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THE GELFAND-LEVITAN AND MARČENKO EQUATIONS VIA TRANSMUTATION

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1. Introduction. The Gelfand-Levitan and Marčenko (GL and M) equations arise in inverse quantum scattering theory as a vital part of the machinery used in recovering the potential (see, e.g., [**33**; **38**; **39**; **41**; **43**; **47**; **48**; **59**; **63**; **64**; **65**; **66**; **67**; **69**; **76**]). They also come up in various forms in other applications of scattering techniques to physical problems (cf. [**7**; **19**; **21**; **22**; **23**; **24**; **53**; **61**; **68**; **71**; **72**; **73**; **74**; **78**; **79**; **80**; **81**; **82**] for example). Moreover various discrete forms of these equations have appeared in studying some relations between scattering theory and the theory of certain orthogonal polynomials (cf. [**27**; **28**; **29**; **30**; **31**; **32**; **50**; **51**; **52**]); this approach thus establishes some connections between scattering theory and certain special functions but, in this respect, it goes in a quite different direction than ours. Our constructions, which are directly phrased in a manner which applies to the context of harmonic analysis on symmetric spaces, can build special functions into the framework from the beginning and leads then to very natural relations between special functions. For example a generalized GL equation is obtained which can be written as a formula in spherical functions (see below and cf. also §4 plus [**8**; **9**; **10**; **11**; **16**; **17**; **18**; **19**; **20**; **25**; **26**]). We begin from §7 and §8 of the survey article on inverse problems in quantum scattering theory by Fadeev [**43**] where he displays certain important operators (U and V) in terms of transmutations. This allows one to give an essentially unified derivation of the GL and M equations which we generalize here in the transmutation framework of Carroll [**8**; **9**; **10**; **11**] (an announcement appears in [**15**]). The link between these two equations (and correspondingly between U and V) is a certain transmutation operator $\tilde{U}$ whose natural generalization $\tilde{\mathscr{B}}$ (sometimes $\hat{\mathscr{B}}$) is determined here in a canonical way. The linking property extends to our more general framework but there are a number of conceptual and technical differences in our constructions, partly because adjointness plays a somewhat different role here (see also [**17**] where adjointness connections are indicated for certain spaces). The operators $\tilde{\mathscr{B}}$ are also a crucial ingredient in providing (together with the corresponding B, $\mathscr{B}$, etc.) a sys-

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tematic abstract context in which to display and study various connection formulas of Riemann-Liouville and Weyl type for special functions (cf. Carroll-Gilbert **[25**; **26]** and cf. also Askey-Fitch **[2]**, Chao **[34]**, Flensted-Jensen **[45]**, Koornwinder **[57]**). The kernels of $\mathscr{B}$ and $\tilde{\mathscr{B}}$ have complementary triangularity properties and one striking feature is that once the correct transmutation $\tilde{\mathscr{B}}$ has been isolated, based on scattering theory arguments linking the GL and M equations, then abstract proofs of theorems such as 4.4, 4.5, and 4.6, etc. can be provided, which give unified information about special functions for which only special derivations had been previously known.

Let us mention here also that equations such as (5.8)–(5.9) can be written explicitly as formulas in spherical functions. Indeed in **[14]** we extend the context to transmutations $B\colon \hat{P} \to \hat{Q}$ etc. (instead of $B_Q\colon D^2 \to \hat{Q}$) and (5.8)–(5.9) can be expressed in a form (notation of text)

$$\text{(1.1)} \qquad \int_0^\infty \langle \varphi_\lambda^P(\xi), \varphi_\lambda^Q(y)\rangle_\nu \Delta_P(\xi)\, T_\xi^x \check{W}(\xi)\, d\xi = \langle \varphi_\lambda^P(x), \varphi_\lambda^Q(y)\rangle_\omega$$

where $T_\xi^x \check{W}(\xi) = \langle \varphi_\lambda^P(\xi), \ \varphi_\lambda^P(x)\rangle_\omega$ ($\check{W}$ differs here by a factor of 2 from the $\check{W}$ of the present paper). Model cases of (1.1) are also studied in [14]. Further use of transmutations connecting $\mathscr{B}$ and $\tilde{\mathscr{B}}$ in **[17]** (cf. also **[54**; **55**; **56]**) in the study of singular pseudodifferential operators leads to an interesting class of transmutations with kernels expressed by Erdélyi-Kober operators (cf. **[18**; **42**; **70**; **84]**). Finally let us mention that our framework and results involving the linking transmutation $\tilde{\mathscr{B}}$ play a crucial role in solving various integral equations which arise in studying inverse problems in geophysics (cf. **[16**; **21**; **22**; **23**; **24**; **62**; **74]**).

It seems virtually mandatory to give in this paper a semidetailed account of some of the background physics material from **[43]** and we have done so in §2 and §3; a resume of some results from **[8**; **9**; **10**; **25**; **26]** is also included. It cannot be assumed that a mathematician interested in special functions has any knowledge of quantum scattering theory and we hope to help provide a motive and points of contact for acquiring such knowledge.

2. Background from scattering theory. Let us recall first some information and procedures from physics following **[33**; **39**; **43**; **66]**. We consider $Ly = -y'' + q(x)y$ in $L^2(0, \infty)$ with eigenfunction equation $Ly = \lambda^2 y$. Take the potential $q(x)$ real here and let us deal with so called regular potentials where $\int_0^\infty x|q(x)|dx < \infty$. The solution $\varphi(x, \lambda)$ of $L\varphi = \lambda^2\varphi$ with $\varphi(0, \lambda) = 0$ and $\varphi'(0, \lambda) = 1$ is called the regular solution while the solutions $\Phi(x, \pm\lambda)$ satisfying the asymptotic conditions $\Phi(x, \pm\lambda) \to \exp(\pm i\lambda x)$, $\Phi'(x, \pm\lambda) \to \pm i\lambda \exp(\pm i\lambda x)$ as $x \to \infty$ are called Jost solutions. For the Wronskian $W(\Phi_+, \Phi_-) = \Phi'_+\Phi_- - \Phi_+\Phi'_-$ we have $W(\Phi_+, \Phi_-)$

$= 2i\lambda$ $(\Phi_\pm = \Phi(x, \pm\lambda))$. The function $\Phi(\lambda) = \Phi(0, \lambda)$ is called the Jost function and $W(\varphi, \Phi_+) = \Phi(\lambda)$ with

$$(2.1) \qquad \varphi(x, \lambda) = \frac{1}{2i\lambda}(\Phi(-\lambda)\Phi(x, \lambda) - \Phi(\lambda)\Phi(x, -\lambda)).$$

If one considers more general initial conditions $w(0, \lambda) = 1$ and $w'(0, \lambda) = h$ as in [59], then $\varphi \sim w_\infty$. The following facts are standard (cf. [33; **43**; **59**]).

LEMMA 2.1. *φ is an entire function in λ of exponential type x, even in λ, and real for $\lambda = k$ real. As $|\lambda| \to \infty$, $\varphi(x, \lambda) - (1/\lambda)$ Sin $\lambda x =$ (exp $|$Im $\lambda|x/|\lambda|)\, o(1)$ and $\int_0^\infty|\varphi(x, k) - (1/k)$ Sin $kx|dk < \infty$. Φ_+ is holomorphic for Im $\lambda > 0$ and continuous and bounded for Im $\lambda \geqq 0$. For $x \to \infty$, $\Phi(x, \lambda) - e^{i\lambda x} = \exp(-x$ Im $\lambda)\, o(1)$, Im $\lambda > 0$, and for $\lambda \to \infty$ with Im $\lambda \geqq 0$, $\Phi(x, \lambda) - e^{i\lambda x} = \{\exp(-x$ Im $\lambda)/|\lambda|\}\, o(1)$, $x > 0$. Further $\Phi(x, k)^- = \Phi(x, -k)$ ($\lambda = k$ real) with $\Phi(k)^- = \Phi(-k)$ and for $\lambda = \sigma + i\tau$, $x > 0, \tau \geqq 0$, one has $\int_{-\infty}^\infty|\Phi(x, \lambda) - e^{i\lambda x}|^2\, d\sigma = O(e^{-2x\tau})$.*

REMARK 2.2. The operator L with the zero initial condition has a self-adjoint realization in L^2 and generally there will be a continuous spectrum on $(0, \infty)$ (i.e., $Ly = \nu y$, $\nu \sim \lambda^2$) with possibly a finite number of simple eigenvalues $\nu_n = -\lambda_n^2$ (bound states) where $\lambda_n = i\gamma_n$; the latter correspond to zeros λ_n of the Jost function $\Phi(\lambda)$. $\Phi(0) = 0$ does not correspond to a bound state and to avoid unnecessary complications we will assume here $\Phi(0) \neq 0$. The function given by $R_\nu(x, y) = \varphi(x, \sqrt{\nu})\Phi(y, \sqrt{\nu})/\Phi(\sqrt{\nu})$, Im $\sqrt{\nu} > 0$, $x < y$, is a resolvent operator kernel defining $(L - \nu I)^{-1}$, by use of which follows the expression

$$(2.2) \qquad \delta(x - y) = \sum c_j\varphi_j(x)\varphi_j(y) + \int_0^\infty \varphi(x, k)\varphi(y, k)d\nu(k)$$

where $\varphi_j(x) = \varphi(x, i\gamma_j) \in L^2$ and the spectral measure is ($k \geqq 0$; $|\Phi(k)|^2 = \Phi(k)\Phi(-k)$).

$$(2.3) \qquad d\nu(k) = \frac{2}{\pi}k^2[\Phi(k)\Phi(-k)]^{-1}\, dk.$$

REMARK 2.3. The properties of $\Phi(x, \lambda)$ indicated in Lemma 2.1 plus a theorem of Titchmarsh [**83**] allow us to write

$$(2.4) \qquad A(x, y) = \frac{1}{2\pi}\int_{-\infty}^\infty [\Phi(x, \lambda) - e^{i\lambda x}]e^{-i\lambda y}\, d\lambda$$

where $A(x, y) = 0$ for $y < x$ so that

$$(2.5) \qquad \Phi(x, \lambda) = e^{i\lambda x} + \int_x^\infty A(x, y)e^{i\lambda y}\, dy.$$

This is the Levin representation (cf. [58]) and is an ingredient for the Marčenko equation; the inverse is written

$$e^{i\lambda x} = \Phi(x, \lambda) + \int_x^\infty \tilde{A}(x, y)\Phi(y, \lambda)dy. \tag{2.6}$$

Similarly the properties of $\varphi(x, \lambda)$ described in Lemma 2.1 plus theorems of Paley-Wiener-Boas (cf. [33]) allow one to write

$$\varphi(x, \lambda) - \frac{1}{\lambda}\operatorname{Sin}\lambda x = \int_{-x}^x \Psi(x, y)e^{i\lambda y}\,dy = 2\int_0^x \Psi(x, y)\operatorname{Cos}\lambda y\,dy. \tag{2.7}$$

Recall that $\varphi(x, \lambda)$ is even in λ so $\Psi(x, -y) = \Psi(x, y)$; further $\Psi(x, \pm x) = 0$ with Ψ continuous. The Povzner-Levitan representation for φ follows from (2.7) upon an integration by parts, namely

$$\varphi(x, \lambda) = \frac{1}{\lambda}\operatorname{Sin}\lambda x + \int_0^x K(x, y)\frac{\operatorname{Sin}\lambda y}{\lambda}\,dy \tag{2.8}$$

where $K(x, y) = -2D_y\Psi(x, y)$. Hence

$$K(x, y) = \frac{2}{\pi}\int_0^\infty \left[\varphi(x, \lambda) - \frac{1}{\lambda}\operatorname{Sin}\lambda x\right]\frac{\operatorname{Sin}\lambda y}{\lambda}\lambda^2\,d\lambda. \tag{2.9}$$

REMARK 2.4. Consider the operators

$$Uf(x) = f(x) + \int_0^x K(x, y)f(y)dy \tag{2.10}$$

$$Vf(x) = f(x) + \int_x^\infty A(x, y)f(y)dy \tag{2.11}$$

working on suitable functions f. U will be a transmutation operator as described in Carroll [**8**; **9**; **10**; **11**] but V is not. The relation between U and V leads to an understanding of the relation between the Gelfand-Levitan and Marčenko equations. This was developed in a very revealing was by Fadeev in §7 of [**43**] and his technique and constructions suggest natural extensions to more general operators admitting a transmutation linkage as in [**8**; **9**; **10**; **11**]. In fact this approach provides an abstract key to handling various problems at ∞ (cf. [**25**; **26**]).

Thus let us extract some material from Fadeev [**43**]. Theorems 2.5, 2.6, and 2.7 appear in [**43**] and we have given a somewhat more detailed derivation in order to display the mathematical properties and features used in obtaining such theorems. Consider $\phi_+(x, k) = \varphi(x, k)/\Phi(k)$ and define

$$T_+g(k) = G(k) = \int_0^\infty g(x)\phi_+(x, k)dx \tag{2.12}$$

for $g \in L^2$ and $G \in L^2_\omega = \{G;\ \int_0^\infty |G(k)|^2k^2dk < \infty\}$. Set then (cf. Remark 2.2)

$$T_+^* G(x) = \frac{2}{\pi}\int_0^\infty G(k)\bar{\psi}_+(x,\ k)k^2\ dk \tag{2.13}$$

so that $T_+T_+^* = I$, $T_+^*T_+ = I_c$, $T_+Lg = k^2\ T_+g$, etc. (here I_c is a projection determined by the continuous spectrum of L). Similarly write

$$T_0 g(k) = \int_0^\infty g(x)\frac{\operatorname{Sin} kx}{k}\,dx \tag{2.14}$$

$$T_0^* G(x) = \frac{2}{\pi}\int_0^\infty G(k)\frac{\operatorname{Sin} kx}{k}\,k^2\ dk \tag{2.15}$$

so that if $L_0 = -D^2$ we have $T_0L_0g = k^2T_0g$, $T_0T_0^* = I$, $T_0^*T_0 = I$, etc. Now given another eigenfunction of L, e.g., φ, one writes

$$T_\varphi g(k) = \int_0^\infty g(x)\varphi(x,\ k)dx \tag{2.16}$$

and then $T_\varphi = N(k)T_+$ where $N(k) = \varphi'(0,\ k)/\psi_+'(0,\ k) = \Phi(k)$. Thus $T_\varphi^* = T_+^*\overline{\Phi(k)} = T_+^*\Phi(-k)$ and $T_\varphi T_\varphi^*\,W(k) = I$ where $W(k) = [\Phi(k)\Phi(-k)]^{-1}$. Similarly $T_\varphi^*\,W(k)T_\varphi = T_+^*T_+ = I_c$.

