ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 2, Spring 1982

GENERALIZED ALTERNATIVE AND MALCEV ALGEBRAS

I.R. HENTZEL AND H.F. SMITH

1. Introduction. As observed in [1], both alternative algebras and Malcev algebras satisfy the flexible law

$$(1) \qquad (x, y, x) = 0,$$

and

$$(2) \qquad (zx, x, y) = -x(z, y, x),$$

where the associator $(a, b, c) = (ab)c - a(bc)$. Algebras satisfying in addition to (1) and (2) the identity

$$(*) \qquad (xz, x, y) = -(z, y, x)x$$

were studied initially by Filippov [1], who showed that a prime algebra of this sort (with characteristic $\neq 2, 3$) must be either alternative, Malcev, or a Jordan nil-algebra of bounded index 3. In this paper we shall consider algebras (with characteristic $\neq 2, 3$) which satisfy only (1) and (2). (Note that algebras opposite to these satisfy instead (1) and (*).) We shall prove that in this variety nil-semisimple algebras are alternative, and that prime algebras are either alternative or nil of bounded index 3. We shall also establish for finite-dimensional algebras the standard Wedderburn principal theorem.

To begin with, there are some elementary consequences of (1) and (2) which need to be noted. We first set

$$T(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) \\ - w(x, y, z) - (w, x, y)z.$$

It can be verified by simply expanding the associators that in any algebra $T(w, x, y, z) = 0$. Also, the linearized form of (2) is

$$(2') \qquad (zx, w, y) + (zw, x, y) = -x(z, y, w) - w(z, y, x),$$

so that

$$F(z, x, w, y) = (zx, w, y) + (zw, x, y) + x(z, y, w) + w(z, y, x) = 0.$$

Received by the editors on May 23, 1980, and in revised form on September 29, 1980.

Then $0 = F(x, x, y, x) - T(x, y, x, x) = 2(x^2, y, x)$, using repeatedly the flexible law and its linearization $(x, y, z) = -(z, y, x)$. Thus we arrive at

$$(x^2, y, x) = 0. \tag{3}$$

Algebras which satisfy (1) and (3) are called noncommutative Jordan. In particular, provided the characteristic $\neq 2$, such algebras are power-associative [5], and as in [2] they satisfy the identity

$$(y, x^2, z) = x \circ (y, x, z), \tag{4}$$

where the symmetric product $a \circ b = ab + ba$.

Next (2) and flexibility imply $(x^2, x, y) = -x(x, y, x) = 0$, that is

$$(x^2, x, y) = 0. \tag{5}$$

Then using (3) and flexibility we have $0 = T(x, x, y, x) = (x, x, yx) - (x, x, y)x$. Hence flexibility, (2), and (4) give $0 = -(yx, x, x) + (y, x, x)x = x(y, x, x) + (y, x, x)x = (y, x^2, x)$. Thus we also have

$$(x, x^2, y) = 0. \tag{6}$$

We now use linearizations of (5) and (3) to obtain $(x^2, y, y) = -(x \circ y, x, y) = (y^2, x, x)$, so that

$$(x^2, y, y) = (y^2, x, x). \tag{7}$$

Using linearized (7), we then see $2(x^3, y, y) = (x^2 \circ x, y, y) = (y^2, x^2, x) + (y^2, x, x^2) = 0$ by (5), (6), and flexibility. Thus we arrive at

$$(x^3, y, y) = 0. \tag{8}$$

Finally, let A be a noncommutative Jordan algebra (with characteristic $\neq 3$), and let $N(A)$ denote the linear span of the set $\{x^3 \mid x \in A\}$. Then as in [1] $N(A)$ is an ideal of A. Hence from (8) actually follows

$$(zx^3, y, y) = 0 = (x^3 z, y, y). \tag{9}$$

2. Main Section. Let A be an algebra which satisfies (1) and (2), and denote by $B(A)$ the linear span of the set $\{(y, x, x) \mid x, y \in A\}$. We shall now establish two identities that imply $B(A)$ is an ideal of A.

