ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 4, Fall 1982

NONOSCILLATION RESULTS FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

ABSTRACT. By means of a change of variable along with appropriate energy functions, criteria are obtained which guarantee that all solutions of the second order nonlinear equation $y'' + p(x)y^{\gamma} = 0$, $p > 0$, $\gamma > 1$, are nonoscillatory. These results strengthen known nonoscillation criteria.

1. Introduction. Consider the nonlinear second order equation

$$y'' + p(x)y^{\gamma} = 0 \tag{1.1}$$

where $p > 0$ is locally integrable on $[a, \infty)$ and $\gamma > 1$ is the quotient of odd positive integers. We shall be interested in obtaining criteria for all nontrivial solutions of (1.1) to be nonoscillatory (i.e., have only finitely many zeros).

It was shown by Atkinson [1] that all solutions are oscillatory if and only if $\int_a^{\infty} xp(x)\, dx = \infty$. We refer to the survey papers [14], [15] and [9] for detailed bibliographies. In contrast to the linear case $\gamma = 1$, however, equation (1.1) permits the coexistence of both oscillatory and nonoscillatory solutions. Thus, while $\int_a^{\infty} xp(x)\, dx < \infty$ guarantees the existence of at least one nontrivial nonoscillatory solution, it remains of interest to find criteria for the existence of an oscillatory solution to (1.1) or conditions which imply that all solutions are nonoscillatory. These have been studied less frequently. For the former we refer to [5], [6], [7] and [8]. Nonoscillation criteria may be found in [4], [12] and [13].

Our technique is to employ a suitable change of variables along with an appropriate energy function. The results obtained strengthen known criteria. For example, Nehari [13] has shown that if $p(x)(x \log x)^{(\gamma+3)/2}$ is nonincreasing on $[a, \infty)$, then equation (1.1) is nonoscillatory. This was subsequently improved by Chiou [4] who showed that if $p(x)x^{(\gamma+3)/2}$ $(\log x)^{\beta}$ is nonincreasing on $[a, \infty)$, where $\beta > (\gamma + 1)/4 - 1/(\gamma + 1)$,

Research supported by NSERC Grants A–7673 and A–7197.
Received by the editors on June 19, 1981.

then (1.1) is nonoscillatory. It was further shown by Nehari **[11]** that $p(x)x^{(\gamma+3)/2}$ nonincreasing with $\lim_{x\to\infty}p(x)x^{(\gamma+3)/2} = 0$ is not in itself sufficient to guarantee that all solutions are nonoscillatory. In Nehari's example $p(x)x^{(\gamma+3)/2}$ is a step function which is constant on intervals of ever increasing length. Here we show that if $p(x)x^{(\gamma+3)/2}$ decreases significantly (in a sense made precise later) on sets of intervals of ever increasing length, then all solutions are nonoscillatory.

The techniques employed by the aforementioned authors and by Coffman and Wong who considered a more general version of (1.1) in **[5]** and **[6]** involve clever use of differential and integral inequalities and identities. We believe the method used below is simpler and gives more geometric insight. This procedure may be readily applied to more general equations (cf. **[6]**) but, for the sake of clarity, the emphasis in this paper is on (1.1).

2. Results. In equation (1.1) we make the change of variables

$$x = e^t,\ y = t^\mu e^{t/2}u,\ \mu \geqq 0 \tag{2.1}$$

which transforms (1.1) into

$$(t^{2\mu}u')' + a(t, u)u = 0 \tag{2.2}$$

where

$$a(t, u) = t^{2\mu}[\sigma(t)u^{\gamma-1} - \lambda(t)]$$
$$\sigma(t) = p(e^t)\, e^{(\gamma+3)t/2}\, t^{\mu(\gamma-1)}$$

and

$$\lambda(t) = \frac{1}{4} + \frac{\mu(1-\mu)}{t^2}.$$

The special case $\mu = 0$ in (2.1) is the standard change of variables used in studying (1.1), for which the transformed equation is

$$u'' + [p(e^t)\, e^{(\gamma+3)t/2}\, u^{\gamma-1} - \tfrac{1}{4}]u = 0. \tag{2.3}$$

It is useful to observe that

$$a(t, u) > 0 \Leftrightarrow |u| > \alpha(t) = \left[\frac{\lambda(t)}{\sigma(t)}\right]^{1/(\gamma-1)}, \tag{2.4}$$

and

$$\int_0^u a(t, s)s\, ds > 0 \Leftrightarrow |u| > \left[\frac{\gamma+1}{2}\right]^{1/(\gamma-1)}\alpha(t). \tag{2.5}$$

PROPOSITION. *Suppose that, for some $\mu \geqq 0$,*

(i) $$d[t^{4\mu}\sigma(t)] \leqq 0$$

and

(ii) $$\liminf_{t\to\infty} \int_t^{t+\delta} d[s^{4\mu}\lambda(s)\alpha(s)^2] > 0$$

for some δ,

$$0 < \delta < 2\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\left[\left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)} - 1\right].$$

Then all solutions of (2.2) *are nonoscillatory.*

The proof of the Proposition is given in §3. Since, from (2.1), nonoscillation of (2.2) is equivalent to nonoscillation of (1.1), this result may be restated as follows with $\beta = \mu(\gamma + 3)$.