Now given an operator A^x in $L^2(0,\ \infty)$ there is an associated operator $A_k = T_0A^xT_0^*$ in L^2_ω. For example $L = -D^2 + q(x)$ has the form

$$L_kF(k) = k^2F(k) + \frac{2}{\pi}\int_0^\infty V(k,\ s)F(s)s^2\ ds \tag{2.17}$$

where

$$V(k,\ s) = \int_0^\infty q(x)\frac{\operatorname{Sin} kx}{k}\ \frac{\operatorname{Sin} sx}{s}\,dx.$$

Consider then the transmutation operator U of (2.10) associated with φ, i.e., $\varphi = U[(\operatorname{Sin} kx/k)]$. One can write $U = T_\varphi^*T_0$ and in L^2_ω, $U_k = T_0T_\varphi^*$, so that $LU = UL_0$ (i.e., U transmutes L_0 into L). It follows from the relations $U_k = T_0T_+^*\Phi(-k)$ and $U = T_+^*\Phi(-k)T_0$ that $U_kW(k)U_k^* = I$ and $U^*UW^x = I$ where $W^x = T_0^*\,W(k)T_0$ (also $UW^xU^* = I$ and $U_k^*U_kW(k) = I$). Now in order to understand V in (2.11) in terms of transmutation. let us assume there is no discrete spectrum for convenience, and let $\tilde{U} = UW^x$, which is the transmutation operator associated with the eigenfunction $\tilde{\varphi}(x,\ k) = \varphi(x,\ k)W(k)$, i.e.,

$$\tilde{\varphi}(x,\ k) = \frac{1}{2ik}\left(\frac{\Phi(x,\ k)}{\Phi(k)} - \frac{\Phi(x,\ -k)}{\Phi(-k)}\right). \tag{2.18}$$

The corresponding kernel $\tilde{K}(x,\ y)$ of $\tilde{U} - I$ is then obtained as before (cf. (2.9))

$$\tilde{K}(x,\ y) = \frac{2}{\pi}\int_0^\infty\left[\tilde{\varphi}(x,\ \lambda) - \frac{\operatorname{Sin}\lambda x}{\lambda}\right]\frac{\operatorname{Sin}\lambda y}{\lambda}\,\lambda^2\ d\lambda \tag{2.19}$$

Now note from (2.7) that $\tilde{K}(x, y)$ can also be written

$$\tilde{K}(x,y) = -2D_y\tilde{\Psi}(x,y) = \frac{i}{\pi}\int_{-\infty}^{\infty}[\lambda\tilde{\varphi}(x, \lambda) - \operatorname{Sin} \lambda x]e^{-i\lambda y}\, d\lambda$$

$$(2.20) \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left(\frac{\Phi((x, \lambda)}{\Phi(\lambda)} - \frac{\Phi(x, -\lambda)}{\Phi(-\lambda)}\right) - (e^{i\lambda x} - e^{-i\lambda x})\right]e^{-i\lambda y}\, d\lambda.$$

In passing observe that $\tilde{K}(x, -y) = -\tilde{K}(x, y)$, but this property is removed in the representation (2.21) below. Now for $x + y > 0$ an integral of the type $\int_{-\infty}^{\infty}\exp[-i\lambda(x + y)]d\lambda$ can be thought of in terms of a contour in the lower half of the λ plane where $\operatorname{Im} \lambda \leqq 0$ and equated to zero. Further one knows that $\Phi(x, \lambda)e^{-i\lambda x}/\Phi(\lambda)$ (resp. $\Phi(x, -\lambda)e^{i\lambda x}/\Phi(-\lambda)$) is analytic and bounded for $\operatorname{Im} \lambda \geqq 0$ (resp. $\operatorname{Im} \lambda \leqq 0$). Hence we can conclude that

$$\int_{-\infty}^{\infty}[\Phi(x, -\lambda)e^{-i\lambda y}/\Phi(-\lambda)]d\lambda = \int_{-\infty}^{\infty}[\Phi(x, -\lambda)e^{i\lambda x}/\Phi(-\lambda)]\exp[-i\lambda(x + y)]d\lambda = 0;$$

from (2.20) then

$$(2.21) \qquad \tilde{K}(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\Phi(x, \lambda)}{\Phi(\lambda)} - e^{i\lambda x}\right]e^{-i\lambda y}\, d\lambda.$$

Thinking of contours now in the halfplane $\operatorname{Im} \lambda \geqq 0$ it follows that $\tilde{K}(x, y) = 0$ for $x > y$. (Note here we do not have $\tilde{K}(x, -y) = -\tilde{K}(x, y)$ any more.) Consequently

$$(2.22) \qquad \tilde{U}f(x) = f(x) + \int_{x}^{\infty}\tilde{K}(x, y)f(y)dy$$

(with $\tilde{U}[(\operatorname{Sin}\ kx)/k] = \tilde{\varphi}(x, k)$ and similarly $\tilde{U}[e^{ikx}] = \Phi(x, k)/\Phi(k)$). Calculating as before one can now show that $\tilde{U}W^{-1}\tilde{U}^* = I(\tilde{U}_kW^{-1}(k)\tilde{U}_k^* = I)$, etc., and in fact $U = (U^*)^{-1}$.

It remains to relate $\tilde{U}$ and V. To do this consider the function

$$(2.23) \qquad \Pi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\Phi(k)} - 1\right]e^{-ikt}\, dk,$$

Since $1/\Phi(k)$ is analytic for $\operatorname{Im} k \geqq 0$, we have $\Pi(t) = 0$ for $t < 0$. Now from (2.21) and (2.5) there results

$$\tilde{K}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iky}\left[\frac{1}{\Phi(k)}\left\{e^{ikx} + \int_{x}^{\infty}A(x, t)e^{ikt}\, dt\right\} - e^{ikx}\right]dk$$

$$(2.24) \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iky}\, e^{ikx}\left[\frac{1}{\Phi(k)} - 1\right]dk + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iky}}{\Phi(k)}\int_{x}^{\infty}A(x, t)e^{ikt}\, dtdk.$$

Hower $1/\Phi(k) = \mathscr{F}\Pi + 1 = \mathscr{F}\Pi + \mathscr{F}\delta$ ($\mathscr{F}$ denoting Fourier transform) and the last term in (2.24) is $\mathscr{F}^{-1}[\mathscr{F}A(x, \cdot)(\mathscr{F}\Pi + \mathscr{F}\delta)] = A(x, \cdot)*(\Pi(\cdot) + \delta(\cdot)) = A(x, \cdot)*\Pi(\cdot) + A(x, y)$. (We use standard distribution terminology from [**49**; **75**].) Since $\Pi(y - t) = 0$ for $t > y$ and $A(x, t) = 0$ for $t < x$, we have $A(x, \cdot)*\Pi(\cdot) = \int_x^y \Pi(y - t)A(x, t)dt$, and consequently

$$\tilde{K}(x, y) = \Pi(y - x) + A(x, y) + \int_x^y \Pi(y - t)A(x, t)dt. \tag{2.25}$$

Define now ($\Xi f = \delta * f + \check{\Pi} * f$; $\check{\Pi}(t) = \Pi(-t)$)

$$\Xi f(x) = f(x) + \int_x^\infty \Pi(y - x)f(y)dy. \tag{2.26}$$

Then Ξ is a bounded operator in L^2 and $\Xi^{-1} = X$ has the form

$$Xf(x) = f(x) + \int_x^\infty \Gamma(y - x)f(y)dy \tag{2.27}$$

($X = \delta * f + \check{\Gamma} * f$) where in fact

$$\Gamma(t) = \frac{1}{2\pi}\int_{-\infty}^\infty (\Phi(k) - 1)e^{-ikt}dk. \tag{2.28}$$

Further from (2.11), (2.22), and (2.25) we obtain easily

$$\tilde{U} = V\Xi = (U^{-1})^* \tag{2.29}$$

and using $\tilde{U}W^{-1}\tilde{U}^* = I$ there follows also $V\Xi W^{-1}\Xi^* V^* = I$.

THEOREM 2.5. *The transmutation $\tilde{U}$ associated with $\tilde{\varphi} = \varphi W$ can be represented as in* (2.22) *and satisfies* $\tilde{U}[(\text{Sin } kx)/k] = \tilde{\varphi}(y, k)$ *with* $\tilde{U}[e^{ikx}] = \Phi(y, k)/\Phi(k)$. *It is related to U by $U = (\tilde{U}^*)^{-1}$ and to V by $\tilde{U} = V\Xi$ with Ξ determined by* (2.26).

REMARK 2.6. The operator $\tilde{U}$ is the crucial ingredient linking U and V. Especially important is the triangular nature of the kernel in (2.22) for $\tilde{U}$ (i.e., $\tilde{K}(x, y) = 0$ for $x > y$). Note that U in (2.10) has a triangular kernel $K(x, y) = 0$ for $y > x$. Upon generalization of the U and $\tilde{U}$ concepts to suitable situations in harmonic analysis on symmetric spaces the corresponding operators $\mathscr{B}$ and $\tilde{\mathscr{B}}$ will also have triangular kernels of the same nature and this leads to a unified way of expressing various connecting formulas for special functions in terms of Riemann-Liouville and Weyl type integrals (see Carroll-Gilbert [**25**; **26**]).

Let us continue now with material from [**43**] and obtain a nicer form for the operator $\Xi W^{-1}\Xi^* = \Delta$. We continue to assume no bound states are present for convenience. First define $S(k) = \Phi(-k)/\Phi(k)$ and note that $|S(k)| = 1$, $S(k)^- = S(-k) = S^{-1}(k)$. Further one knows that $S(k) - 1 =$

$\int_{-\infty}^{\infty}\mathscr{S}(t)e^{-ikt}dt$ with $\mathscr{S}\in L^1$. Now in keeping with the stipulation $A^* = T_0^*A_kT_0$ one can write in L^2, $W^{-1}f(x) = f(x) + \int_0^\infty \tilde{W}(x,y)f(y)dy$, where

$$\text{(2.30)} \qquad \begin{aligned}\tilde{W}(x,y) &= \frac{2}{\pi}\int_0^\infty\left(\frac{1}{W(k)}-1\right)\frac{\operatorname{Sin} kx}{k}\,\frac{\operatorname{Sin} ky}{k}\,k^{2}dk \\ &= \tilde{W}(x-y) - \tilde{W}(x+y)\end{aligned}$$

where the even function $\tilde{W}(t)$ has the form

$$\text{(2.31)} \qquad \tilde{W}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{W(k)}-1\right)e^{-ikt}\,dk.$$

We observe here that the generalized translation $W(x,y)$ arises in the form

$$\text{(2.32)} \qquad W(x,y) = \frac{2}{\pi}\int_{-\infty}^{\infty}(W(k)-1)\frac{\operatorname{Sin} kx}{k}\,\frac{\operatorname{Sin} ky}{k}\,k^2\,dk$$

and appears in the Gelfand-Levitan equation below. In [9] we used the notation $F(x,y)$ for $W(x,y)$ and showed that $W(x,y) = S_x^y\ell(x) = S_x^yL(x,0)$ where S_x^y is an appropriate generalized translation associated with Q and $L(x,y)$ is the kernel in $Bf(x) = \mathscr{Q}\mathfrak{B}f(x) = f(x) + \int_0^x L(x,y)f(y)dy$. Let now $\tilde{W}_1$ (resp. $\tilde{W}_2$) be the operator with kernel $\tilde{W}(x-y)$ (resp. $-\tilde{W}(x+y)$). Then the identity $(1/W(k))(1/\Phi(k)) = \Phi(-k)$ can be expressed as $(1+\Pi)(I+\tilde{W}_1) = I + \Gamma^*$ (cf. (2.23) and (2.28)). Here we should spell out the action however since $(I+\Pi)f$ means $\delta*f + \check{\Pi}*f$ in (2.26). Let us observe $\mathscr{F}(\Pi+\delta) = 1/\Phi(k)$, $\mathscr{F}(\Gamma+\delta) = \Phi(k)$, $\mathscr{F}(\tilde{W}_1+\delta) = 1/W(k)$, $\mathscr{F}(\check{\Gamma}+\delta) = \Phi(-k)$, and $\mathscr{F}(f*\check{g}) = \mathscr{F}f\mathscr{F}\check{g} = \mathscr{F}f\overline{\mathscr{F}g}$ with $\mathscr{F}\check{g}(k) = \overline{\mathscr{F}g(k)} = \mathscr{F}g(-k)$. Thus $(I+\tilde{W}_1)(I+\Pi)$ means $(\delta+\tilde{W}_1)*(\delta+\check{\Pi}) = \mathscr{F}^{-1}(1/W(k)\Phi(-k)) = \mathscr{F}^{-1}\Phi(k) = \Gamma+\delta$ acting by convolution; but $I+\Gamma^* = (\delta+\check{\Gamma})^* = \delta+\Gamma$. Hence

$$\text{(2.33)} \qquad \begin{aligned}\Xi W^{-1}\Xi^* &= (I+\Pi)(I+\tilde{W}_1+\tilde{W}_2)(I+\Pi^*) \\ &= [I+\Gamma^*+(I+\Pi)\tilde{W}_2](I+\Pi^*) \\ &= I+(I+\Pi)\tilde{W}_2(I+\Pi^*) = I+\mathscr{W}\end{aligned}$$

(since $(I+\Gamma^*)(I+\Pi^*) = I$ by (2.26)–(2.27)). The kernel of $\mathscr{W}$ is now of interest. First note that $\tilde{W}_2$ acts as $-\tilde{W}(x+y)$ under convolution and $\tilde{W}(t)$ is even, so one has $-\int\tilde{W}(x+y)f(y)dy = -\int\tilde{W}(-x-y)f(y)dy = -(\tilde{W}*f)^{\vee}$. Further $\mathscr{F}(\tilde{W}*g)^{\vee} = \overline{\mathscr{F}\tilde{W}\,\mathscr{F}g} = \mathscr{F}\tilde{W}\,\overline{\mathscr{F}g} = \mathscr{F}\tilde{W}\mathscr{F}g(-k) = \mathscr{F}\tilde{W}\mathscr{F}\check{g}$. Now $\mathscr{W}$ was the form of a convolution $-(\delta+\check{\Pi})*(\tilde{W}*(\delta+\Pi))^{\vee}$ with Fourier transform $-\Phi^{-1}(-k)(1/W(k)-1)\,\Phi^{-1}(-k) = -(S(k) - 1/\Phi^2(k))^{-} = -\tilde{H}(k)^{-} = -\tilde{H}(-k)$. Let

$$\text{(2.34)} \qquad \tilde{\mathscr{S}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{H}(-k)e^{-ikt}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{H}(k)e^{ikt}\,dk$$

so that $\mathscr{W}$ involves acting with $-\tilde{\mathscr{S}}(t)$ by convolution and in fact for

$t > 0$, $\tilde{\mathscr{S}}(t) = \mathscr{S}(t) = (1/2\pi)\int_{-\infty}^{\infty}(S(k) - 1)\exp ikt\, dk$. This follows upon considering a contour integral in the upper halfplane $\operatorname{Im} k \geqq 0$ where $(1/\Phi^2(k))e^{ikt}$ is analytic and bounded under our assumptions. Now associate with $\mathscr{S}(t)$ the operator

$$\mathscr{S}f(x) = \int_0^\infty \mathscr{S}(x+y)f(y)dy. \tag{2.35}$$

Note that Vf involves convolution with $\delta + A$ where $A(x, y) = (1/2\pi)\int_{-\infty}^{\infty}[\Phi(x,k) - e^{ikx}]e^{-iky}\,dk$. Hence $\mathscr{S}(\cdot)*(\delta + A)$ or $\mathscr{S}(\cdot)*(\delta + A^*)$ has to be thought of as $\mathscr{S}(\cdot)*(\delta + A)^{\vee}$ for example. But $\int\mathscr{S}(x-\xi)\check{f}(\xi)d\xi = \int\mathscr{S}(x-\xi)f(-\xi)d\xi = \int\mathscr{S}(x+y)f(y)dy$ and since the $f(y)$ which arise are only defined on $(0, \infty)$ we arrive at (2.35). Consequently we have the important relation

$$V(I - \mathscr{S})V^* = I. \tag{2.36}$$

THEOREM 2.6. *The operator* $\Xi W^{-1}\Xi^* = \Delta$ *expressing the lack of unitarity of* V *by* $V\Delta V^* = I$ *has the form* $\Delta = I - \mathscr{S}$ *where* $S(k) - 1 = \int_{-\infty}^{\infty}\mathscr{S}(t)e^{-ikt}\,dt$ *and* $\mathscr{S}$ *is given by* (2.35).