PROPOSITION 1. *If A is a flexible algebra (with characteristic $\neq 2$) which satisfies (2), then the following identities hold in A:*

$$\begin{aligned} z(y, x, x) = &- (yz, x, x) - \{(yx, z, x) + (yx, x, z)\} \\ &- \{(z, x, yx) + (z, yx, x)\} + \{(xz, y, x) + (xz, x, y)\} \\ &+ \{(x, xz, y) + (x, y, xz)\} + \{(x, xy, z) + (x, z, xy)\}, \end{aligned} \tag{10}$$

$$\begin{aligned}(11)\quad (z, x, x)y = {} & 2\{(x, xz, y) + (x, y, xz)\} + 2\{(x, xy, z) + (x, z, xy)\} \\ & + \{(y, z, x^2) + (y, x^2, z)\} - (yz, x, x) \\ & - \{(yx, z, x) + (yx, x, z)\} - \{(z, x, yx) + (z, yx, x)\} \\ & + \{(xz, y, x) + (xz, x, y)\}.\end{aligned}$$

PROOF. First (2) implies $x(y, x, x) = -(yx, x, x)$. Then linearization of this identity gives

$$\begin{aligned}\text{(i)}\qquad z(y, x, x) = {} & -(yz, x, x) - \{(yx, z, x) + (yx, x, z)\} \\ & - \{x(y, z, x) + x(y, x, z)\}.\end{aligned}$$

Next using (2) and flexibility we obtain $0 = x(y, z, x) + (yx, x, z) - T(x, y, x, z) = \{x(y, z, x) + x(y, x, z)\} + \{(yx, x, z) + (x, yx, z)\} - (xy, x, z) + (xz, y, x)$. From this by flexibility one has

$$\begin{aligned}\text{(ii)}\qquad 0 = {} & \{x(y, z, x) + x(y, x, z)\} - \{(z, x, yx) + (z, yx, x)\} \\ & + \{(xz, y, x) + (xz, x, y)\} - (xz, x, y) - (xy, x, z).\end{aligned}$$

Now by flexibility $0 = T(x, y, x, y) = (xy, x, y) - (x, yx, y) + (x, y, xy) = (xy, x, y) - (x, y \circ x, y) + \{(x, xy, y) + (x, y, xy)\}$. But $0 = (y, x^2, y) = -(x, y \circ x, y)$ by flexibility and linearized (6). Substituting this in the preceding equation, we have

$$0 = (xy, x, y) + \{(x, xy, y) + (x, y, xy)\}.$$

Linearization of this last identity gives

$$\begin{aligned}\text{(iii)}\qquad 0 = {} & (xz, x, y) + (xy, x, z) + \{(x, xz, y) + (x, y, xz)\} \\ & + \{(x, xy, z) + (x, z, xy)\}.\end{aligned}$$

If we now add equations (i)–(iii), we arrive at (10).

To prove (11), we first use linearized (1) repeatedly to show

$$\begin{aligned}(z, x, x)y &= (z, x, x)y + T(z, x, x, y) \\ &= (zx, x, y) - (z, x^2, y) - (xy, x, z) + z(y, x, x) \\ &= (zx, x, y) + (x^2, z, y) - (xy, x, z) \\ &\quad - \{(x^2, z, y) + (z, x^2, y)\} + z(y, x, x) \\ &= (zx, x, y) + (x^2, z, y) - (xy, x, z) \\ &\quad + \{(y, z, x^2) + (y, x^2, z)\} + z(y, x, x).\end{aligned}$$

Now by linearized (5) $(zx, x, y) + (x^2, z, y) = -(xz, x, y)$. Substituting this in the preceding equation gives

$$\begin{aligned}\text{(iv)}\qquad (z, x, x)y = {} & -(xz, x, y) - (xy, x, z) \\ & + \{(y, z, x^2) + (y, x^2, z)\} + z(y, x, x).\end{aligned}$$

Then adding equations (iii) and (iv) we obtain

$$\text{(v)}\quad (z, x, x)y = \{(x, xz, y) + (x, y, xz)\} + \{(x, xy, z) + (x, z, xy)\} + \{(y, z, x^2) + (y, x^2, z)\} + z(y, x, x).$$

If we now use the already established (10) to substitute for $z(y, x, x)$ in (v), we arrive at (11).

