then the roles of $t = 0$ and $t = 1$ are reversed, and so the solution of (3.5) is

$$y(t, \varepsilon) = A + O(|A - B|\exp[-|k|(1 - t)\varepsilon^{-1}]),$$

that is, y has a boundary layer at $t = 1$.) Let us hasten to point out that the limit $u \equiv B$ is the solution of the reduced equation $-ku' = 0$ which satisfies the right-hand boundary condition $u(1) = B$. (If k is negative, then the limit of the solution away from $t = 1$ is $u \equiv A$, which satisfies the reduced equation and the left-hand boundary condition $u(0) = A$.)

For the general problem (3.1) with $F(t, y, y') = f(t, y, y')$ Brish assumed the following:

(1) the reduced problem $f(t, u, u') = 0$, $a < t < b$, $u(b) = B$, has a solution $u = u(t)$ of class $C^{(2)}[a, b]$;

(2) the function u is "stable" in the sense that there exists a positive constant k such that

$$f_{y'}(t, u(t), u'(t)) \leqq -k < 0 \text{ in } [a, b];$$

(3) the function u is "stable in the boundary layer" at $t = a$ in the sense that if $u(a) \neq A$, then for all z in $\mathbf{R}$,

$$f_{y'}(a, \lambda, z) \leqq -k < 0$$

for all values of λ between $u(a)$ and A;
(4) $f(t, y, z) = O(|z|^2)$ as $|z| \to \infty$, for (t, y) in the domain $\mathscr{D}$ which is the union of the sets

$$\{a \leqq t \leqq a + \delta,\ |y - u(t)| \leqq |A - u(a)|\exp[-k(t - a)\varepsilon^{-1}] + c\varepsilon\}$$

and

$$\{a + \delta \leqq t \leqq b,\ |y - u(t)| \leqq c\varepsilon\},$$

where $0 < \delta < b - a$ and c is a known positive constant depending on f and u.

Under these assumptions Brish showed that there exists an $\varepsilon_0 > 0$ such that for each ε in $(0, \varepsilon_0]$ the problem (3.1) with $F = f$ has a solution $y = y(t, \varepsilon)$ satisfying in $[a, b]$

$$y(t, \varepsilon) = u(t) + O(|A - u(a)|\exp[-k(t - a)\varepsilon^{-1}]) + O(\varepsilon).$$

Nagumo's result is applicable because of Assumption (4), and Brish constructed a $C^{(2)}$-bounding pair of functions for (3.1) of the form (in the case where $u(a) \geqq A$, for example):

$$\alpha(t, \varepsilon) = u(t) - (u(a) - A)\exp[\lambda_1(t - a)] - \varepsilon\gamma\ell^{-1}(\exp[\lambda_2(t - b)] - 1)$$

and

$$\beta(t, \varepsilon) = u(t) + \varepsilon\gamma\ell^{-1}(\exp[\lambda_2(t - b)] - 1).$$

Here $\|f_y(t, y, u'(t))\|_\infty \leqq \ell$, for (t, y) in $\mathscr{D}$, $\lambda_1 = -k\varepsilon^{-1} + \ell k^{-1} + O(\varepsilon)$ and $\lambda_2 = -\ell k^{-1} + O(\varepsilon)$ are the negative zeros of the characteristic polynomial $\varepsilon\lambda^2 + k\lambda + \ell$ of the linearized equation (provided $\varepsilon < k^2(4\ell)^{-1}$), and $\gamma = \|u''\|_\infty$.

For the study of the Robin problem (3.2) Brish made use of Nagumo's result on the perturbed initial value problem (2.2) by arguing as follows. If $p_2 > 0$ in the left-hand boundary condition, then this relation can be replaced by the two initial conditions

$$\begin{aligned} y(a, \varepsilon) &= u(a) + \lambda \\ y'(a, \varepsilon) &= -p_2^{-1}[p_1(u(a) + \lambda) - A] \end{aligned} \tag{3.6}$$

where λ is an adjustable (shooting) parameter. Thus, a solution of the differential equation $\varepsilon y'' = f(t, y, y')$ satisfying the initial conditions (3.6) depends on the parameter λ as well as on ε, that is, $y = y(t, \varepsilon; \lambda)$. If there exists a value of λ, say λ^*, such that $q_1 y(b, \varepsilon; \lambda^*) + q_2 y'(b, \varepsilon; \lambda^*) = B$, then the function $y = y^*(t, \varepsilon) \equiv y(t, \varepsilon; \lambda^*)$ is a solution of the original problem (3.2). Under the appropriate assumptions on the function f, this solution y^* behaves like the solution of (2.2) near $t = a$, in that for t in $[a, b]$

$$y^*(t, \varepsilon) = u(t) + O(\varepsilon k^{-1}|A - p_1 u(a) + p_2 u'(a)|\exp[-k(t - a)\varepsilon^{-1}] + O(\varepsilon).$$