We shall continue to assume that there are no bound states for convenience. Recall now that $UWU^* = I$ and write this as $UW = (U^*)^{-1} = \tilde{U}$. Recall the kernel experssions $\tilde{K}(x, y)$ of (2.19) for $\tilde{U} - 1$ and $K(x, y)$ of (2.9) for $U - I$, while $W - I$ will have kernel $W(x, y)$ given by (2.32); further $\tilde{K}(x, y) = 0$ for $x > y$. It follows that $(I + K)(I + W) = I + \tilde{K}$ in an obvious notation or $K + W + KW = \tilde{K}$ which written out in terms of kernels is

$$K(x,y) + W(x,y) + \int_0^x K(x,t)W(t,y)dt = \tilde{K}(x,y). \tag{2.37}$$

For $x > y$ the right side vanishes and we obtain the Gelfand-Levitan equation

$$K(x, y) + W(x, y) + \int_0^x K(x, t)W(t, y)dt = 0. \tag{2.38}$$

We write out now $V(I - \mathscr{S})V^* = I$ from (2.36) as $V(I - \mathscr{S}) = (V^*)^{-1}$ where $V - I$ has kernel A from (2.11) and $\mathscr{S}$ has the kernel of (2.35). We do not need an explicit kernel for $(V)^{*-1}$ here; it suffices to note that from $(V - I)f(x) = \int_x^\infty A(x, y)f(y)dy$ we have

$$\begin{aligned} ((V - I)f, g) &= \int_0^\infty\int_x^\infty A(x,y)f(y)g(x)dydx \\ &= \int_0^\infty\int_0^y A(x,y)g(x)f(y)dxdy \end{aligned} \tag{2.39}$$

which is $(f, (V - I)^*g)$. Consequently $(Vf, g) = (f, V^*g) = (f, g) + (f, \hat{A}g)$ in an obvious notation where $\hat{A}$ has a kernel $\hat{A}(x, y) = A(y, x) = 0$ for $y > x$ (here $\hat{A}g(x) = \int_0^x \hat{A}(x, y)g(y)dy$). From $V^* = I + \hat{A}$ we get $V^{*-1} = I + \tilde{A}$ where $\tilde{A}$ is determined by a similar kernel $\tilde{A}(x, y)$ with $\tilde{A}(x, y) = 0$ for $y > x$. Now (2.36) becomes $(I + A)(I - \mathscr{S}) = I + \tilde{A}$ or $A - \mathscr{S} - A\mathscr{S} = \tilde{A}$ which implies

$$(2.40) \qquad A(x, y) = \mathscr{S}(x + y) + \int_x^\infty A(x, t)\mathscr{S}(y + t)dt$$

for $y > x$, and this is the Marčenko equation.

THEOREM 2.7. *The Gelfand-Levitan and Marčenko equations* (2.38) *and* (2.40) *respectively arise naturally from the above formulation as indicated.*

REMARK 2.8. The derivation of (2.38) above is more or less standard and so is that of (2.40), once (2.36) is known (cf. **33**; **59**]). The "novelty" here which Fadeev displays in [**43**], and which does not seem to have been exploited in the physics literature (?), is the linking between U and V expressed via a transmutation $\tilde{U}$. We will see that some of the arguments used above in discussing $\tilde{U}$ are particular to the given Schrödinger type operator L but similar properties (e.g., triangularity) of the natural generalization $\tilde{\mathscr{B}}$ of $\tilde{U}$ can be obtained by other auguments. This suggests that a generalized Marčenko equation can be derived in the framework of [**8**; **9**; **10**; **11**] and we will give a unified derivation below of Marčenko and Gelfand-Levitan type equations by generalizing the above procedures (a general Gelfand-Levitan equation also can be obtained as in [9]—see Theorem 5.7).

3. Preliminary constructions. Let us recall first some basic facts and notation from [**8**; **9**; **10**]. Thus we deal first with a situation where the following constructions make sense.

$$(3.1) \quad P(D)H = \mu H;\ H(0, \mu) = 1;\ H'(0, \mu) = 0,$$

$$(3.2) \quad Q(D)\Theta = \mu\Theta;\ \Theta(0, \mu) = 1;\ \Theta'(0, \mu) = 0,$$

$$(3.3) \quad P^*(D)\Omega = \mu\Omega;\ Q^*(D)W = \mu W,$$

$$(3.4) \quad \langle \Omega(x, \mu), 1\rangle_\nu = \delta(x);\ \langle W(y, \mu), 1\rangle_\omega = \delta(y),$$

$$(3.5) \quad \mathfrak{P}f(\mu) = \hat{f}(\mu) = \langle \Omega(x, \mu), f(x)\rangle,$$

$$(3.6) \quad \mathscr{P}f(\mu) = \hat{\hat{f}}(\mu) = \langle H(x, \mu), f(x)\rangle,$$

$$(3.7) \quad \boldsymbol{\mathfrak{P}}F(x) = \langle F(\mu), H(x, \mu)\rangle_\nu,$$

(3.8) $\mathscr{P}F(x) = \langle F(\mu), H(x, \mu)\rangle_\omega$,

(3.9) $\boldsymbol{P}F(x) = \langle F(\mu), \Omega(x, \mu)\rangle_\nu$,

(3.10) $\mathfrak{Q}g(\mu) = \tilde{g}(\mu) = \langle W(y, \mu), g(y)\rangle$,

(3.11) $\mathscr{Q}g(\mu) = \breve{g}(\mu) = \langle \Theta(y, \mu), g(y)\rangle$,

(3.12) $\boldsymbol{\mathfrak{Q}}G(y) = \langle G(\mu), \Theta(y, \mu)\rangle_\omega$,

(3.13) $\boldsymbol{\mathscr{Q}}G(y) = \langle G(\mu), \Theta(y, \mu)\rangle_\nu$,

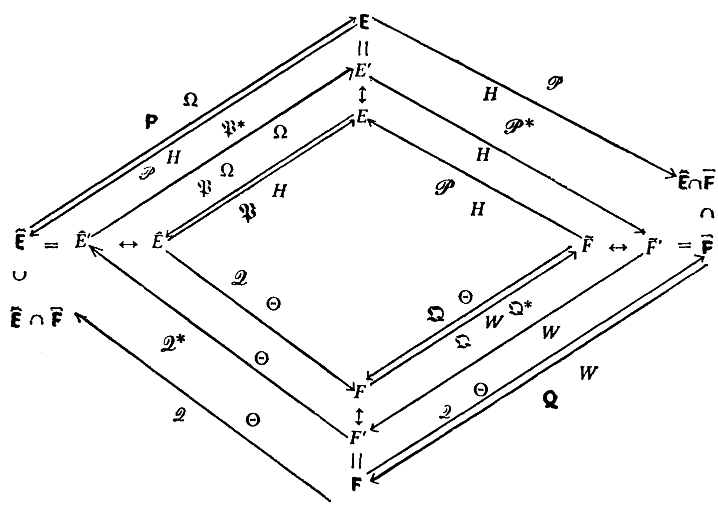
(3.14) $\boldsymbol{Q}G(y) = \langle G(\mu), W(y, \mu)\rangle_\omega$,

We based our framework on a model problem where $P(D) = D^2 + ((2m + 1)/x)D = P_m(D)(m \geqq -1/2 \text{ real})$ and $Q(D) = D^2$ with $\Theta(y, \mu) = \operatorname{Cos} \lambda y \quad (\mu = -\lambda^2)$, $W(y, \mu) = (2/\pi) \operatorname{Cos} \lambda y$, and

$$H(x, \mu) = 2^m\Gamma(m + 1)(\lambda x)^{-m}J_m(\lambda x);\ \Omega(x, \mu) = 2^{-2m}\Gamma(m + 1)^{-2}(\lambda x)^{2m+1}H(x, \mu). \tag{3.15}$$

It was then natural to introduce spaces E, $\hat{E}$, $\boldsymbol{E} = E'$, $\hat{\boldsymbol{E}} = \hat{E}'$, etc. so that the subscripts ν and ω in (3.7)–(3.9) and (3.12)–(3.14) refer to natural duality pairings and the transmutation operators $B = \boldsymbol{\mathscr{Q}}\mathfrak{P}$ and $\mathscr{B} = B^{-1} = \boldsymbol{\mathscr{P}}\mathfrak{Q}$ are displayed in (cf. [19] for a more refined diagram):

(3.16)



THEOREM 3.1. *The diagram* (3.16) *indicates the relations* $\boldsymbol{\mathfrak{P}} = \mathfrak{P}^{-1}$, $\boldsymbol{\mathfrak{Q}} = \mathfrak{Q}^{-1}$, $\boldsymbol{P} = \mathscr{P}^{-1}$, $\boldsymbol{Q} = \mathscr{Q}^{-1}$, $\mathfrak{P}^* = \boldsymbol{P}$, $\mathfrak{Q}^* = \boldsymbol{Q}$, $\boldsymbol{\mathscr{P}}^* = \mathscr{P}$, $\boldsymbol{\mathscr{Q}}^* = \mathscr{Q}$, $B^* = (\boldsymbol{\mathscr{Q}}\mathfrak{P})^* = \boldsymbol{P}\mathscr{Q}$, *and* $\mathscr{B}^* = (\boldsymbol{\mathscr{P}}\mathfrak{Q})^* = \boldsymbol{Q}\mathscr{P}$. *Here* $D(\boldsymbol{\mathscr{Q}}) = D(\boldsymbol{\mathscr{P}}) =$

$\hat{E} \cap \tilde{F}$ and $R(\mathscr{Q}^*)$, $R(\mathscr{P}^*) \subset \hat{\tilde{E}} \cap \tilde{\bar{F}} = \hat{E}' \cap \tilde{F}'$. From $\mathscr{B} = \mathscr{P}\mathfrak{Q} = B^{-1} = (\mathscr{Q}\mathfrak{P})^{-1} = \mathfrak{P}^{-1}\mathscr{Q}^{-1}$ we obtain $\mathscr{Q}^{-1} = \mathfrak{P}\mathscr{P}\mathfrak{Q}$ and $\mathscr{P}^{-1} = \mathfrak{Q}\mathscr{Q}\mathfrak{P}$.

It is shown in [**8**; **9**] that $Bf(y) = \langle\beta(y, x), f(x)\rangle$ and $\mathscr{B}g(x) = \langle\gamma(x, y), g(y)\rangle$ with

$$(3.17)\qquad \beta(y, x) = \langle\Omega(x, \mu), \Theta(y, \mu)\rangle_\nu;\ \gamma(x, y) = \langle H(x, \mu), W(y, \mu)\rangle_\omega.$$

These kernels are related to Marčenko's K and L by $\gamma(x, y) \sim \delta(x - y) + K(x, y)$ and $\beta(y, x) \sim \delta(x - y) + L(y, x)$ (note K and/or L in our case may be distributions). A slightly different choice of Ω in (3.15) (factoring our λ^{2m+1}) is in certain ways more natural and we will indicate this later (cf. [**9**]).

Now we cannot expect to duplicate the procedure exactly nor perhaps produce as "tight" a theory in our framework as is possible in §2 since $E \neq F$ in general and our operators are not selfadjoint. Recall however that the constructions are based on spreading out a selfadjoint situation (cf. [**9**]) and we will expand upon this here. It is exactly the type of situation which arises in working in the context of harmonic analysis on symmetric spaces (cf. [**12**; **13**; **35**; **36**; **37**; **44**; **45**; **57**]). Thus consider

$$(3.18)\qquad P(D)u = (Au')'/A$$

where $A(x)$ generally will have properties so that $-P(D)$ can be the radial part of the Laplace-Beltrami operator on a noncompact Riemannian symmetric space of rank one. Thus assume $A \in C^\infty(0, \infty)$, $A(0) = 0$, $A(x) > 0$ for $x > 0$, $A'/A = \alpha/x + B(x)$ for $B \in C^0(0, \infty)$, $A(x)$ is increasing with $A(x) \to \infty$ as $x \to \infty$, and A'/A is decreasing with $\rho = \rho_A = \lim (1/2)A'(x)/A(x)$ as $x \to \infty$.

EXAMPLE 3.2. For $A(x) = x^{2m+1}$ we have our previous model operator where $A'/A = (2m + 1)/x$; this corresponds to a Euclidean situation in group theory. Other examples which we will treat later in more detail are (cf. [**45**; **57**]) $A(x) = \Delta_{\alpha\beta}(x) = (e^x - e^{-x})^{2\alpha+1}(e^x + e^{-x})^{2\beta+1}$ and $A(x) = \Delta(x) = (e^x - e^{-x})^p (e^{2x} - e^{-2x})^q$. The case $A(x) = sh^{2m+1}\, x$ with $A'/A = (2m + 1)$ coth x arises in working in $SL(2, \boldsymbol{R})/SO(2)$ and is particularly useful for illustrative purposes (cf. [**8**; **11**; **12**; **13**]).

Somewhat more refined hypotheses on $A(x)$ can be made later as needed, following [**37**], for $P(D)$ and suitable $P_q(D) = P(D) - q(x)$. Now Chebli considers $P(D)$ as a selfadjoint operator in $L^2(Adx)$ but we prefer to work with it in $L^2(dx)$ so that

$$(3.19)\qquad P^*(D)\psi = [A(\psi/A)']'$$

is the formal adjoint. Note that we will want a nonselfadjoint context for complex $q(x)$ in $P_q(D)$ in any event. For domain $D(P)$ in $L^2(Adx)$ one takes

$D(P) = \{u;\ u,\ u',\ Pu \in L^2(Adx);\ A(x)u'(x) \to 0 \text{ as } x \to 0\}$. Let us observe that if we take $H^1 = \{u,\ u' \in L^2(Adx)\}$, then $D(P)$ can be alternatively characterized as the set of $u \in H^1$ such that $v \to \int_0^\infty u'v'A(x)dx: H^1 \to \boldsymbol{C}$ is continuous in the topology of $L^2(Adx)$ (integrate by parts). The space E for the model problem based on $A = x^{2m+1}$ involved $E = \{f;\ x^{m+1/2} f \in L^2\}$. But $u \in L^2(Adx) \sim A^{1/2}u \in L^2$ so take $E_A = \{f;\ A^{1/2} f \in L^2\}$ and recast $D(P)$ in E_A as $D(P) = \{u,\ u',\ Pu \in E_A;\ Au' \to 0 \text{ as } x \to 0\}$. On the other hand following the model set $E'_A = \{f;\ A^{-1/2} f \in L^2\}$ and consider

$$\text{(3.20)}\qquad \begin{aligned}\langle P(D)u, \psi\rangle &= \int_0^\infty \frac{1}{A}(Au')'\psi\,dx \\ &= \left[Au'\frac{\psi}{A} - uA\left(\frac{\psi}{A}\right)'\right]_0^\infty + \int_0^\infty u\left[A\left(\frac{\psi}{A}\right)'\right]' dx.\end{aligned}$$

We will use the notation $W(f, g) = f'g - fg'$ so that the bracket [ ] in (3.20) is $AW(u, \psi/A)|_0^\infty$. At $x = \infty$ we have $AW = u'\psi - u\psi' + u\psi(A'/A)$ which vanishes for $u, u' \in E_A$ and $\psi, \psi' \in E'_A$ since $A'/A \to 2\rho$. For $D(P^*) \subset E'_A$ consider then $\psi,\ \psi',\ P^*(D)\psi = [A(\psi/A)']' \in E'_A$ with $AW(u, \psi/A) = 0$ at $x = 0$ for $u \in D(P)$. This can be refined somewhat but there is no need here (cf. [37]).