THEOREM. *Suppose that, for some* $\beta \geqq 0$

(i) $$d[p(x)x^{(\gamma+3)/2}(\log x)^{\beta}] \leqq 0$$

and

(ii) $$\liminf_{x\to\infty} \int_x^{\kappa x} d[(\log s)^{2\beta/(\gamma+3)}/(p(s)\, s^{(\gamma+3)/2})^{2/(\gamma-1)}] > 0,$$

where $\kappa = e^{\delta}$ *and* δ *is as in the proposition. Then all solutions of* (1.1) *are nonoscillatory.*

Corollaries 1 and 2 give concrete conditions for nonoscillation.

COROLLARY 1. *If* $p(x) > 0$ *is locally absolutely continuous and, for some* $k > 0$,

$$(p(x)\, x^{(\gamma+3)/2})' \leqq \frac{-k}{x}(p(x)\, x^{(\gamma+3)/2})^{(\gamma+1)/(\gamma-1)},$$

then all solutions of (1.1) *are nonscillatory.*

COROLLARY 2. *If* $\beta > 0$ *and* η *are such that* $2\beta/(\gamma + 3) + 2\eta/(\gamma - 1) \geqq 1$,

(i) $p(x)\, x^{(\gamma+3)/2}(\log x)^{\beta}$ *is nonincreasing*

and

(ii) $p(x)\, x^{(\gamma+3)/2}(\log x)^{\eta}$ *is bounded,*

then all solutions of (1.1) *are nonoscillatory.*

Corollary 1 follows from the theorem with $\beta = 0$ since the condition given implies that $p(x)\, x^{(\gamma+3)/2}$ is nonincreasing and

$$\int_x^{\kappa x} d(p(s)\, s^{(\gamma+3)/2})^{-2/(\gamma-1)}$$
$$= -\frac{2}{\gamma-1}\int_x^{\kappa x}(p(s)\, s^{(\gamma+3)/2})'\,(p(s)\, s^{(\gamma+3)/2})^{-(\gamma+1)/(\gamma-1)}\, ds$$
$$\geqq \frac{2k}{\gamma-1}\int_x^{\kappa x}\frac{ds}{s} = \frac{2k}{\gamma-1}\log\kappa > 0.$$

Thus both conditions of the theorem are satisfied.

To verify that the conditions of Corollary 2 imply the conditions of the theorem, observe that from (i)

$$d[(\log s)^{2\beta/(\gamma+3)}(p(s)\, s^{(\gamma+3)/2})^{-2/(\gamma-1)}]$$
$$\geqq [p(s)\, s^{(\gamma+3)/2}(\log s)^{\beta}]^{-2/(\gamma-1)}\, d(\log s)^{2\beta/(\gamma+3)+2\beta/(\gamma-1)}$$
$$= [p(s)\, s^{(\gamma+3)/2}(\log s)^{\eta}]^{-2/(\gamma-1)}(\log s)^{2(\eta-\beta)/(\gamma-1)}\, d(\log s)^{2\beta/(\gamma+3)+2\beta/(\gamma-1)}$$
$$\geqq M^{-2/(\gamma-1)}\left(\frac{4\beta(\gamma+1)}{(\gamma+3)(\gamma-1)}\right)(\log s)^{2\beta/(\gamma+3)+2\eta/(\gamma-1)-1}\,\frac{1}{s}\, ds$$

if $p(s)\, s^{(\gamma+3)/2}(\log s)^{\eta} \leqq M$. Therefore,

$$\int_x^{\kappa x} d[(\log s)^{2\beta/(\gamma+3)}(p(s)\, s^{(\gamma+3)/2})^{-2/(\gamma-1)}] \geqq C\log\kappa > 0$$

for some C if $2\beta/(\gamma+3) + 2\eta/(\gamma-1) \geqq 1$.

The special case $\eta = \beta$ in Corollary 2 shows that (1.1) is nonoscillatory if $p(x)x^{(\gamma+3)/2}(\log x)^{\beta}$ is nonincreasing and $\beta \geqq (\gamma+1)/4 - 1/(\gamma+1)$, which is Chiou's result **[4]**. Note that, as pointed out by Nehari **[11]**, Chiou's result in which the condition is given as $\beta > 0$ is incorrect and his proof requires the condition cited here on β.