Here $u = u(t)$ is the solution of the corresponding reduced problem $f(t, u, u') = 0$, $a < t < b$, $q_1u(b) + q_2u'(b) = B$. A sufficient condition for the existence of such a value λ^* is that $q_2f_y(b, u(b), u'(b)) - q_1f_{y'}(b, u(b), u'(b)) \neq 0$.

Since the appearance of Brish's note several writers on singular perturbations have used his approach coupled with Nagumo's theory to study the boundary value problems (3.1) and (3.2). We mention only the work of Dorr, Parter and Shampine [**4**], Habets and Laloy [**8**] and the author [**10–12**; **14**], which contain many references and applications.

4. Extensions to systems. The basic results of Nagumo in §2 on the scalar boundary value problem (2.1) and the vector initial value problem (2.5) can be combined to yield an existence and comparison theorem for the initial-boundary value problem in (a, b)

$$\boldsymbol{x}' = \boldsymbol{g}(t, \boldsymbol{x}, y, y'),\ \boldsymbol{x}(a) = \boldsymbol{\xi}. \tag{4.1}$$

$$y'' = f(t, \boldsymbol{x}, y, y'),\ y(a) = A,\ y(b) = B, \tag{4.2}$$

where $\boldsymbol{x}$, $\boldsymbol{g}$ and $\boldsymbol{\xi}$ are in $\mathbf{R}^n$, and the functions $\boldsymbol{g}$ and f are continuous on their respective domains. Namely, suppose there exist vector-valued functions $\boldsymbol{w}$, $\boldsymbol{W}$ and scalar functions α, β such that $\{\boldsymbol{w}, \boldsymbol{W}\}$ is a $C^{(1)}$-bounding pair for (4.1) uniformly with respect to (y, y') in $[\alpha, \beta] \times \mathbf{R}$ and $\{\alpha, \beta\}$ is a $C^{(2)}$-bounding pair for (4.2) uniformly with respect to $\boldsymbol{x}$ in $[\boldsymbol{w}, \boldsymbol{W}]$. Suppose also that $f(t, \boldsymbol{x}, y, z) = O(|z|^2)$ as $|z| \to \infty$ for $(t, \boldsymbol{x}, y)$ in $[a, b] \times [\boldsymbol{w}, \boldsymbol{W}] \times [\alpha, \beta]$. Then it is not difficult to show (by combining Nagumo's original arguments) that the problem (4.1), (4.2) has a smooth solution $(\boldsymbol{x}, y) = (\boldsymbol{x}(t), y(t))$ satisfying $\boldsymbol{w}(t) \leqq \boldsymbol{x}(t) \leqq \boldsymbol{W}(t)$ and $\alpha(t) \leqq y(t) \leqq \beta(t)$ in $[a, b]$; cf. [**19**]. This result applies, in particular, to the scalar n-th order differential equation

$$z^{(n)} = f(t, z, z', \ldots, z^{(n-1)}),\ a < t < b,$$

with boundary conditions of the form

$$z^{(j)}(a) = A_j,\ z^{(n-2)}(b) = B_{n-2}$$

or

$$z^{(n-2)}(a) = A_{n-2},\ z^{(j)}(b) = B_j$$

for $j = 0, \ldots, n - 2$.

One frequently encounters in applications boundary value problems for the perturbed equation

$$\varepsilon z^{(n)} = f(t, z, z', \ldots, z^{(n-1)})$$

(cf. for example similarity solutions of the Navier-Stokes equations at high Reynolds number [**6**]). So the above differential inequality theorem can provide sharp estimates on the behavior of solutions and their derivatives as $\varepsilon \to 0^+$. The idea is the same as before: use the solution of an appropriate reduced problem to construct bounding functions which reflect the asymptotic behavior of the solution. Some results in this direction are also contained in [19] together with existence and comparison results for the problem (4.1), (4.2) with (4.2) replaced by an analogous second-order system.