We consider now functions φ_λ^P satisfying $P(D)\varphi_\lambda^P = (-\lambda^2 - \rho_A^2)\varphi_\lambda^P$ where $\rho_P = \rho_A = (1/2)\lim\,(A'/A)$ as above and $\varphi_\lambda^P(0) = 1$ with $D_x\varphi_\lambda^P(0) = 0$ (set also $A = \Delta_P$). Such functions exist and correspond to spherical functions in geometrical situations (cf. [35; 37; 45; 57]). Now in order to link operators of this sort via transmutation as in Theorem 3.1 set $\hat{P}(D) = P(D) + \rho_P^2$ Then if $\hat{Q}(D) = Q(D) + \rho_Q^2$ for a similar operator $Q(D)$ as in (3.18) we can speak of transmuting $\hat{P}$ into $\hat{Q}$ with formulas as before where $\mu = -\lambda^2$ is common to both $\hat{P}$ and $\hat{Q}$. Thus $\hat{P}(D)H = \mu H$ with $\mu = -\lambda^2$ corresponds to $H(x, \mu) = \varphi_\lambda^P(x)$. We set further $\Omega(x, \mu) = A(x)H(x, \mu) = \Omega_\lambda^P(x)$ $(= \Delta_P(x)\,\varphi_\lambda^P(x))$ so that $\hat{P}^*(D)\,\Omega = \mu\Omega$. Now let $\mathscr{D}_0$ denote even C^∞ functions on $\boldsymbol{R}$ with compact support and note that $\mathscr{D}_0$ is dense in $D(P) \subset L^2(Adx)$ in graph norm. Define (cf. (3.5))

$$\text{(3.21)}\qquad \hat{f}(\mu) = \langle \Omega(x, \mu), f(x)\rangle = \int_0^\infty f(x)H(x, \mu)A(x)dx$$

for, say, $f \in \mathscr{D}_0$; we will often use the notation $\hat{f}(\lambda)$ for $\mathfrak{P}f = \langle \Omega(x, \mu), f(x)\rangle$ later for convenience. More generally (cf. [37]) one works with $P_q(D) = P(D) - q(x)$ where $q(x) \geqq q_0$ ($q_0 \leqq 0$ possibly negative but finite) and $q(x)$ has the form $\beta^2/x^2 + \tilde{q}(x)$ near $x = 0$ with $q(x) = \beta_1^2/x^2 + \hat{q}(x)$ near $x = \infty$ ($\tilde{q}$ and $\hat{q}$ suitable). Then the spectrum $\sigma(-P_q)$ will have a continuous part on $[\rho^2, \infty)$ (i.e., $\rho^2 \leqq \lambda^2 + \rho^2 < \infty$ or $0 \leqq \lambda^2 < \infty$) and possibly a finite number of eigenvalues in $[q_0, \rho^2]$ (i.e., $q_0 \leqq \lambda^2 + \rho^2 \leqq \rho^2$ or $q_0 - \rho^2 \leqq \lambda^2 \leqq 0$). Let $\lambda_j = is_j$ denote these eigenvalues so $\mu_j = s_j^2$

and recall $H(x, \mu) = \varphi_\lambda^P(x)$. The inversion formula for (3.21) has the form

$$(3.22)\qquad f(x) = \int_0^\infty \hat{f}(\mu)H(x, \mu)d\nu(\lambda) + \sum_{j=1}^n \hat{f}(\mu_j)\frac{\varphi_{\lambda_j}^P(x)}{\|\varphi_{\lambda_j}^P(x)\|^2}$$

where $\|\ \|$ is the $L^2(Adx)$ norm. Generally this kind of formula, with even more complicated terms arising from a generalized spectral function R (cf. [9; 59]), is what one understands by an expression (3.7) (i.e., $\mathfrak{P}\hat{f}(x) = \langle \hat{f}(\mu), H(x, \mu)\rangle_\nu$). However for now let us assume the discrete spectrum is absent (e.g., $q(x) = 0$) and treat situations where $\mathfrak{P}f(x) = \int_0^\infty \hat{f}(\mu)H(x, \mu)d\nu(\lambda)$ for an absolutely continuous positive measure $d\nu(\lambda) = \hat{\nu}(\lambda)d\lambda$; further let us consistently write now $\hat{f}(\lambda)$ for $\hat{f}(\mu)$, etc.

Thus we shall deal first with two operators $P(D)$ and $Q(D)$ of the above form (3.18) relative to weight functions $A(x) = \Delta_P(x)$ and $B(x) = \Delta_Q(x)$. This will be sufficient to produce many results for special functions and to provide a general perspective (cf. also [25, 26]). We have then for $H(x, \mu) = \varphi_\lambda^P(x)$, $\Theta(x, \mu) = \varphi_\lambda^Q(x)$, etc. (cf. (3.5) etc).

$$(3.23)\qquad \begin{aligned} \hat{f}(\lambda) = \mathfrak{P}f(\lambda) &= \int_0^\infty f(x)\varphi_\lambda^P(x)\Delta_P(x)\,dx;\ f(x) = \mathfrak{P}\hat{f}(x) \\ &= \int_0^\infty \hat{f}(\lambda)\varphi_\lambda^P(x)\,d\nu, \end{aligned}$$

$$(3.24)\qquad \begin{aligned} \tilde{g}(\lambda) = \mathfrak{Q}g(\lambda) &= \int_0^\infty g(x)\varphi_\lambda^Q(x)\Delta_Q(x)\,dx;\ g(x) = \mathfrak{Q}\tilde{g}(x) \\ &= \int_0^\infty \tilde{g}(\lambda)\varphi_\lambda^Q(x)d\omega. \end{aligned}$$

Now consider $\Delta_P f = h$ in (3.23) and write

$$(3.25)\qquad \begin{aligned} \hat{\hat{h}}(\lambda) = \mathscr{P}h(\lambda) &= \int_0^\infty h(x)\varphi_\lambda^P(x)\,dx;\ h(x) = P\hat{\hat{h}}(x) \\ &= \int_0^\infty \hat{\hat{h}}(\lambda)\varphi_\lambda^P(x)\Delta_P(x)d\nu. \end{aligned}$$

Similarly, associated with (3.24) we have

$$(3.26)\qquad \begin{aligned} \tilde{\tilde{k}}(\lambda) = \mathscr{Q}k(\lambda) &= \int_0^\infty k(x)\varphi_\lambda^Q(x)dx;\ k(x) = Q\tilde{\tilde{k}}(x) \\ &= \int_0^\infty \tilde{\tilde{k}}(\lambda)\varphi_\lambda^Q(x)\Delta_Q(x)d\omega. \end{aligned}$$

Note that if one writes $\delta_A(x) = \delta(x)/\Delta_P(x) = \delta(x)/A(x)$ (which must be considered as acting on suitable functions), then formally (cf. [9])

$$(3.27)\qquad \begin{aligned} &\hat{\delta}_A(\lambda) = H(0, \mu) = 1;\ \tilde{\delta}_B(\lambda) = \Theta(0, \mu) = 1; \\ &\delta_A(x) = \int_0^\infty \varphi_\lambda^P(x)d\nu;\ \delta_B(x) = \int_0^\infty \varphi_\lambda^Q(x)d\omega. \end{aligned}$$

The operators $\mathscr{P}$ and $\mathscr{Q}$ can be exhibited in a similar manner following (3.8) and (3.13).

As a first step in generalizing the Fadeev constructions let us take P and Q as indicated and view Q as the base object (e.g., $Q \sim D^2$). Since ν and ω are absolutely continuous relative to λ, it is convenient to take $d\lambda$ as a basic measure in some general space G_λ of λ functions on which two measures ν and ω are distinguished. We will set $d\nu = W(\lambda)d\omega$ (in order to identify $W(\lambda)$ with $W(k)$ in §2). Let us think of $\mathfrak{Q}\colon F \to G_\lambda$ in place of T_0 but for T_0^* we take $\mathfrak{Q}^{-1} = \mathbf{\mathfrak{Q}}\colon G_\lambda \to F$ (the domains of all operators working out of G_λ must of course be specified). If we associate $\mathfrak{P}$ with T_φ and $\mathscr{P}$ with T_φ^*, then $\mathfrak{P}\colon E \to G_\lambda$ and $\mathscr{P}\colon G_\lambda \to E$ (on a suitable domain). The operator $U = T_\varphi^* T_0$ then corresponds to $\mathscr{B} = \mathscr{P}\mathfrak{Q}$ and $T_\varphi^* W T_\varphi = I$ corresponds to $\mathscr{P}W\mathfrak{P} = I$ where $W = W(\lambda)$. Recall that all transmutations involve $\hat{P}$ and $\hat{Q}$ while $\mu = -\lambda^2$. Note that ($H(x, \mu) = \varphi_\lambda^P(x)$, etc.)

$$\tag{3.28} \begin{aligned} \mathscr{P}WF(x) &= \int_0^\infty F(\lambda)H(x, \mu)W(\lambda)d\omega(\lambda) \\ &= \int_0^\infty F(\lambda)H(x, \mu)d\nu(\lambda) = \mathbf{\mathfrak{P}}F(x). \end{aligned}$$

Recall also that $H = \mathscr{B}\Theta = \mathscr{P}\mathfrak{Q}\Theta$. Now let $h(x, \mu) = h(\lambda)H(x, \mu)$ be another eigenfunction of $\hat{P}(D)$ ($h(0, \mu) = h(\lambda)$; $\mu = -\lambda^2$). Define

$$\tag{3.29} \mathscr{P}_h(x) = \langle F(\lambda), h(x, \mu)\rangle_\omega$$

and set $\mathscr{B}_h = \mathscr{P}_h\mathfrak{Q}$. Then $\mathscr{B}_h\Theta = h$ since $\mathscr{B}_h\Theta(\cdot, \mu)(x) = \langle h(x, \zeta), \langle\Theta(y, \mu), W(y, \zeta)\rangle\rangle_\omega$ for $W(y, \zeta) = \Delta_Q(y)\varphi_z^Q(y)$ ($\mu = -\lambda^2$, $\zeta = -z^2$) and from $\mathfrak{Q}\mathbf{\mathfrak{Q}} = I$ we know that $\langle\Theta(y, \mu), W(y, \zeta)\rangle = \delta_\omega(\lambda - z) = \delta(\lambda - z)/\hat{\omega}(z)$ for $d\omega(z) = \hat{\omega}(z)dz$. Formally $\mathscr{B}_h$ has kernel

$$\tag{3.30} \gamma_h(x, y) = \langle h(\lambda)H(x, \mu), W(y, \mu)\rangle_\omega$$

so $\hat{P}(D_x)\gamma_h = \hat{Q}^*(D_y)\gamma_h$ and $\mathscr{B}_h f(x) = \langle\gamma_h(x, y), f(y)\rangle$ satisfies $\mathscr{B}_h\hat{Q} = \hat{P}\mathscr{B}_h$ acting on suitable objects f. Thus formally we have the following proposition.

PROPOSITION 3.3. *Let G_λ be a basic spectral space on which two measures $d\omega(\lambda) = \hat{\omega}(\lambda)d\lambda$ and $d\nu(\lambda) = \hat{\nu}(\lambda)d\lambda$ are distinguished. For $h(x, \mu) = h(\lambda)H(x, \mu)$ define $\mathscr{P}_h$ by* (3.29) *and set $\mathscr{B}_h = \mathscr{P}_h\mathfrak{Q}$ with kernel γ_h given by* (3.30). *Then $\mathscr{B}_h$ is a transmutation $\hat{Q} \to \hat{P}$ (i.e., $\mathscr{B}_h\hat{Q} = \hat{P}\mathscr{B}_h$) and $\mathscr{B}_h\Theta = h$.*

In particular consider $\tilde{h}(x, \mu) = W(\lambda)H(x, \mu)$ (which will correspond to $\tilde{\varphi}$ in §2) and write $\gamma_{\tilde{h}} = \tilde{\gamma}$ with $\mathscr{B}_{\tilde{h}} = \tilde{\mathscr{B}}$. Then from (3.29) (cf. also (3.28)) $\mathscr{P}_{\tilde{h}}F(x) = \langle F(\lambda), W(\lambda)H(x, \mu)\rangle_\omega = \langle F(\lambda), H(x, \mu)\rangle_\nu = \mathbf{\mathfrak{P}}F(x)$ so that $\tilde{\mathscr{B}} = \mathbf{\mathfrak{P}}\mathfrak{Q}$. Further $\mathscr{B}$ and $\tilde{\mathscr{B}}$ map $F \to E$ and if we set $W^x = \mathbf{\mathfrak{Q}}W(\lambda)\mathfrak{Q}\colon F \to F$, then (cf. (3.28))

(3.31) $$\mathscr{B}W^x = \mathscr{P}\mathfrak{Q}\mathfrak{Q}W(\lambda)\,\mathfrak{Q} = \mathscr{P}W\mathfrak{Q} = \tilde{\mathscr{B}}.$$

The kernel $\tilde{\gamma}$ of $\tilde{\mathscr{B}}$ is given by (3.30) as

(3.32) $$\tilde{\gamma}(x, y) = \langle H(x, \mu), W(y, \mu)\rangle_\nu = \langle \varphi_\lambda^P(x), \Delta_Q(y)\varphi_\lambda^Q(y)\rangle_\nu$$

Because of its importance later we summarize this in the following theorem

THEOREM 3.4. *The transmutation $\tilde{\mathscr{B}}$ determined by $\tilde{h} = W(\lambda)H(x, \mu)$ (i.e., $W(\lambda)H(x, \mu) = \tilde{\mathscr{B}}\Theta(\cdot, \mu)(x)$) has the form $\tilde{\mathscr{B}} = \mathfrak{P}\mathfrak{Q}$ with kernel* (3.32) *and $\tilde{\mathscr{B}} = \mathscr{B}W^x$.*

For comparison recall that $\mathscr{B} = \mathscr{P}\mathfrak{Q}$ has kernel $\gamma(x, y) = \langle H(x, \mu), W(y, \mu)\rangle_\omega = \langle \varphi_\lambda^P(x), \Delta_Q(y)\varphi_\lambda^Q(y)\rangle_\omega$ while $B = \mathscr{Q}\mathfrak{P}$ has kernel $\beta(y, x) = \langle \Omega(x, \mu), \Theta(y, \mu)\rangle_\nu = \langle \Delta_P(x)\varphi_\lambda^P(x), \varphi_\lambda^Q(y)\rangle_\nu$ (cf. (3.17)). Hence we have the following corollary.

COROLLARY 3.5. $\tilde{\gamma} = \ker \tilde{\mathscr{B}}$ *and* $\beta = \ker B$ *are related by* $\Delta_P^{-1}(x)\beta(y, x) = \Delta_Q^{-1}(y)\tilde{\gamma}(x, y)$.

Now let us recall that $U[e^{ikx}] = \Phi(x, k)/\Phi(k)$ in §2 and as a measure of the correctness of our construction of a putative analogue $\tilde{\mathscr{B}}$ of $\tilde{U}$ we will show that a corresponding property will hold. In order to do this we must deal with the correct analogue of the Jost functions and Jost solutions. We will want to normalize somewhat differently than is customary in physics in order to have a nice connection with standard objects on symmetric spaces so we recall briefly the physics in order to indicate the linkage. Let $P_m(D) = D^2 + ((2m + 1)/x)D$ and $\tilde{P}_m(D) = D^2 - (m^2 - 1/4)/x^2$ as in [9] so that $\tilde{P}_m(D)[x^{m+1/2}f] = x^{m+1/2}P_m(D)f$. The operator $\tilde{P}_m(D)$ has been extensively studied in physics where $\ell = m - 1/2$ is an angular momentum (cf. [**33**; **38**; **43**; **66**]); the hookup here with physics occurs for $m = 1/2$ (not $m = -1/2$). It is common to take as Jost solution for P_m

(3.33) $$\tilde{\Phi}(x, \lambda) = ie^{i\pi(m-1/2)}\left(\frac{\pi\lambda x}{2}\right)^{1/2} H_m^1(\lambda x)$$

($\tilde{P}_m(D)\tilde{\Phi} = -\lambda^2\tilde{\Phi}$ here and H_m^1 denotes the Hankel function). Recalling that $H_m^1(-z) = e^{-im\pi}H_m^2(z)$ (also $\bar{H}_m^1(z) = H_m^2(z)$ for real z) we see that the other Jost solution $\tilde{\Phi}(x, -\lambda)$ involves $H_m^2(\lambda x)$. On the other hand the regular solution is chosen so that

(3.34) $$\frac{2^{2m}\Gamma(m + 1)}{\sqrt{\pi}\,x^{m+1/2}}\,\tilde{\varphi}(x, \lambda) \to 1$$

as $x \to 0$. This leads to

(3.35) $$\hat{\varphi}(x, \lambda) = \lambda^{-m-1/2}\left(\frac{\pi\lambda x}{2}\right)^{1/2} J_m(\lambda\, x).$$

Now since H for $P(D) = P_m(D)$ has the form (3.15), we see that

$$H(x, \mu) = \sqrt{2/\pi}\, 2^m \Gamma(m+1)\, x^{-m-1/2}\, \tilde{\varphi}(x, \lambda). \tag{3.36}$$

for $m = 1/2$, $\tilde{\varphi} = \lambda^{-1} \operatorname{Sin} \lambda x$ since $J_{1/2}(\lambda x) = (2/\pi\lambda x)^{1/2} \operatorname{Sin} \lambda x$ while H becomes $H = (\lambda x)^{-1} \operatorname{Sin} \lambda x$. Thus initial conditions $H(0, \mu) = 1$ and $H'(0, \mu) = 0$ do in fact reflect the regular solution type of conditions $\tilde{\varphi}(0, \lambda) = 0$ and $\tilde{\varphi}'(0, \lambda) = 1$ for $m = 1/2$.