3. Proof of the proposition. We will assume throughout this proof that σ is locally absolutely continuous on its domain. When this condition is not satisfied, the same results may be obtained by an integration-by-parts procedure replacing differentiation of energy functions (cf. **[10]**).

Suppose u is an oscillatory solution of (2.2) and consider, for $u = u(t)$,

$$E(t) = (t^{2\mu}u')^2 + 2t^{2\mu}\int_0^u a(t, s)\, s\, ds$$
$$= (t^{2\mu}u')^2 + \frac{2}{\gamma+1}\, t^{4\mu}\sigma(t)u^{\gamma+1} - t^{4\mu}\lambda(t)u^2.$$

The function E is nonincreasing because

$$E'(t) = \frac{2}{\gamma+1}(t^{4\mu}\sigma)'u^{\gamma+1} - (t^{4\mu}\lambda)'u^2 \leqq 0 \tag{2.6}$$

since $(t^{4\mu}\sigma)' \leqq 0$ is given and $(t^{4\mu}\lambda)' \geqq 0$ from $\mu \geqq 0$. Since $E = (t^{2\mu}u'(t))^2 > 0$ if $u(t) = 0$, it follows that $E(t) > 0$ for all t if u is an oscillatory solution, because E is nonincreasing. Thus, whenever $u'(t) = 0$, $E(t) = 2t^{2\mu}\int_0^{u(t)} a(t, s)\, s\, ds > 0$ so that, from (2.5),

$$|u(t)| > \left[\frac{\gamma + 1}{2}\right]^{1/(\gamma-1)} (\alpha(t))$$

at any local extremum of u. Thus, between successive zeros of u, there is an interval in which $|u| > \alpha$. We next show that the length of these intervals is ultimately greater than δ if

$$\delta < 2\left(\frac{\gamma + 1}{\gamma - 1}\right)^{1/2}\left[\left(\frac{\gamma + 1}{2}\right)^{1/(\gamma-1)} - 1\right].$$

For fixed t, the minimum value of $2t^{2\mu}\int_0^u a(t, s)s\, ds$ is achieved if $|u| = \alpha(t)$ and equals

$$\begin{aligned} 2t^{2\mu}\int_0^{\pm\alpha(t)} a(t, s)\, s\, ds &= \frac{2}{\gamma + 1}\, t^{4\mu}\sigma\left(\frac{\lambda}{\sigma}\right)^{(\gamma+1)/(\gamma-1)} - t^{4\mu}\lambda\left(\frac{\lambda}{\sigma}\right)^{2/(\gamma-1)} \\ &= -\frac{\gamma - 1}{\gamma + 1}\, t^{4\mu}\lambda\left(\frac{\lambda}{\sigma}\right)^{2/(\gamma-1)} \qquad (2.7) \\ &= -\frac{\gamma - 1}{\gamma + 1}\, t^{4\mu}\lambda\alpha^2. \end{aligned}$$

Conditions (i), (ii) of the proposition imply α is nondecreasing and

$$\lim_{t\to\infty} t^{4\mu}\lambda(t)\alpha(t)^2 = \infty. \tag{2.8}$$

Now suppose t_n, and t_{n+1} are successive zeros of u and $t_n < r_n < \tau_n < s_n < t_{n+1}$ where $|u(r_n)| = \alpha(r_n)$, $|u(s_n)| = \alpha(s_n)$, $|u| > \alpha$ on (r_n, s_n) and $u'(\tau_n) = 0$ so that

$$|u(\tau_n)| > \left(\frac{\gamma + 1}{2}\right)^{1/(\gamma-1)} (\alpha(\tau_n)). \tag{2.9}$$

We further assume that $u > 0$ on (t_n, t_{n+1}); the same argument may be applied to $-u$ if $u < 0$. Since $a(t, u(t)) > 0$ if $r_n \leqq t \leqq s_n$, it follows from (2.2) that $t^{2\mu}u'(t)$ is decreasing on this interval. Therefore,

$$\begin{aligned} (\tau_n - r_n)u'(r_n) &\geqq \int_{r_n}^{\tau_n}\left(\frac{t}{r_n}\right)^{2\mu} u'(t)\, dt \\ &\geqq \int_{r_n}^{\tau_n} u'(t)\, dt \\ &= u(\tau_n) - u(r_n) \\ &> \left(\frac{\gamma + 1}{2}\right)^{1/(\gamma-1)} \alpha(\tau_n) - \alpha(r_n), \end{aligned}$$

from (2.9) and

$$(2.10) \qquad (\tau_n - r_n)u'(r_n) > \left[\left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)} - 1\right]\alpha(r_n)$$

since α is nondecreasing. To estimate $u'(r_n)$ observe that $E(r_n) \leqq E_0 = E(t_0)$, if $r_n \geqq t_0$, and so

$$(r_n^{2\mu}u'(r_n))^2 - \frac{\gamma-1}{\gamma+1} r_n^{4\mu}\lambda(r_n)\alpha(r_n)^2 \leqq E_0$$