5. Concluding remarks. The above theory has been extended to several classes of singularly perturbed systems of the form

$$\text{(5.1)} \qquad \begin{aligned} &\varepsilon \boldsymbol{y}'' = F(t, \boldsymbol{y})\boldsymbol{y}' + \boldsymbol{H}(t, \boldsymbol{y}),\ a < t < b, \\ &\boldsymbol{y}(a, \varepsilon) = \boldsymbol{A},\ \boldsymbol{y}(b, \varepsilon) = \boldsymbol{B}, \end{aligned}$$

where $\boldsymbol{y}$, $\boldsymbol{H}$, $\boldsymbol{A}$, and $\boldsymbol{B}$ are n-vectors, and F is an $(n \times n)$-matrix; cf. [17], [20], [23]. One can study these problems for small values of $\varepsilon > 0$ by deriving estimates for the norm of the difference between $\boldsymbol{y}$ and the solution of an appropriate reduced problem. As an illustration, suppose that in (5.1) F is the zero matrix and $\boldsymbol{H}(t, \boldsymbol{0}) \equiv \boldsymbol{0}$ in $[a, b]$. If the Jacobian matrix $J = \partial \boldsymbol{H}/\partial \boldsymbol{y}$ is positive definite along the reduced solution $\boldsymbol{0}$ and in the boundary layers at $t = a$ and $t = b$ (say $\boldsymbol{y}\cdot J\boldsymbol{y} \geqq m\|\boldsymbol{y}\|^2$ for $\|\boldsymbol{y}\| = (\boldsymbol{y}\cdot\boldsymbol{y})^{1/2} = (\sum_{i=1}^n y_i^2)^{1/2}$ and m a positive constant), then (recall Brish's treatment of the scalar analog of (5.1) in §3) the problem (5.1) with $F \equiv 0$ has for each sufficiently small $\varepsilon > 0$ a solution $\boldsymbol{y} = \boldsymbol{y}(t, \varepsilon)$ satisfying in $[a, b]$

$$\begin{aligned} \|\boldsymbol{y}\|(t, \varepsilon) = {} & O(\|\boldsymbol{A}\|\exp[-(m\varepsilon^{-1})^{1/2}(t - a)]) \\ & + O(\|\boldsymbol{B}\|\exp[-(m\varepsilon^{-1})^{1/2}(b - t)). \end{aligned}$$

Finally, we note that the theory of §2 for the boundary value problem (2.1) has been generalized by Nagumo and others (cf. [31], [1], [25]) to Dirichlet and Robin problems for the elliptic equation in $\Omega \subset \mathbf{R}^N$

$$L[u] = \mathfrak{F}(\boldsymbol{x}, u, \boldsymbol{\partial u}).$$

Here L is the linear, uniformly elliptic, second-order operator in N variables $\boldsymbol{x} = (x_1, \ldots, x_N)$, $\boldsymbol{\partial u}$ is the gradient of u, and Ω is a bounded domain whose boundary Γ is a smooth $(N - 1)$ dimensional manifold. By means of these results it is then possible to discuss the existence and the asymptotic behavior as $\varepsilon \to 0^+$ of solutions of the perturbed elliptic problems

$$\text{(5.2)} \qquad \begin{aligned} &\varepsilon L[u] = \mathfrak{F}(\boldsymbol{x}, u, \boldsymbol{\partial u}),\ \boldsymbol{x} \text{ in } \Omega, \\ &\alpha \frac{\partial u}{\partial n} + \beta u|_\Gamma \text{ prescribed}, \end{aligned}$$

under various assumptions on the characteristic curves of the first-order equation $\mathfrak{F} = 0$; cf. [7], [13], [24]. The differential inequality approach of Nagumo allows one to embellish the classical results of Levinson [28], Vishik and Liusternik [32] and Eckhaus and de Jager [5] on the linear Dirichlet problem (5.2), and to extend their theory to the case of nonlinear functions $\mathfrak{F}$ and Robin boundary conditions.

In conclusion let us note that many other interesting singular perturbation problems await solution (cf. [18]). We have every hope that the differential inequality techniques of Nagumo and Jackson can be reapplied or adapted to accommodate these new situations.

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