A standard way of introducing the Jost functions $\tilde{F}(\lambda)$ is to write

$$\tilde{\varphi} = \frac{i\lambda^{-m-1/2}}{2} [\tilde{F}(\lambda)\tilde{\Phi}(x, -\lambda) - (-1)^{-m+1/2}\, \tilde{F}(-\lambda)\tilde{\Phi}(x, \lambda)]. \tag{3.37}$$

This is easily established using the fact that $\tilde{\varphi}$ is even in λ (note however $(-1)^{m-1/2} \neq (-1)^{-m+1/2}$ for general $\ell = m - 1/2$ not integral). One has $W(\tilde{\Phi}_+, \tilde{\Phi}_-) = (-1)^{m-1/2}\, 2i\lambda$ $\;W(f, g) = f'g - fg')$ and $W(\tilde{\varphi}, \tilde{\Phi}_+) = (-1)^{m-1/2}\, \lambda^{-m+1/2}\, \tilde{F}(\lambda)$ (here $\tilde{\Phi}_+ = \tilde{\Phi}(x, \lambda)$ and $\tilde{\Phi}_- = \tilde{\Phi}(x, -\lambda)$). Using properties of the form $\tilde{\varphi}^* = \tilde{\varphi}$ $\;(\tilde{\varphi}^* = \tilde{\varphi}^-)$ and $\tilde{\Phi}(x, \lambda)^* = (-1)^{m-1/2}\, \tilde{\Phi}(x, -\lambda)$ for λ real, one obtains $\tilde{F}^*(\lambda) = \exp(m - 1/2)\, 2i\pi\, \tilde{F}(-\lambda)$ which is nice for $m - 1/2$ integral but otherwise is unpleasant. Here as $x \to \infty$ the normalization for $\tilde{\Phi}_+$ is

$$\tilde{\Phi}_+ \sim e^{i\pi(m-1/2)/2}\, e^{i\lambda x}, \tag{3.38}$$

and if different Jost solutions $\tilde{\Psi}$ are chosen, normalized by

$$\tilde{\Psi}_+ \sim e^{i\lambda x};\ \tilde{\Psi}'_+ \sim i\lambda e^{i\lambda x} \tag{3.39}$$

as $x \to \infty$, then $W(\tilde{\Psi}_+, \tilde{\Psi}_-) = 2i\lambda$, $\tilde{\Psi}^*_+ = \tilde{\Psi}_-$, and one can write

$$\tilde{\varphi} = \frac{1}{2\lambda i} [\tilde{\mathscr{F}}(\lambda)\tilde{\Psi}_- - \tilde{\mathscr{F}}(-\lambda)\tilde{\Psi}_+] \tag{3.40}$$

where $\tilde{\mathscr{F}}(\lambda) = W(\tilde{\Psi}_+, \tilde{\varphi})$ and it follows that $\tilde{\mathscr{F}}^*(\lambda) = \tilde{\mathscr{F}}(-\lambda)$. This normalization leads to a more appropriate form for the transported equation for $P_m(D)$. Indeed from (3.36) $H = k_m x^{-m-1/2}\tilde{\varphi}$ and if we set $\Psi = x^{-m-1/2}\tilde{\Psi}$, then

$$H = \frac{k_m}{2\lambda i} [\tilde{\mathscr{F}}(\lambda)\Psi_- - \tilde{\mathscr{F}}(-\lambda)\Psi_+] \tag{3.41}$$

where $\tilde{\mathscr{F}}(\lambda) = k_m^{-1} x^{2m+1}\, W(\Psi_+, H)$. This is not quite what we want yet but let us summarize the above as a proposition ((3.40) might conceivably be more useful than (3.37) in physics).

PROPOSITION 3.6. *If one chooses Jost solutions for* $\tilde{P}_m(D)y = -\lambda^2 y$ *normalized by* (3.39), *then* (3.40) *holds with* $\tilde{\mathscr{F}}(\lambda) = W(\tilde{\Psi}_+, \tilde{\varphi})$ *and* $\tilde{\mathscr{F}}^*(\lambda) = \tilde{\mathscr{F}}(-\lambda)$ *for* λ *real. The corresponding* $\Psi = x^{-m-1/2}\tilde{\Psi}$ *satisfy* (3.41) *with* $\tilde{\mathscr{F}}(\lambda) = k_m^{-1} x^{2m+1}\, W(\Psi_+, H)$.

Now referring to [37], and modifying slightly the notation, if we consider $P(D)$ of the form (3.18) for suitable $A(x)$, then $P(D)y = -\lambda^2 y$ will have two linearly independent solutions $\Psi_\pm$ satisfying as $x \to \infty$

$$\Psi_\pm \sim A^{-1/2}(x)e^{\pm i\lambda x} \qquad (\lambda \neq 0). \tag{3.42}$$

Taking $A(x) = x^{2m+1}$ we obtain our $\Psi_\pm$ of Proposition 3.6. The function H can be expressed as

$$H(x, \mu) = c(\lambda)\Psi_+ + c(-\lambda)\Psi_- \tag{3.43}$$

which will be a standard form for harmonic analysis on symmetric spaces (cf. [25; 26; 45; 57]). One has here $x^{2m+1}\, W(\Psi_+, \Psi_-) = 2i\lambda$ and $x^{2m+1}\, W(\Psi_+, H) = 2i\lambda c(-\lambda)$. Comparing with Proposition 3.6 we have for λ real

$$k_m\tilde{\mathscr{F}}(\lambda) = 2i\lambda c(-\lambda). \tag{3.44}$$

Now recall that $H = 2^m\Gamma(m + 1)(\lambda x)^{-m}J_m(\lambda x)$ and $\Psi_+ = ie^{\pi i(m-1/2)/2}$ $x^{-m-1/2}(\pi\lambda x/2)^{1/2}\, H^1_m(\lambda x)$ so that near $x = 0$ we have $\Psi_+ \sim e_m\lambda^{-m+1/2}\, x^{-2m}$ where

$$e_m = \sqrt{\frac{\pi}{2}}\,\frac{2^m e^{i\pi(m-1/2)/2}}{\Gamma(1-m)\operatorname{Sin} m\pi}$$

and correspondingly $\Psi'_+ \sim -2me_m\lambda^{-m+1/2}\, x^{-2m-1}$. Recalling that $\Gamma(m)/\pi = 1/\Gamma/1 - m)\operatorname{Sin} m\pi$, we obtain as $x \to 0$, $x^{2m+1}\, W(\Psi_+, H) \to \lim x^{2m+1}\Psi'_+H$ $= -2me_m\lambda^{-m+1/2}$ from which follows

$$c(-\lambda) = ime_m\lambda^{-m-1/2} = \frac{1}{\sqrt{2\pi}}\, 2^m\Gamma(m + 1)\lambda^{-m-1/2}e^{i\pi(m+1/2)/2}. \tag{3.45}$$

Since $\overline{c(\lambda)} = c(-\lambda)$ for λ real, we get then

$$\frac{1}{2\pi|c(\lambda)|^2} = c_m^2\lambda^{2m+1} = R_0 = \hat{\nu}(\lambda) \tag{3.46}$$

where $c_m = 1/2^m\Gamma(m + 1)$ and R_0 is the spectral function of [9; 59] which in the present case is the measure $d\nu(\lambda) = \hat{\nu}(\lambda)d\lambda$. Thus we have the following theorem.

THEOREM 3.7. *The Jost function* $\tilde{\mathscr{F}}(\lambda)$ *for* $P_m(D)$ *is related to the Harish-Chandra function* $c(-\lambda)$ *by* (3.44) *for* λ *real and the spectral function* $R_0 = c_m^2\lambda^{2m+1} = \hat{\nu}(\lambda)$ *is given by* (3.46) *as* $1/2\pi|c(\lambda)|^2$.

4. Harmonic analysis and symmetric spaces. In order to have good background information available for our functions φ^P_λ, φ^Q_λ, etc., let us take $A(x) = \Delta_P(x)$ and $B(x) = \Delta_Q(x)$ of the form indicated in Example 3.2. Thus (cf. [25; 26; 34; 45; 57]).

EXAMPLE 4.1. Let $A(x) = \Delta(x) = (e^x - e^{-x})^p(e^{2x} - e^{-2x})^q$ (cf. [45]).

Then $\rho_A = (p + 2q)/2 = \rho$ and the spherical function φ_λ^P satisfying $\omega_{pq}\varphi_\lambda^P = -(\lambda^2 + \rho_A^2)\varphi_\lambda^P = \Delta^{-1}(x)D_x(\Delta(x)D_x\varphi_\lambda^P)$, $\varphi_\lambda^P(0) = 1$, and $D_x\varphi_\lambda^P(0) = 0$ is entire in λ for $t > 0$, even in λ, and $\bar{\varphi}_\lambda(t) = \varphi_{\bar{\lambda}}(t)$. For Im $\lambda \geqq 0$ there is a unique solution $\Phi_\lambda^P(t)$ of $\omega_{pq}\Phi_\lambda^P = -(\lambda^2 + \rho^2)\Phi_\lambda^P$ satisfying

$$\Phi_\lambda^P(t) = e^{(i\lambda-\rho)t}[1 + o(1)] \tag{4.1}$$

as $t \to \infty$. $\Phi_\lambda^P(t)$ is analytic in λ for $t > 0$ and $\lambda \in \{C/ - iN\} = \Omega$. For $\lambda \neq 0$ such that $\lambda, -\lambda \in \Omega$ there is a second linearly independent solution $\Phi_{-\lambda}^P(t) \sim e^{-(i\lambda+\rho)t}$ as $t \to \infty$ and one can write

$$\varphi_\lambda^P(t) = c_P(\lambda)\Phi_\lambda^P(t) + c_P(-\lambda)\Phi_{-\lambda}^P(t). \tag{4.2}$$

Here $\bar{\Phi}_\lambda^P(t) = \Phi_{-\bar{\lambda}}^P(t)$ and $\bar{c}_P(\lambda) = c_P(-\bar{\lambda})$. For $f \in L^2(\Delta dt)$ one defines $\hat{f}(\lambda)$ as in (3.23) and $f \leftrightarrow \hat{f}$ is a linear isometry $L^2(\Delta dt) \leftrightarrow L^2(d\nu)$ where $d\nu(\lambda) = (1/2\pi|c_P(\lambda)|^2)d\lambda$; the inversion is given by $\mathfrak{P}$ as in (3.23). We note also that (cf. §3) $\Delta(x)W(\Phi_\lambda^P, \varphi_\lambda^P) = 2i\lambda c_P(-\lambda)$.

EXAMPLE 4.2. A related example is developed in [57] in taking $A = \Delta_{\alpha\beta} = (e^t - e^{-t})^{2\alpha+1}(e^t + e^{-t})^{2\beta+1}$ (with $\rho = \rho_A = \alpha + \beta + 1$) and considering

$$\Delta_{\alpha\beta}^{-1}D_t[\Delta_{\alpha\beta}u'] = -(\lambda^2 + \rho_A^2)u. \tag{4.3}$$

The solution

$$\varphi_\lambda^{\alpha\beta}(t) = F\left(\frac{\rho + i\lambda}{2}, \frac{\rho - i\lambda}{2}, \alpha + 1, -sh^2 t\right) = \varphi_\lambda^P(t) \tag{4.4}$$

satisfying $\varphi_\lambda^{\alpha\beta}(0) = 1$ with $D_t\varphi_\lambda^{\alpha\beta}(0) = 0$ is called a Jacobi function of the first kind. For $\lambda \neq -i, -2i, -3i, \ldots$, another solution is given by

$$\Phi_\lambda^{\alpha\beta}(t) = (e^t - e^{-t})^{i\lambda-\rho} F\left(\frac{\beta-\alpha+1-i\lambda}{2}, \frac{\beta+\alpha+1-i\lambda}{2}, 1 - i\lambda, -sh^{-2}t\right) \tag{4.5}$$

and is called a Jacobi function of the second kind. One can write for nonintegral λ

$$\frac{2\sqrt{\pi}}{\Gamma(\alpha + 1)}\varphi_\lambda^{\alpha\beta}(t) = \hat{c}_P(\lambda)\Phi_\lambda^{\alpha\beta}(t) + \hat{c}_P(-\lambda)\Phi_\lambda^{\alpha\beta}(t) \tag{4.6}$$

where for real α, β, λ, $\bar{c}_P(\lambda) = \hat{c}_P(-\lambda)$. Since $\Phi_\lambda^{\alpha\beta}(t) = e^{(i\lambda-\rho)t}[1 + o(1)]$ as $t \to \infty$ the function $\Phi_\lambda^{\alpha\beta}$ correspond to the Φ_λ^P of Example 4.1 and $c_P(\lambda) \sim \Gamma(\alpha + 1)\hat{c}_P(\lambda)/2\sqrt{\pi}$. Explicit formulas for such $c_P(\lambda)$ are indicated below. One notes that

$$\begin{gathered} \varphi_{2\lambda}^{\alpha\alpha}(t) = \varphi_\lambda^{\alpha,-1/2}(2t);\ \Phi_{2\lambda}^{\alpha\alpha}(t) = \Phi_\lambda^{\alpha,-1/2}(2t); \\ \Delta_{\alpha\alpha}(t) = \Delta_{\alpha,-1/2}(2t);\ c_{\alpha\alpha}(2\lambda) = c_{\alpha,-1/2}(\lambda). \end{gathered} \tag{4.7}$$

REMARK 4.3. It will be useful to give here a more complete description of certain properties of such typical φ_λ^P, φ_λ^P, $c_P(\lambda)$, etc. In the case of Example 4.1 we have for example ($\rho = \rho_A = \rho_P$)

$$c_P(\lambda) = \frac{2^{\rho-i\lambda}\Gamma((p+q+1)/2)\Gamma(i\lambda)}{\Gamma((\rho+i\lambda)/2)\Gamma((p+1+i\lambda)/2)}. \tag{4.8}$$

Then $\lambda c_P(-\lambda)$ is analytic in Ω with zeros in the set $-i[\varepsilon, \infty)$ ($\varepsilon > 0$). For any $\varepsilon > 0$ there exists K_ε such that for all $\lambda = \xi + i\eta$ with $\eta \geqq -\varepsilon|\xi|$

$$\begin{aligned} |\lambda c_P(-\lambda)| &\leqq K_\varepsilon(1 + |\lambda|)^{1-(p+q)/2}; \\ |c_P(-\lambda)|^{-1} &\leqq K_\varepsilon(1 + |\lambda|)^{(p+q)/2}. \end{aligned} \tag{4.9}$$

The entire functions $\varphi_\lambda^P(t)$ has estimates

$$\begin{aligned} |D_t^n \varphi_\lambda^P(t)| &\leqq K_n(1 + |\lambda|)^n (1 + t)\, e^{(|\eta|-\rho)t}; \\ |D_t^n \varphi_\lambda^P(t)| &\leqq \tilde{K}_n(1 + t)^{n+1}\, e^{(|\eta|-\rho)t} \end{aligned} \tag{4.10}$$

for $\lambda = \xi + i\eta$, $t \in [0, \infty)$. For $\eta \geqq -\varepsilon|\xi|$ and $t \in [c, \infty)$ (any c, ε fixed)

$$\Phi_\lambda^P(t) = e^{(i\lambda-\rho)t}[1 + e^{-2t}\Phi(\lambda, t)] \tag{4.11}$$

where $|D_t^n \Phi(\lambda, t)| \leqq K_n$. In particular in such a region $|\Phi_\lambda^P(t)| \leqq ce^{-(\eta+\rho)t}$. Standard properties of the gamma function show that $c_P(\lambda) = 0$ for $\lambda = i(2m + \rho)$ and $c_P(\lambda) \to \infty$ at values $\lambda = in$ where $n \neq 2m + \rho + 1$.