COROLLARY. *If A is a flexible algebra (with characteristic $\neq 2$) which satisfies (2), then $B(A)$ is an ideal of A.*

PROOF. This follows immediately from (10) and (11), since

$$\{(a, b, c) + (a, c, b)\} = (a, b + c, b + c) - (a, b, b) - (a, c, c).$$

PROPOSITION 2. *If A is a flexible algebra (with characteristic $\neq 2, 3$) which satisfies (2), then the following identity holds in A:*

$$y^3 \circ (z, x, x) = 0. \tag{12}$$

PROOF. To facilitate notation let us set $t = y^3$. We shall now proceed to justify a series of equations whose sum will imply (12). First, from (11) we have

$$\begin{aligned}\text{(vi)}\quad -(z, x, x)t = {} & -2\{(x, xz, t) + (x, t, xz)\} - 2\{(x, xt, z) + (x, z, xt)\} \\ & - \{(t, z, x^2) + (t, x^2, z)\} + (tz, x, x) \\ & + \{(tx, z, x) + (tx, x, z)\} + \{(z, x, tx) + (z, tx, x)\} \\ & - \{(xz, t, x) + (xz, x, t)\}.\end{aligned}$$

We also need

$$\begin{aligned}\text{(vii)}\quad 0 &= F(z, t, x, x) - T(x, x, t, z) \\ &= (zt, x, x) + (zx, t, x) + t(z, x, x) + x(z, x, t) \\ &\quad - (x^2, t, z) + (x, xt, z) - (x, x, tz) + x(x, t, z) + (x, x, t)z.\end{aligned}$$

By (2) one has

$$\text{(viii)}\quad 0 = \{(zx, x, t) + x(z, t, x)\} - \{(tx, x, z) + x(t, z, x)\}.$$

From linearized (3) we obtain

$$\begin{aligned}\text{(ix)}\quad 0 = {} & -\{(x^2, t, z) + (xz, t, x) + (zx, t, x)\} \\ & -2\{(x^2, z, t) + (xt, z, x) + (tx, z, x)\},\end{aligned}$$

and from linearized (6)

$$\begin{aligned}\text{(x)}\quad 0 = {} & \{(t, x^2, z) + (x, tx, z) + (x, xt, z)\} \\ & + \{(z, x^2, t) + (x, zx, t) + (x, xz, t)\}.\end{aligned}$$

Linearized (7) implies

$$\text{(xi)} \qquad 0 = 2\{(x^2, z, t) + (x^2, t, z) - (tz, x, x) - (zt, x, x)\}.$$

Since $t = y^3$, from (8) and its linearization we have

$$\begin{aligned}\text{(xii)} \qquad 0 = {} & (t, x, x)z + \{(t, x, zx) + (t, zx, x)\} \\ & + \{x(t, x, z) + x(t, z, x)\} - \{(t, x, xz) + (t, xz, x)\} \\ & + \{(t, x^2, z) + (t, z, x^2)\}.\end{aligned}$$

Analogously from (9) follows

$$\text{(xiii)} \qquad 0 = (zt, x, x) + \{(tx, x, z) + (tx, z, x)\}.$$

If we now add equations (vi)–(xiii) and use the linearized flexible law to make repeated cancellations in addition to immediate ones, we arrive at $-(z, x, x)t = t(z, x, x)$ which is (12).

COROLLARY. *If A is a flexible algebra (with characteristic $\neq 2, 3$) which satisfies (2), then $B(A)$ is a nil ideal of bounded index 4.*

PROOF. Of course $B(A)$ is an ideal by the corollary to Proposition 1. Let

$$a = \sum_{i=1}^{n} \alpha_i(y_i, x_i, x_i) \in B(A).$$

Then

$$2a^4 = a^3 \circ a = a^3 \circ \sum_{i=1}^{n} \alpha_i(y_i, x_i, x_i) = \sum_{i=1}^{n} \alpha_i(a^3 \circ (y_i, x_i, x_i)) = 0$$

by (12), whence $a^4 = 0$.

We can now prove the following theorems.