from (2.7). Thus

$$u'(r_n) \leqq \left[E_0 r_n^{-4\mu} + \frac{\gamma-1}{\gamma+1}\lambda(r_n)\alpha(r_n)^2\right]^{1/2}$$

and, if $c > ((\gamma-1)/(\gamma+1))^{1/2}/2$, it follows from (2.8) that

$$(2.11) \qquad u'(r_n) < c\,\alpha(r_n)$$

for n sufficiently large. From (2.10) and (2.11),

$$(\tau_n - r_n)c\,\alpha(r_n) > \left[\left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)} - 1\right]\alpha(r_n)$$

and hence

$$(2.12) \qquad s_n - r_n > \tau_n - r_n > c^{-1}\left[\left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)} - 1\right] = \delta.$$

Next we estimate E' in the interval (r_n, s_n) as follows. Since $|u| > \alpha$ in this interval, it follows from (2.6) that

$$\begin{aligned} E' &\leqq \frac{2}{\gamma+1}(t^{4\mu}\sigma)'\alpha^{\gamma+1} - (t^{4\mu}\lambda)'\alpha^2 \\ &= \frac{2}{\gamma+1}(t^{4\mu}\sigma)'\left(\frac{\lambda}{\sigma}\right)^{(\gamma+1)/(\gamma-1)} - (t^{4\mu}\lambda)'\left(\frac{\lambda}{\sigma}\right)^{2/(\gamma-1)} \\ &= \frac{2}{\gamma+1}g'\left(\frac{f}{g}\right)^{(\gamma+1)/(\gamma-1)} - f'\left(\frac{f}{g}\right)^{2/(\gamma-1)} \\ &= -\frac{\gamma-1}{\gamma+1}\frac{d}{dt}\left[f\left(\frac{f}{g}\right)^{2/(\gamma-1)}\right], \end{aligned}$$

where $f(t) = t^{4\mu}\lambda(t)$ and $g(t) = t^{4\mu}\sigma(t)$. Therefore,

$$(2.13) \qquad dE(t) \leqq -\frac{\gamma-1}{\gamma+1}d[t^{4\mu}\lambda(t)\alpha(t)^2],\ r_n \leqq t \leqq s_n.$$

From (2.6) and (2.13),

$$\begin{aligned} E(\infty) - E(t_0) &= \int_{t_0}^{\infty} dE(t) \\ &\leqq \sum_n \int_{r_n}^{s_n} dE(t) \end{aligned}$$

$$\leqq -\frac{\gamma-1}{\gamma+1}\sum_n \int_{r_n}^{s_n} d[t^{4\mu}\lambda(t)\alpha(t)^2]$$
$$= -\infty,$$

from condition (ii). This contradicts $E(t) > 0$ for an oscillatory solution and so no such solution exists.

4. Concluding remarks. As observed in the introduction, we have for purposes of simplicity confined our attention to equation (1.1). However, the above techniques would apply equally well (with appropriate modifications) to the more general equation

$$(r(x)y')' + p(x)y^\gamma = 0 \tag{4.1}$$

where $r > 0$. We leave the details to the interested reader. Perhaps of more interest is the question of nonoscillation of (1.1) if $p(x)\ x^{(\gamma+3)/2}(\log x)^\beta$ is non-increasing and $\beta > 0$. As noted above this was originally claimed by Chiou in [4] because of a computational error. Corollary 2 shows that the claim is true provided condition (ii) also holds where η and β satisfy the inequality given in the assumptions of the corollary.

Of interest also is the application of the above change-of-variable techniques to the sublinear case of (1.1), where $0 < \gamma < 1$. This has been done as far as the existence of oscillatory solutions is concerned (cf. [3], [6], [7], [8]). However, it appears that the nonoscillation problem in the sublinear case is more difficult. It was claimed in [6, Corollary 6] that (1.1) is nonoscillatory if $0 < \gamma < 1$ and if for some $\beta \geqq (5-\gamma)/2$, $p(x)x^{(\gamma+3)/2}(\log x)^\beta$ is nondecreasing and bounded above. However, the validity of this claim as well as that of [6, Corollary 4] is still undecided since both resulted from a computational error in the application of Lemma 4 of [6]. Thus, as far as the authors are aware, there are as yet no known criteria, similar to those in the superlinear case, involving monotoneity assumptions on the expression $p(x)\ x^{(\gamma+3)/2}(\log x)^\beta$, $\beta > 0$. Therefore, the result of Belohorec [2], which states that (1.1) is nonoscillatory if $0 < \gamma < 1$ and $x^{(\gamma+3)/2+\delta}\,p(x)$ is non-decreasing and bounded above, where $0 < \delta < (1-\gamma)/2$, has yet to be improved upon.

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