Using these properties the following theorem is shown in Carroll-Gilbert **[25, 26]** (cf. Theorem 3.4 and Corollary 3.5).

THEOREM 4.4. *For P and Q of the type above $\gamma(x, y) = \ker \mathscr{B} = \langle \varphi_\lambda^P(x), \Delta_Q(y)\varphi_\lambda^Q(y)\rangle_\omega$ $\mathscr{B} = \mathscr{P}\mathfrak{Q}$: $\hat{Q} \to \hat{P}$) satisfies $\gamma(x, \cdot) \in \mathscr{E}'_y$ and $\gamma(x, y) = 0$ for $y > x$. Similarly $\tilde{\gamma}(x, y) = \langle \varphi_\lambda^P(x), \Delta_Q(y)\varphi_\lambda^Q(y)\rangle_\nu = \ker \tilde{\mathscr{B}}$ ($\tilde{\mathscr{B}} = \mathfrak{P}\mathfrak{Q}$: $\hat{Q} \to \hat{P}$) satisfies $\beta(y, \cdot) = \Delta_P(\cdot)\tilde{\gamma}(\cdot, y)\Delta_Q^{-1}(y) \in \mathscr{E}'_x$ and $\tilde{\gamma}(x, y) = 0$ for $x > y$.*

The triangularity properties in Theorem 4.4 follow from the analyicity and growth properties indicated in Examples 4.1–4.2 and Remark 4.3 plus the Paley-Winer type theorems of **[45; 57]** (see **[25; 26]** for details).

Now recall from Theorem 3.1 that $\mathscr{B}^* = \mathcal{Q}\mathscr{P}$ and define an operator

$$\hat{\mathscr{B}} = \mathfrak{Q}\mathfrak{P}. \tag{4.12}$$

This operator $\hat{\mathscr{B}}$ will replace U^* in the Fadeev formulation of §2. It is obvious first that

$$\hat{\mathscr{B}} = \tilde{\mathscr{B}}^{-1} = (\mathfrak{P}\mathfrak{Q})^{-1}. \tag{4.13}$$

Further we have (cf. (3.23)–(3.26)

$$\hat{\mathscr{B}}f(y) = \langle \varphi_\lambda^Q(y), \langle f(x), \varphi_\lambda^P(x)\Delta_P(x)\rangle\rangle_\omega, \tag{4.14}$$

(4.15) $$\mathscr{B}^*f(y) = \langle \varphi_\lambda^Q(y)\Delta_Q(y), \langle f(x), \varphi_\lambda^P(x)\rangle\rangle_\omega.$$

Consequently, setting $\hat{\beta}(y, x) = \ker \hat{\mathscr{B}}$,

(4.16) $$\hat{\beta}(y, x) = \langle \varphi_\lambda^Q(y), \varphi_\lambda^P(x)\Delta_P(x)\rangle_\omega$$

while from $\mathscr{B}^*f(y) = \langle \gamma(x, y), f(x)\rangle = \langle \gamma^*(y, x), f(x)\rangle$, or (4.15), we have $\gamma^*(y, x) = \ker \mathscr{B}^* = \langle \varphi_\lambda^Q(y)\Delta_Q(y), \varphi_\lambda^P(x)\rangle_\omega = \gamma(x, y)$ (cf. (3.17), Corollary 3.5, etc.). Recall also Theorem 3.4 where $\tilde{\mathscr{B}} = \mathscr{B}W^x$ so the analogue of $U^*UW^* = I$ in §2 becomes $\hat{\mathscr{B}}\mathscr{B}W^x = \mathfrak{Q}\mathfrak{P}\tilde{\mathscr{B}} = \mathfrak{Q}\mathfrak{P}\mathfrak{P}\mathfrak{Q} = I$. Let us summarize this in the following theorem.

THEOREM 4.5. *Define* $\hat{\mathscr{B}} = \mathfrak{Q}\mathfrak{P}: E \to F$. *Then* $\hat{\mathscr{B}}\mathscr{B}W^x = I$ *and* $\hat{\mathscr{B}} = \tilde{\mathscr{B}}^{-1}$. *The kernel* $\hat{\beta}(y, x)$ *of* $\hat{\mathscr{B}}$ *is given by* (4.16) *and*

(4.17) $$\gamma^*(y, x) = \gamma(x, y) = \Delta_Q(y)\Delta_P^{-1}(x)\hat{\beta}(y, x).$$

Hence $\hat{\beta}(y, x) = 0$ *for* $y > x$ *by Theorem* 4.4.

Let us cite one more result from Carroll-Gilbert [**25**; **26**] which is needed below. Take $P = D^2$ so $\Delta_P(x) = 1$ and set $\mathscr{B} = \mathscr{B}_Q, \hat{\mathscr{B}} = \hat{\mathscr{B}}_Q$, etc. Then from Theorem 4.5, $\mathscr{B}_Q^* = \Delta_Q(y)\hat{\mathscr{B}}_Q$ in fact (i.e., $\mathscr{B}_Q^*f(y) = \Delta_Q(y)\hat{\mathscr{B}}_Qf(y)$ from (4.14)–(4.15)). Using analyticity and growth properties of $\Phi_\lambda^Q(y)$, $c_Q(\lambda)$, etc. (from Examples 4.1–4.2 and Remark 4.3) one proves in [**25**; **26**] the following theorem.

THEOREM 4.6. *Take* $P = D^2$ *and put* Q *subscripts on all operators and kernels to indicate this. Then it can be proved sbstractly that*

(4.18) $$\frac{e^{i\lambda x}}{1/2} = \tilde{\mathscr{B}}_Q\left(\frac{\Phi_\lambda^Q(y)}{c_Q(-\lambda)}\right)(x).$$

We have emphasized the word "abstractly" in Theorem 4.6 since similar theorems can also be proved using properties of hypergeometric functions (cf. [**25**; **26**]) whereas the proof of Theorem 4.6 relies only on abstract techniques.

REMARK 4.7. Recall is §2 we used $Q = D^2$ as a base operator while Theorem 4.6 was stated for $P = D^2$ in conformity with the notation of [**25**, **26**]. We had previously introduced $\tilde{\mathscr{B}}$ then to correspond to $\tilde{U}$ (when $Q = D^2$) but now let us take $P = D^2$ (as the easiest way to resolve notational differences) so $U \sim B_Q: D^2 \to \hat{Q}$ $(\hat{Q}B_Q = B_QD^2)$ and $\tilde{U} \sim \tilde{B}_Q$ where in fact $\tilde{B}_Q = \tilde{\mathscr{B}}_Q^{-1}$. Indeed set $\tilde{W}(\lambda) = |c_P(\lambda)/c_Q(\lambda)|^2 = 1/W(\lambda)$ $(d\nu_P = W(\lambda)d\omega_Q$ and $d\nu_P = (1/2\pi|c_P(\lambda)|^2)d\lambda$ etc. from Example 4.1 so $d\omega_Q = \tilde{W}(\lambda)d\nu_P)$; here for $P = D^2$, $\varphi_\lambda^P(t) = \operatorname{Cos} \lambda t$, $\Delta_P(t) = 1$, $c_P(\lambda) = 1/2$, $d\nu_P = (2/\pi)d\lambda$, and $\tilde{W}(\lambda) = 1/4|c_Q(\lambda)|^2$. The notational clarifications are immediate (cf. Proposition 3.3, Theorem 3.4, etc.). Thus $P = D^2$, $B_Q =$

$\mathfrak{Q}\mathfrak{P}$ and $\Theta = BH$ so for $\tilde{\Theta} = \tilde{W}\Theta = \tilde{B}_Q H$ (definition of $\tilde{B}_Q$) we have $\tilde{B}_Q = \mathfrak{Q}\mathfrak{P}$. Indeed since $H \sim \varphi_\lambda^P$ and $\Theta \sim \varphi_\lambda^Q$, we have

$$\begin{aligned} \mathfrak{Q}\mathfrak{Q}\varphi_\lambda^P(x) &= \langle \varphi_z^Q(x), \langle \Delta_P(\xi)\varphi_z^P(\xi), \varphi_\lambda^P(\xi)\rangle\rangle_\omega \\ &= \langle \tilde{W}(z)\varphi_z^Q(x), \delta_\nu(\lambda - z)\rangle_\nu = \tilde{W}(\lambda)\varphi_\lambda^Q(x) \end{aligned} \tag{4.19}$$

(note $\mathfrak{Q}\tilde{g}(x) = \mathfrak{Q}\tilde{W}(\lambda)\tilde{g}(x)$). Thus the $\sim$ construction $B_Q \to \tilde{B}_Q$ gives $\tilde{B}_Q = \mathfrak{Q}\mathfrak{P}$ which happens to be $\hat{\mathscr{B}}_Q = \tilde{\mathscr{B}}_Q^{-1}$ from Theorem 4.5. Set now $\tilde{W}^x = \mathfrak{P}\tilde{W}(\lambda)\mathfrak{P}$ so

$$B_Q\tilde{W}^x = \mathfrak{Q}\mathfrak{P}\mathfrak{P}\tilde{W}(\lambda)\mathfrak{P} = \mathfrak{Q}\tilde{W}(\lambda)\mathfrak{P} = \mathfrak{Q}\mathfrak{P} = \tilde{B}_Q \tag{4.20}$$

and writing $\hat{B}_Q = \tilde{B}_Q^{-1}$ $(= \hat{\mathscr{B}}_Q)$ we have

$$\hat{B}_Q B_Q \tilde{W}^x = I; \; \hat{B}_Q = \mathfrak{P}\mathfrak{Q}. \tag{4.21}$$

When (4.21) in the form $B_Q\tilde{W}^x = \hat{B}_Q^{-1} = \tilde{B}_Q$ (as in (4.20)) is written out in terms of kernels it will be our version of the Gelfand-Levitan equation (cf. (2.37)–(2.38)).

Now referring back to (2.5) which says $\Phi(x, \lambda) = V[e^{iky}](x)$ we seek an analogue $\tilde{\mathscr{A}}_Q$ of V^{-1} in the form

$$e^{i\lambda x} = \tilde{\mathscr{A}}_Q[\Phi_\lambda^Q(y)](x) = \langle \tilde{A}_Q(x, y), \Phi_\lambda^Q(y)\rangle. \tag{4.22}$$

Setting $\hat{\mathscr{A}}_Q = \tilde{\mathscr{A}}_Q^{-1}(\hat{\mathscr{A}}_Q \sim V)$ this becomes

$$\begin{aligned} \Phi_\lambda^Q(y) &= \hat{\mathscr{A}}_Q[e^{i\lambda x}](y) = \langle \hat{A}_Q(y, x), e^{i\lambda x}\rangle \\ &= \int_{-\infty}^{\infty} \hat{A}_Q(y, x)e^{i\lambda x}\, dx = \mathscr{F}\hat{A}_Q(y, \cdot) \end{aligned} \tag{4.23}$$

and consequently

$$\hat{A}_Q(y, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\lambda^Q(y)\, e^{-i\lambda x}\, d\lambda. \tag{4.24}$$

We write integrals for Fourier transforms even when dealing with distribution pairings. Next $\hat{\mathscr{B}}_Q = \tilde{\mathscr{B}}_Q^{-1}$ has kernel $\hat{\beta}_Q$ given by (4.16) (with $\Delta_P(x) = 1$, $\varphi_\lambda^P(x) = \operatorname{Cos}\lambda x$, etc.) and $\hat{\beta}_Q(y, x) = 0$ for $y > x$. Hence from (4.18)

$$\begin{aligned} \frac{\Phi_\lambda^Q(y)}{c_Q(-\lambda)} &= \hat{\mathscr{B}}_Q\left(\frac{e^{i\lambda x}}{1/2}\right)(y) = 2\langle \hat{\beta}_Q(y, x), e^{i\lambda x}\rangle \\ &= 2\int_y^{\infty} \hat{\beta}_Q(y, x)\, e^{i\lambda x}\, dx. \end{aligned} \tag{4.25}$$

Now (cf. [**25**; **26**; **45**; **57**] and Remark 4.3) $\Phi_\lambda^Q(y)$ is analytic in λ for say $\operatorname{Im} \lambda > 0$ and for $\eta \geqq -\varepsilon|\xi|$ $(\lambda = \xi + i\eta)$ with $y \in [c, \infty)$ $(c, \varepsilon$ fixed) we have $\Phi_\lambda^Q(y) \sim e^{(i\lambda - \rho)y} (\rho = \rho_Q)$. Thus in (4.24) for $\eta \geqq 0$ the integrand is bounded by $e^{-\rho y}e^{-\eta(y-x)}$ and referring to a contour integral in the halfplane $\operatorname{Im}\lambda \geqq 0$ we obtain $\hat{A}_Q(y, x) = 0$ for $y > x$. Hence in (4.23) we obtain

$$\Phi_\lambda^Q(y) = \int_y^\infty \hat{A}_Q(y, x)\, e^{i\lambda x}\, dx. \tag{4.26}$$

Now using (4.25) and (4.26) we have

$$\begin{aligned} \hat{\beta}_Q(y, x) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\Phi_\lambda^Q(y)}{c_Q(-\lambda)}\, e^{-i\lambda x}\, d\lambda \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{c_Q(-\lambda)} \left(\int_y^\infty \hat{A}_Q(y, \xi) e^{i\lambda\xi}\, d\xi\right) d\lambda. \end{aligned} \tag{4.27}$$

This is an analogue to formula (2.24) of §2. If we can write now

$$\frac{1/2}{c_Q(-\lambda)} = \mathscr{F}\Psi_Q = \int_{-\infty}^{\infty} \Psi_Q(\xi)\, e^{i\lambda\xi}\, d\xi, \tag{4.28}$$

then from (4.27) we will have

$$\hat{\beta}_Q(y, x) = [\Psi_Q(\cdot) * \hat{A}_Q(y, \cdot)](x) = \int_y^\infty \Psi_Q(x - \xi)\hat{A}_Q(y, \xi)\, d\xi. \tag{4.29}$$

In this connection we record

LEMMA 4.8. *$c_Q^{-1}(-\lambda) \in \mathscr{S}'$ and $\Psi_Q \in \mathscr{S}'$ determined by* (4.28) *has support in* $[0, \infty)$.