THEOREM 1. *Let A be an algebra (with characteristic $\neq 2, 3$) which satisfies (1) and (2). If A is without nonzero nil ideals, then A is alternative.*

PROOF. Since by the preceding corollary $B(A)$ is a nil ideal of A, $B(A) = (0)$ by our assumption. Hence $(y, x, x) = 0 = (x, x, y)$ using flexibility, and such an algebra is alternative by definition.

THEOREM 2. *Let A be an algebra (with characteristic $\neq 2, 3$) which satisfies (1) and (2). If A is prime, then A is either alternative or nil of bounded index 3.*

PROOF. First let C be any ideal such that $(C, y, y) = 0$ for all $y \in A$. Linearizing this identity, for any ideal D one has

$$\text{(xiv)} \qquad (cd)a = -(ca)d + c(a \circ d),$$

where $c \in C$, $d \in D$, $a \in A$. This shows CD is a right ideal of A. Since also $(y, y, C) = 0$ by flexibility, analogously DC is a left ideal. In particular, by (8) we can let $C = N(A)$ and $D = B(A)$. Then since by Proposition 2 the elements of $N(A)$ and $B(A)$ anti-commute, it follows that $K = N(A)B(A)$ is an ideal of A.

Now $K \subseteqq N(A)$, so also $(K, y, y) = 0$ for all $y \in A$. Thus likewise K^2 is an ideal of A. Furthermore, the elements of K also anti-commute. Thus if c, d, $a \in K$, then $(cd)a = -(ca)d$ by (xiv). In particular, $(dc)a = -(cd)a = (ca)d = -d(ca)$, which shows the elements of K anti-associate. But then for $x, y, z, w \in K$ we have

$$\begin{aligned}(xy)(zw) &= -x[y(zw)] = x[(yz)w] \\ &= -[x(yz)]w = [(xy)z]w = -(xy)(zw).\end{aligned}$$

Hence it follows that K^2 is an ideal which squares to zero. Since A is prime, this means $K^2 = (0)$, and so in turn $K = (0)$. Thus either $B(A) = (0)$ or $N(A) = (0)$, which completes the proof of the theorem.

THEOREM 3 (WEDDERBURN PRINCIPAL THEOREM). *Let A be a finite-dimensional algebra which satisfies* (1) *and* (2) *over a field with characteristic $\neq 2, 3$; and let N be the nil radical of A. If A/N is separable, then $A = S + N$ (vector space direct sum) where S is a subalgebra of A such that $S \cong A/N$.*

PROOF. If A is without nonzero nil ideals, then by Theorem 1 A is alternative. Thus for such a nil-semisimple A we know $A = A_1 \oplus \cdots \oplus A_n$, where each A_i is simple with an identity element [6]. Also, since A is noncommutative Jordan, from [3] we know that if e is an idempotent in A then $A_e(i)A_e(i) \subseteqq A_e(i)$ for $i = 0, 1$. This means that by [4] we can now reduce consideration to the case when A itself has an identity element 1. But then $0 = F(1, x, x, y) = 2(x, x, y)$, whence $(x, x, y) = 0 = (y, x, x)$. Thus A is alternative, and so the result follows in this case from [6].

The authors would like to thank the referee for pointing out Theorem 2. They would also like to express appreciation to their colleague Leslie Hogben, who helped design and implement a computer program utilized in this work.

ACKNOWLEDGEMENT. This research was partially supported by the Iowa State University Sciences and Humanities Research Institute.

REFERENCES

1. V.T. Filippov, *On a generalization of alternative algebras and Malcev algebras* (Russian), Algebra *i* Logika **17** (1978), 102–117.

2. F.G. Florey, *A generalization of noncommutative Jordan algebras*, J. Algebra **23** (1972), 502–518.

3. K. McCrimmon, *Structure and representations of noncommutative Jordan algebras*, Trans. Amer. Math. Soc. **121** (1966), 187–199.

4. D.J. Rodabaugh, *On the Wedderburn principal theorem*, Trans. Amer. Math. Soc. **138** (1969), 343–361.

5. R.D. Schafer, *Noncommutative Jordan algebras of characteristic O*, Proc. Amer. Math. Soc. **6** (1955), 472–475.

6. ———, *An Introduction to Nonassociative Algebras*, Pure and Appl. Math., **22**, Academic Press, New York, 1966.

DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IA 50011