PROOF. From Remark 4.3, etc., one knows that for $\lambda = \xi + i\eta$, $\eta \geqq -\varepsilon\,|\xi|$, $c_Q(-\lambda)$ is analytic with $|c_Q(-\lambda)|^{-1} \leqq p_\varepsilon(|\lambda|)$ (here $p_\varepsilon(|\lambda|) = K_\varepsilon(1 + |\lambda|)^\gamma$, $\gamma = (p + q)/2$, for Δ_Q as in Example 4.1). Now from (4.28) we define

$$\Psi_Q(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} c_Q^{-1}(-\lambda)\, e^{-i\lambda x}\, d\lambda. \tag{4.30}$$

But $-i\lambda x = x(\eta - i\xi)$ so for $x < 0$ we consider (4.30) as the limit of contour integrals in the half plane $\operatorname{Im}\lambda \geqq 0$. For example approximate first a large semicircular contour C by a sequence C_δ with base lines $\eta = \delta|\xi|$ so that the polynomial growth of $c_Q^{-1}(-\lambda)$ at ∞ is controlled by $\eta > 0$ in the exponent. More rigorously (cf. [**25**; **26**]) set $c_Q^{-1}(-\lambda) = 4\pi\mathscr{F}\Psi_Q$ and work in $\mathscr{S}'$ with the Parseval formula for $\hat{\varphi} \in \mathscr{S}$ and $\mathscr{F}\hat{\varphi} = \varphi \in \mathscr{S}$.

$$\langle c_Q^{-1}(-\lambda), \hat{\varphi}(\lambda)\rangle = \langle 4\pi\Psi_Q(x), \varphi(x)\rangle = \int_{-\infty}^{\infty} c_Q^{-1}(-\lambda)\hat{\varphi}(\lambda) d\lambda. \tag{4.31}$$

Now the integral makes sense for real λ by standard growth features of $\hat{\varphi} \in \mathscr{S}$ and if we take $\varphi \in \mathscr{D}$ with supp $\varphi \subset [-R, -\delta]$, then for $\eta =$

$\operatorname{Im}\lambda \geqq 0$ on a semicircle $|\lambda| = \hat{R}$, $|\hat{\varphi}(\lambda)| \leqq ce^{-\delta\eta}$, and $|c_Q^{-1}(-\lambda)\hat{\varphi}(\lambda)| \leqq p(|\lambda|)e^{-\eta\delta}$. Consequently for such φ and λ integral in (4.31) vanishes, so $\langle \Psi_Q(x), \varphi(x)\rangle = 0$ and hence the distribution $\Psi_Q(x)$ has support in $[0, \infty]$.

Using now Lemma 4.8 we can write (4.29) in the form

$$\hat{\beta}_Q(y, x) = \int_y^x \Psi_Q(x - \xi)\hat{A}_Q(y, \xi)d\xi \tag{4.32}$$

(the integral is formal of course) and this yields again $\mathscr{B}_Q(y, x) = 0$ for $y > x$. This formula is the analogue of (2.25) in §2 and we summarize this in

THEOREM 4.9. *The kernels $\hat{\beta}_Q$ and $\hat{A}_Q$ are related by* (4.32).

Now define, in anlogy with §2, an operator

$$\Xi_Q f(\xi) = \int_\xi^\infty \Psi_Q(x - \xi)f(x)dx. \tag{4.33}$$

Then, writing out the $\hat{\beta}_Q$ action from (4.32) we have

$$\begin{aligned} \langle\hat{\beta}_Q(y, x), f(x)\rangle &= \int_y^\infty \hat{\beta}_Q(y, x)f(x)dx \\ &= \int_y^\infty f(x)\int_y^x \Psi_Q(x - \xi)\hat{A}_Q(y, \xi)d\xi dx \\ &= \int_y^\infty \hat{A}_Q(y, \xi)\int_y^\infty \Psi_Q(x - \xi)f(x)dxd\xi. \end{aligned} \tag{4.34}$$

Consequently one has the following corollary to Theorem 4.8.

COROLLARY 4.10. $\hat{\mathscr{B}}_Q = \hat{\mathscr{A}}_Q\Xi_Q$.

This is of course analogous to $\tilde{U} = V\Xi$ in the quantum situation.

REMARK 4.11. We mention in passing the natural connection of generalized convolution with Parseval formulas (cf. [**9**; **46**]). Use the notation of [**9**] and write $T_x^y f(x) = \langle H(y, \mu), \hat{f}(\lambda)H(x, \mu)\rangle_\nu$ for $\hat{f}(\lambda) = \mathfrak{P}f(\lambda)$, so that $T_x^y f(x) = \langle f(\xi), \beta(x, y, \xi)\rangle$ with $\beta(x, y, \xi) = \int \Omega(\xi, \mu)H(x, \mu)H(y, \mu)d\nu_P$. We shall omit the factor of $1/\sqrt{2\pi}$ used in [**46**] (which gets absorbed in our $d\nu_P$) and set

$$(f*g(y) = \int f(x)T_x^y g(x)\Delta_P(x)dx. \tag{4.35}$$

We note then by an easy calculation that

$$(4.36)\qquad \begin{aligned}(f*g(y) &= \int f(x)\langle H(y,\mu), \hat{g}(\lambda)H(x,\mu)\rangle_\nu \Delta_P(x)dx \\ &= \langle H(y,\mu), \hat{g}(\lambda)\int f(x)H(x,\mu)\Delta_P(x)dx\rangle_\nu \\ &= \langle H(y,\mu), \hat{g}(\lambda)\hat{f}(\lambda)\rangle_\nu = \mathfrak{P}(\hat{g}\hat{f}),\end{aligned}$$

so that $\mathfrak{P}(f*g) = \hat{g}\hat{f}$. Hence

$$(4.37)\qquad (f*g)(0) = \int f(x)g(x)\Delta_P(x)dx = \int \hat{g}(\lambda)\hat{f}(\lambda)d\nu_P.$$

This is identical with the Parseval formulas of [9] if we write $f = \breve{f}/\Delta_P$, $g = \breve{g}/\Delta_P$ so that $\hat{f} = \mathfrak{P}f = \mathscr{P}\breve{f}$ and $\hat{g} = \mathscr{P}\breve{g}$ while $\int fg\Delta_P dx = \int \breve{f}\Delta_P^{-1/2}\breve{g}\Delta_P^{-1/2}\,dx$.

5. The Gelfand-Levitan and Marčenko equations. First recall that the Gelfand-Levitan equation will result from (4.20)–(4.21) when everything is expressed in terms of kernels. Now for $P = D^2$ we have $\mathfrak{P} = \mathscr{F}_C$ (Fourier cosine transform) and $\boldsymbol{\mathfrak{P}} = \mathscr{F}_C^{-1}$ Thus

$$(5.1)\qquad \begin{aligned}\tilde{W}^x f(y) &= \frac{2}{\pi}\int_0^\infty \operatorname{Cos}\lambda y\, \tilde{W}(\lambda)\int_0^\infty \operatorname{Cos}\lambda x f(x)dxd\lambda \\ &= \langle \tilde{W}(y,x), f(x)\rangle\end{aligned}$$

where $\tilde{W}(\lambda)$ is even and

$$(5.2)\qquad \begin{aligned}\tilde{W}(y,x) &= \frac{2}{\pi}\int_0^\infty \tilde{W}(\lambda)\operatorname{Cos}\lambda x\operatorname{Cos}\lambda y\, d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{W}(\lambda)[e^{i\lambda(x+y)} + e^{i\lambda(x-y)}]d\lambda\end{aligned}$$

which is $\breve{W}(x+y) + \breve{W}(x-y)$. Here envidently $\breve{W}(t)$ is even where

$$(5.3)\qquad \breve{W}(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{W}(\lambda)\,e^{i\lambda t}\,d\lambda,$$

Further we recall that $\beta_Q = \ker B_Q$ is given by (3.17) as

$$(5.4)\qquad \beta_Q(y,x) = \langle \Omega_\lambda^P(x), \varphi_\lambda^Q(y)\rangle_\nu = \frac{2}{\pi}\int_0^\infty \varphi_\lambda^Q(y)\operatorname{Cos}\lambda x\,d\lambda$$

and $\beta_Q(y,x) = 0$ for $x > y$ (cf. Theorem 4.4). Note here $\beta_Q(y,x)\Delta_Q(y) = \tilde{\gamma}_Q(x,y)$. Now in (4.20) we have

$$(5.5)\qquad \begin{aligned}B_Q\tilde{W}^x f(y) &= \langle \beta_Q(y,\xi), \langle \tilde{W}(\xi,x), f(x)\rangle\rangle \\ &= \langle f(x), \langle \beta_Q(y,\xi), \tilde{W}(\xi,x)\rangle\rangle\end{aligned}$$

and the kernel $K(y,x)$ of $B_Q\tilde{W}^x$ is

(5.6)
$$K(y, x) = \int_0^y \beta_Q(y, \xi)\tilde{W}(\xi, x)d\xi,$$

On the other hand the kernel $\tilde{\beta}_Q(y, x)$ of $\tilde{B}_Q = \hat{\mathscr{B}}_Q$ is $\hat{\beta}_Q(y, x)$, so from (4.16)

(5.7)
$$\begin{aligned}\tilde{\beta}_Q(y, x) &= \langle \varphi_\lambda^Q(y), \Omega_\lambda^Q(x)\rangle_\omega \\ &= \frac{1}{2\pi}\int_0^\infty \varphi_\lambda^Q(y) \operatorname{Cos} \lambda x \, d\lambda/|c_Q(\lambda)|^2\end{aligned}$$

and $\tilde{\beta}_Q(y, x) = 0$ for $y > x$. Hence (4.20) gives a Gelfand-Levitan type equation

(5.8)
$$\int_0^y \beta_Q(y, \xi)\tilde{W}(\xi, x)d\xi = 0$$

for $y > x$, while for $x > y$ we have

(5.9)
$$\int_0^y \beta_Q(y, \xi)\tilde{W}(\xi, x)d\xi = \tilde{\beta}_Q(y, x)$$

(note $\langle \tilde{\beta}_Q(y, x), f(x)\rangle = \int_y^\infty \tilde{\beta}_Q(y, x)f(x)dx$). In the quantum situation one writes all kernels $K(x, y)$ as $\delta(x - y) + k(x, y)$ so that (5.8) will kick out terms $\beta_Q(y, x)$ and $\tilde{W}(y, x)$. Let us leave this aspect for the moment however and summarize in the following theorem.

THEOREM 5.1. *Equation* (5.8), *with* (5.9), *is the Gelfand-Levitan equation associated with* (4.20)–(4.21).

Now we cannot really treat (4.24) or (4.32) as a kind of Marčenko equation. Indeed recalling $\hat{\mathscr{B}}_Q = \tilde{B}_Q$ we can write in Corollary 4.10, $\tilde{B}_Q = \hat{\mathscr{A}}_Q \Xi_Q$, which in terms of kernels becomes (cf. (4.29), (4.32), (5.7))

(5.10)
$$\tilde{\beta}_Q(y, x) = \int_y^x \Psi_Q(x - \xi)\hat{A}_Q(y, \xi)d\xi.$$

But $\tilde{B}_Q(y, x) = 0$ for $y > x$ doesn't provide an equation. Consider $\tilde{B}_Q = \hat{\mathscr{A}}_Q \Xi_Q$ in conjunction with $B_Q \tilde{W}^x = \tilde{B}_Q$ to get for example

(5.11)
$$B_Q = \hat{\mathscr{A}}_Q \Xi_Q \tilde{W}^{-1}.$$

Let us examine the operator $\Xi_Q \tilde{W}^{-1} = \Gamma_Q$. Recall $\tilde{W}^x = \mathfrak{P}\tilde{W}(\lambda)\mathfrak{P}$ and $\mathfrak{P} = \mathscr{F}_C$ (Fourier cosine transform). We will write $\tilde{W}^{-1}(\lambda) = W(\lambda) = 4|c_Q(\lambda)|^2$ and $W = \tilde{W}^{-1}$. As in (5.1)–(5.2) one has

(5.12)
$$\begin{aligned}\tilde{W}^{-1}f(y) &= \langle W(y, x), f(x)\rangle = W^x f(y); \\ W(y, x) &= \frac{2}{\pi}\int_0^\infty W(\lambda) \operatorname{Cos} \lambda x \operatorname{Cos} \lambda y \, d\lambda,\end{aligned}$$

Write as in (5.3)

$$\hat{W}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(\lambda)\, e^{i\lambda t}\, d\lambda \tag{5.13}$$

and then $W(y,x) = \hat{W}(x+y) + \hat{W}(x-y)$. Note here $W(\lambda)$ is even in λ so $\hat{W}(t)$ is even in t. Now go to (5.11) and write $W = W_1 + W_2$ where $\ker W_1 = \hat{W}(x-y) = \hat{W}(y-x)$ and $\ker W_2 = \hat{W}(x+y)$. For W_1 use $\hat{W}(x-y)$ to obtain

$$\begin{aligned} \mathscr{F} W_1 f &= \int_{-\infty}^{\infty} e^{i\lambda y} \int_0^{\infty} \hat{W}(x-y) f(x)dx\, dy \\ &= W(\lambda)\int_0^{\infty} f(x)e^{i\lambda x}dx = W(\lambda)\mathscr{F}f \end{aligned} \tag{5.14}$$

where some decision about f should be made. Thus we could work with even f for example and extend the x integral over $(-\infty, \infty)$ or simply set $f(x) = 0$ for $x < 0$ (the latter choice will recommend itself as in §2—cf. [43]). In either case we consider $\int_{-\infty}^{\infty}$ as meaningful.

Now from (4.33) written as $\Xi_Q f(\xi) = \int_{-\infty}^{\infty} \Psi_Q(x-\xi)f(x)dx = \check{\Psi}_Q * f(\check{g}(x) = g(-x))$ and (4.28) stating that $1/2/c_Q(\lambda) = 2\pi\mathscr{F}^{-1}\Psi_Q$ we have

$$\mathscr{F}\Xi_Q f = \mathscr{F}[\check{\Psi}_Q * f] = \mathscr{F}\check{\Psi}_Q\mathscr{F}f = \overline{\mathscr{F}\Psi_Q}\,\mathscr{F}f = \frac{1/2}{c_Q(\lambda)}\mathscr{F}f \tag{5.15}$$

(note $\mathscr{F}\check{g} = \overline{\mathscr{F}g} = 2\pi\mathscr{F}^{-1}g$). In (5.11) now we multiply by $\check{\Xi}_Q$ to get

$$B_Q\check{\Xi}_Q = \hat{\mathscr{A}}_Q\Xi_Q W\check{\Xi}_Q \tag{5.16}$$

where $\check{\Xi}_Q f = \Psi_Q * f$, i.e.,

$$\check{\Xi}_Q f(\xi) = \int_{-\infty}^{\infty} \check{\Psi}_Q(x-\xi)f(x)dx = \int_{-\infty}^{\infty} \Psi_Q(\xi - x)f(x)dx. \tag{5.17}$$

This "intuitive" step will lead to a formulation very close to that of §2. Consider now (cf. (5.14)–(5.15))

$$\begin{aligned} \Gamma_Q^1 &= \Xi_Q W_1\check{\Xi}_Q = \mathscr{F}^{-1}\left(\frac{1/2}{c_Q(\lambda)}\mathscr{F}W_1\check{\Xi}_Q\right) \\ &= \mathscr{F}^{-1}\left(\frac{W(\lambda)}{4c_Q(\lambda)c_Q(-\lambda)}\mathscr{F}\right) = \mathrm{I} \end{aligned} \tag{5.18}$$

since $\mathscr{F}\check{\Xi}_Q f = \mathscr{F}(\Psi_Q * f) = \mathscr{F}\Psi_Q\mathscr{F}f = (1/2/c_Q(-\lambda))\mathscr{F}f$. On the other hand, recalling that $\hat{W}(t)$ is even,

$$\begin{aligned} W_2 f(y) &= \int \hat{W}(x+y)f(x)dx \\ &= \int \hat{W}(-x-y)f(x)dx = (\hat{W} * f)^{\vee}(y) \end{aligned} \tag{5.19}$$

where $\check{f}(y) = f(-y)$. Hence

$$\mathscr{F} W_2 f = \mathscr{F}(\hat{W} * f)^{\vee} = \overline{\mathscr{F}\hat{W}}\, \overline{\mathscr{F} f} = \mathscr{F}\hat{W}\, \overline{\overline{\mathscr{F} f}} \tag{5.20}$$
$$= \mathscr{F}\hat{W}\mathscr{F}\check{f} = W(\lambda)\mathscr{F}\check{f}.$$

Therefore we have first

$$\Xi_Q W_2 f = \mathscr{F}^{-1}\left(\frac{1/2}{c_Q(\lambda)}\right)\mathscr{F} W_2 f = \mathscr{F}^{-1}\left(\frac{W(\lambda)}{2c_Q(\lambda)}\right)\mathscr{F}\check{f} \tag{5.21}$$
$$= \mathscr{F}^{-1}\left(\frac{c_Q(-\lambda)}{1/2}\right)\mathscr{F}\check{f}$$

and then from above we obtain

$$\Gamma_Q^2 f = \Xi_Q W_2 \check{\Xi}_Q f = \mathscr{F}^{-1}\left(\frac{c_Q(-\lambda)}{1/2}\right)\mathscr{F}(\check{\Xi}_Q f)^{\vee}$$
$$= \mathscr{F}^{-1}\left(\frac{c_Q(-\lambda)}{1/2}\right)(\mathscr{F}\check{\Xi}_Q f)^{-} \tag{5.22}$$
$$= \mathscr{F}^{-1}\left(\frac{c_Q(-\lambda)}{c_Q(\lambda)}\right)\overline{\mathscr{F} f} = \mathscr{F}^{-1}S(-\lambda)\mathscr{F}\check{f}.$$

Therefore we have shown ($W = \tilde{W}^{-1}$)

$$\Xi_Q W \check{\Xi}_Q f = f + \mathscr{F}^{-1}S(-\lambda)\mathscr{F}\check{f} \tag{5.23}$$

where $S(\lambda)$ is a "scattering" term. Now set

$$\mathscr{S}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\lambda)\, e^{i\lambda t}\, d\lambda \tag{5.24}$$

so that

$$S(-\lambda) = \int_{-\infty}^{\infty} \mathscr{S}(t)\, e^{i\lambda t}\, dt = \mathscr{F}\mathscr{S} \tag{5.25}$$

and hence

$$\mathscr{F}^{-1} S(-\lambda)\mathscr{F}\check{f} = \mathscr{F}^{-1}\mathscr{F}\,[\mathscr{S} * \check{f}] = \int_0^{\infty} \mathscr{S}(y + x) f(x) dx. \tag{5.26}$$

Here we write $\int_0^\infty$ in (5.26) following [**43**] and §2; this involves thinking of f defined only on $[0, \infty)$. Consequently, writing

$$\mathscr{S} f(y) = \int_0^{\infty} \mathscr{S}(y + x) f(x) dx \tag{5.27}$$

we have the following theorem (cf. (5.16)).

THEOREM 5.2. *The equation* $B_Q = \hat{\mathscr{A}}_Q \Xi_Q \tilde{W}^{-1}$ *of* (5.11) *becomes* (5.16), *or* $B_Q\check{\Xi}_Q = \hat{\mathscr{A}}_Q(\Xi_Q W \check{\Xi}_Q)$, *which in turn can be written as*

$$B_Q\check{\Xi}_Q = \hat{\mathscr{A}}_Q[I + S]. \tag{5.28}$$

This formula will produce a version of the Marčenko equation which is quite parallel to the quantum situation of **[43]** as outlined in §2; the scattering term $\mathscr{S}$ arises in much the same manner. We consider now the kernels in (5.28) so note first

$$B_Q\check{\Xi}_Q f(y) = \langle(\beta_Q(y, \xi), \int_{-\infty}^{\infty} \Psi_Q(\xi - x)f(x)dx\rangle \tag{5.29}$$
$$= \int_0^y \beta_Q(y, \xi)\left[\int_{-\infty}^{\xi} \Psi_Q(\xi - x)f(x)dx\right]d\xi.$$

Consequently we have ker $B_Q\check{\Xi}_Q = K_Q$ where

$$K_Q(y, x) = \int_x^y \beta_Q(y, \xi)\Psi_Q(\xi - x)d\xi \tag{5.30}$$

and $K_Q(y, x) = 0$ for $y < x$. On the other hand

$$\hat{\mathscr{A}}_Q[I + \mathscr{S}]f(y) = \int_y^{\infty} \hat{A}_Q(y, \xi)\left[f(\xi) + \int_0^{\infty} \mathscr{S}(\xi + x)f(x)dx\right]d\xi$$
$$= \int_y^{\infty} \hat{A}_Q(y, x)f(x)dx \tag{5.31}$$
$$+ \int_0^{\infty} f(x) \int_y^{\infty} \hat{A}_Q(y, \xi)\mathscr{S}(\xi + x)d\xi dx.$$

Hence (from (5.28)) for $y < x$

$$0 = \hat{A}_Q(y, x) + \int_y^{\infty} \hat{A}_Q(y, \xi)\mathscr{S}(\xi + x)d\xi, \tag{5.32}$$

while for $y > x$

$$\int_x^y \beta_Q(y, \xi)\Psi_Q(\xi - x)d\xi = \int_y^{\infty} \hat{A}_Q(y, \xi)\mathscr{S}(\xi + x)d\xi. \tag{5.33}$$

THEOREM 5.3. *The Marčenko equation associated with Theorem* 5.2 *can be written as* (5.32), *with* (5.33) *as a supplement.*

REMARK 5.4. Equations (5.8) and (5.32) take on a more familiar from if one writes for example $\beta_Q(y, x) = \delta(x - y) + L(y, x)$, $\hat{A}_Q(y, x) = \delta(x - y) + \hat{A}(y, x)$, etc. Thus we get first

$$\tilde{W}(y, x) + \int_0^y L(y, \xi)\tilde{W}(\xi, x)d\xi = 0 \tag{5.34}$$

for $x < y$, while for $y < x$

$$\hat{A}(y, x) + \mathscr{S}(y + x) + \int_y^{\infty} \hat{A}(y, \xi)\mathscr{S}(\xi + x)d\xi = 0 \tag{5.35}$$

The latter is of the same form as (2.40) except that $\mathscr{S}$ is replaced by $-\mathscr{S}$.

This arises because we are dealing with $W(y, x)$ of the form (5.12) whereas in §2 $W(y, x)$ involves sine functions instead of cosines (cf. (2.32)); thus W_2 and $-W_2$ are interchanged basically. If we work with $\check{\mathscr{W}}(t) = \check{W}(t) - \delta(t)$ (cf. (5.3)), then $\tilde{W}(\xi, x) = \tilde{\mathscr{W}}(\xi, x) + \delta(x + \xi) + \delta(x - \xi) \equiv \tilde{\mathscr{W}}(\xi, x) + \delta(x - \xi)$ and (5.34) becomes $(x < y)$

$$(5.36) \qquad \tilde{\mathscr{W}}(y, x) + L(y, x) + \int_0^y L(y, \xi)\tilde{\mathscr{W}}(\xi, x)d\xi = 0$$

which corresponds to (2.38). However such decompositions are not in general what is wanted (see below).

REMARK 5.5. Let us modify an argument in [9] to shed further light upon the Gelfand-Levitan equation. Thus we think of $P = D^2$ as known and recall (cf. (5.2)) for $\hat{W}(\lambda) = |c_P(\lambda)/c_Q(\lambda)|^2 = 1/4|c_Q(\lambda)|^2$, $\tilde{W}(y, x) = (2/\pi)\int_0^\infty \hat{W}(\lambda)$ Cos λx Cos $\lambda y\, d\lambda = \int_0^\infty$ Cos λx Cos $\lambda y\, d\omega_Q(\lambda)$. Since $\hat{W}(\lambda) = \int_{-\infty}^{\infty} \check{W}(t)e^{-i\lambda t}\, dt = 2\int_0^\infty \check{W}(t)$ Cos $\lambda t\, dt = \mathfrak{P}[2\check{W}(t)]$, we can write (note that $\tilde{W}(y, x)$ is a distribution in general)

$$(5.37) \qquad \tilde{W}(y, x) = T_x^y[2\check{W}(x)]$$

where T_x^y is the generalized translation associated with $D^2 = P$ (this agrees with (5.2)). Now consider (cf. [9])

$$(5.38) \qquad S_x^y\delta_Q(x) = \langle R, \Theta(x, \mu)\Theta(y, \mu)\rangle_\nu = \langle R, \varphi_\lambda^Q(x)\varphi_\lambda^Q(y)\rangle_\nu,$$

where S_x^y denotes the generalized translation associated with $\hat{Q}$ (or equivalently Q) and R is a generalized spectral function. Here $\delta_Q(x) = \langle R, \varphi_\lambda^Q(x)\rangle_\nu = \mathscr{Q}R$ and given that formally $S_x^y\delta_Q(x) = \delta(x - y)/\Delta_Q(y)$ we have a Parseval formula

$$(5.39) \qquad \begin{aligned} \iint f(x)g(y)S_x^y\delta_Q(x)dxdy &= \langle \Delta_Q^{-1/2}(y)f(y), \Delta_Q^{-1/2}(y)g(y)\rangle \\ &= \langle R, \mathscr{Q}f\mathscr{Q}g\rangle_\nu. \end{aligned}$$

Actually the argument in [9] (Part II, §4) based on Theorem 3.6 is essentially sufficient to justify (5.39). Now operate with $\mathscr{B}_Q = \mathscr{P}\mathfrak{Q}$ on $\delta_Q = \mathscr{Q}R$ to get $\mathscr{B}_Q\delta_Q = \mathscr{P}\mathfrak{Q}\mathscr{Q}R = \mathfrak{P}R$ (since $\mathscr{P}^{-1} = \mathfrak{Q}\mathscr{Q}\mathfrak{P}$) and then $R = \mathfrak{P}\mathscr{B}_Q\delta_Q$.

REMARK 5.6. It is tempting to work with decompositions $\beta_Q = \delta + L$ and correspondingly $\gamma_Q(x, y) = \delta(x - y) + K(x, y)$ as in the quantum mechanical situation; however in general the L and K which then appear will not be functions. This is easily seen from the kernels exhibited in [8; 9] for example and is studied in detail in [14]. In general in transmuting from $P = D^2$ to $\hat{Q} = Q + \rho_Q^2$ with ρ_Q as before we will expect $\gamma_Q(x, y) = \langle$Cos $\lambda x, \Delta_Q(y)\varphi_\lambda^Q(y)\rangle_\omega$ to be a distribution of order $\geqq 1$ and $\beta_Q(y, x) = \langle$Cos $\lambda x, \varphi(y)_\lambda^Q\rangle_\nu$ will be a function already (no $\delta(x - y)$ term).

Now observe that (continuing Remark 5.5)

$$\begin{aligned}\mathscr{B}_Q\delta_Q &= \langle \gamma_Q(x,y), \delta_Q(y)\rangle = \langle\langle \varphi_\lambda^P(x), \Delta_Q(y)\varphi_\lambda^Q(y)\rangle_\omega, \delta_Q(y)\rangle \\ &= \langle \varphi_\lambda^P(x), 1\rangle_\omega = \int_0^\infty \operatorname{Cos} \lambda x \, d\omega(\lambda) \\ &= \int_0^\infty \tilde{W}(\lambda) \operatorname{Cos} \lambda x \, d\lambda = \mathfrak{P}\tilde{W} \qquad (= 2\check{W}(x)).\end{aligned} \tag{5.40}$$

Hence we operate with $\mathfrak{P}$ to display R formally as the function

$$R = \mathfrak{P}\mathscr{B}_Q\delta_Q = 2\mathfrak{P}\check{W} = \tilde{W}(\lambda). \tag{5.41}$$

Consider φ_λ^P as a distribution multiplier in λ for example and we obtain using (5.37)

$$\begin{aligned}\phi(y,x) &= \mathfrak{P}[R\varphi_\lambda^P(x)] = \langle \varphi_\lambda^P(y), \varphi_\lambda^P(x)\langle \varphi_\lambda^P(\xi), 2\check{W}(\xi)\rangle\rangle_\nu \\ &= T_x^y[2\check{W}(x)] = \tilde{W}(y,x)\end{aligned} \tag{5.42}$$

(recall $\Delta_P(\xi) = 1$ and ker T_x^y has the form $\langle \varphi_\lambda^P(y)\varphi_\lambda^P(x), \varphi_\lambda^P(\xi)\rangle_\nu$ from **[9]** Part I). Hence (note also $\phi(y,x) = \phi(x,y)$)

$$\begin{aligned}R\varphi_\lambda^Q(y) &= R\langle \beta_Q(y,t), \varphi_\lambda^P(t)\rangle = \langle \beta_Q(y,t), \mathfrak{P}_x\phi(x,t)\rangle \\ &= \mathfrak{P}_x\langle \beta_Q(y,t), \phi(x,t)\rangle.\end{aligned} \tag{5.43}$$

On the other hand set $G(\lambda) = \mathscr{Q}\mathscr{B}^*g = \mathscr{P}g$ (cf **[9]**) and in (5.39) take $f = \delta(x-y)$ and $\mathscr{Q}f = \varphi_\lambda^Q(y)$ (formally) to obtain

$$\begin{aligned}\langle G(\lambda), R\varphi_\lambda^Q(y)\rangle_\nu &= \langle \mathscr{Q}\mathscr{B}^*g, R\varphi_\lambda^Q(y)\rangle_\nu = \Delta_Q^{-1}(y)\mathscr{B}^*g(y) \\ &= \langle g(x), \Delta_Q^{-1}(y)\gamma_Q(x,y)\rangle.\end{aligned} \tag{5.44}$$

We can also write (cf. Theorem 3.1)

$$\begin{aligned}\langle G(\lambda), R\varphi_\lambda^Q(y)\rangle_\nu &= \langle \mathscr{P}g, R\varphi_\lambda^Q(y)\rangle_\nu = \langle g, \mathscr{P}^*R\varphi_\lambda^Q(y)\rangle \\ &= \langle g(x), \mathscr{P}^*\{R\varphi_\lambda^Q(y)\}(x)\rangle \\ &= \langle g(x), \mathfrak{P}\{R\varphi_\lambda^Q(y)\}(x)\rangle\end{aligned} \tag{5.45}$$

so that from (5.44) and (5.45) one has $\mathfrak{P}\{R\varphi_\lambda^Q(y)\}(x) = \Delta_Q^{-1}(y)\gamma_Q(x,y)$ or

$$R\varphi_\lambda^Q(y) = \mathfrak{P}_x\Delta_Q^{-1}(y)\gamma_Q(x,y). \tag{5.46}$$

Eequating (5.46) and (5.43) we obtain (after $\mathfrak{P}$ action)

$$\Delta_Q^{-1}(y)\gamma_Q(x,y) = \int_0^y \beta_Q(y,t)\phi(x,t)dt \tag{5.47}$$

which for $y > x$ becomes the Gelfand-Levitan equation (5.8) since $\phi(x,y) = \phi(y,x) = \tilde{W}(y,x)$ (cf. also (5.9) for $y < x$). We summarize this in the following theorem.

THEOREM 5.7. *The Gelfand-Levitan equation* (5.8) *can be derived as indicated following procedures of* [9] *and written in the form*

$$\int_0^y \beta_Q(y, \xi)\, T_\xi^x \tilde{W}(\xi) d\xi = 0. \tag{5.48}$$

REMARK 5.8. Let us take $\beta_Q(y, x)$ as in (5.4) and $\tilde{W}(y, x)$ as in (5.2) and examine the Gelfand-Levitan equation (5.48) or (5.8). Thus (5.8) becomes

$$\begin{aligned}
&\int_0^y \beta_Q(y, \xi)\, \tilde{W}(\xi, x) d\xi \\
&= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty\int_0^y \varphi_\lambda^Q(y)\, \tilde{W}(z) \operatorname{Cos} xz \operatorname{Cos} \lambda\xi \operatorname{Cos} z\xi \, d\xi\, dz d\lambda \\
&= \frac{2}{\pi^2}\int_0^\infty\int_0^\infty \varphi_\lambda^Q(y)\, \tilde{W}(z) \operatorname{Cos} xz \left(\frac{\operatorname{Sin}(\lambda+z)y}{\lambda+z} + \frac{\operatorname{Sin}(\lambda-z)y}{\lambda-z}\right) dz d\lambda \\
&= \frac{2}{\pi^2}\int_0^\infty \tilde{W}(z) \operatorname{Cos} xz \int_{-\infty}^\infty \varphi_\lambda^Q(y) \frac{\operatorname{Sin}(\lambda-z)y}{\lambda-z} d\lambda dz = 0
\end{aligned} \tag{5.49}$$

(recall also $W(z)dz = (\pi/2)d\omega_Q(z)$). This formula should be exploitable and we will return to it in **[14]**.

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