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ON AN OCTUPLE-PRODUCT IDENTITY

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ABSTRACT. The author represents an eightfold infinite product in two complex variables by a double series, which is subsequently simplified by infinite products.

1. Introduction. For each complex number x such that $|x| < 1$, the following identities are valid.

$$\prod_{n=1}^{\infty}(1 - x^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n + 1)x^{n(n+1)/2}, \tag{1}$$

$$\prod_{n=1}^{\infty}(1 - x^n)^3(1 - x^{2n-1})^2 = \sum_{n=-\infty}^{\infty}(6n + 1)x^{n(3n+1)/2}. \tag{2}$$

The first identity is a celebrated result due to Jacobi [**3**, p. 285], while the second is apparently due to Basil Gordon [**2**, p. 285]. These identities are respectively derived in similar fashion from the Gauss-Jacobi triple-product identity and G. N. Watson's quintuple-product identity, below stated as identities (3) and (4).

$$\prod_{n=1}^{\infty}(1 - x^n)(1 - ax^n)(1 - a^{-1}x^{n-1}) = \sum_{-\infty}^{\infty}(-1)^n x^{n(n+1)/2}a^n \tag{3}$$

$$\prod_{n=1}^{\infty}(1 - x^n)(1 - ax^n)(1 - a^{-1}x^{k-1})(1 - a^2x^{2n-1})(1 - a^{-2}x^{2n-1}) \tag{4}$$
$$= \sum_{-\infty}^{\infty} x^{n(3n+1)/2}(a^{3n} - a^{-3n-1}).$$

Both (3) and (4) are valid for each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$. For a proof of (3) see [**3**, p. 282], and for proof of (4) see [**1**, pp. 42–43]. By multiplying identities (1) and (2) we can obviously express the infinite produce $\Pi(1 - x^n)^6(1 - x^{2n-1})^2$ as a double series.

In this paper we express the product in terms of a different double series, which apparently is not a trivial transformation of the former. This result is here deduced as a corollary of the following theorem.

Key words and phrases: Octuple-product identity, Gauss-Jacobi triple-product identity, G.N. Watson's quintuple-product identity.

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THEOREM 1. *For each pair of complex numbers a, x, with $a \neq 0$ and $|x| < 1$,*

$$\prod_{n=1}^{\infty} (1 - x^n)^2(1 - ax^n)(1 - a^{-1}x^n)(1 - ax^{n-1})(1 - a^{-1}x^{n-1})$$

$$(1 - a^2x^{2n-1})(1 - a^{-2}x^{2n-1}) \tag{5}$$

$$= 2P(x) \sum_{n=-\infty}^{\infty} (-1)^n x^{2n^2} a^{4n} - Q(x) \sum_{n=-\infty}^{\infty} (-1)^n x^{2n^2} a^{4n}(ax^n + a^{-1}x^{-n})$$

where

$$P(x) = \prod_{n=1}^{\infty} (1 - x^{4n}),$$

$$Q(x) = \prod_{n=1}^{\infty} (1 - x^{12n})(1 - x^{12n-7})(1 - x^{12n-5})$$

$$+ x \prod_{n=1}^{\infty} (1 - x^{12n})(1 - x^{12n-11})(1 - x^{12n-1}).$$

2. Proof of Theorem 1. In identity (3) we replace a by a^{-1} and multiply the resulting identity by identity (4) to get

$$\prod_{n=1}^{\infty}(1 - x^n)^2(1 - ax^n)(1 - a^{-1}x^n)(1 - ax^{n-1})(1 - a^{-1}x^{n-1})$$

$$(1 - a^2x^{2n-1})(1 - a^{-2}x^{2n-1})$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n a^n x^{(n^2-n)/2} \sum_{m=-\infty}^{\infty} x^{m(3m+1)/2}(a^{3m} - a^{-3m-1})$$

$$= \sum_{n,m=-\infty}^{\infty} (a^{3m+n} - a^{-3m+n-1})(-1)^n x^{(n^2-n+3m^2+m)/2}$$

$$= \sum_{s=-\infty}^{\infty} a^s \sum_{m=-\infty}^{\infty} (-1)^{s-3m} x^{[(s-3m)^2-(s-3m)+3m^2+m]/2}$$

$$- \sum_{s=-\infty}^{\infty} a^s \sum_{m=-\infty}^{\infty} (-1)^{s+3m+1} x^{[(s+3m+1)^2-(s+3m+1)+3m^2+m]/2}$$

$$= \sum_{s=-\infty}^{\infty} (-1)^s x^{s^2/8} a^s \sum_{m=-\infty}^{\infty} (-1)^m x^{6(m-s/4)^2+2(m-s/4)}$$

$$+ \sum_{s=-\infty}^{\infty} (-1)^s x^{s^2/8} a^s \sum_{m=-\infty}^{\infty} (-1)^m x^{6(m+s/4)^2+2(m+s/4)}$$

$$= \sum_{j=-1}^{2} \left\{ \sum_{t=-\infty}^{\infty} (-1)^{4t+j} x^{2t^2+tj+j^2/8} a^{4t+j} \sum_{k=-\infty}^{\infty} (-1)^{k+t} x^{6(k-j/4)^2+2(k-j/4)} \right.$$

$$\left. + \sum_{t=-\infty}^{\infty} (-1)^{4t+j} x^{2t^2+tj+j^2/8} a^{4t+j} \sum_{k=-\infty}^{\infty} (-1)^{k-t} x^{6(k+j/4)^2+2(k+j/4)} \right\}$$

$$= -x \sum_{t=-\infty}^{\infty} (-1)^t x^{2t^2} a^{4t}(a^{-1}x^{-t}) \sum_{k=-\infty}^{\infty} (-1)^k x^{6k^2+5k}$$

$$- \sum_{t=-\infty}^{\infty} (-1)^t x^{2t^2} a^{4t}(a^{-1}x^{-t}) \sum_{k=-\infty}^{\infty} (-1)^k x^{6k^2-k}$$

$$+ \sum_{t=-\infty}^{\infty} (-1)^t x^{2t^2} a^{4t} \sum_{k=-\infty}^{\infty} (-1)^k x^{6k^2+2k}$$

$$+ \sum_{t=-\infty}^{\infty} (-1)^t x^{2t^2} a^{4t} \sum_{k=-\infty}^{\infty} (-1)^k x^{6k^2+2k}$$

$$- \sum_{t=-\infty}^{\infty} (-1)^t x^{2t^2} a^{4t}(ax^t) \sum_{k=-\infty}^{\infty} (-1)^k x^{6k^2-k}$$

$$- x \sum_{t=-\infty}^{\infty} (-1)^t x^{2t^2} a^{4t}(ax^t) \sum_{k=-\infty}^{\infty} (-1)^k x^{6k^2+5k}$$

$$+ x \sum_{t=-\infty}^{\infty} (-1)^t x^{2t^2} a^{4t}(a^2x^{2t}) \sum_{k=-\infty}^{\infty} (-1)^k x^{6k^2-6k}$$

$$+ x^3 \sum_{t=-\infty}^{\infty} (-1)^t x^{2t^2} a^{4t}(a^2x^{2t}) \sum_{k=-\infty}^{\infty} (-1)^k x^{6k^2+6k}$$

We now use the triple-product identity (3) to simplify the summations over k (while realizing that the last two k-sums vanish), and collect like terms to obtain the desired conclusion.

COROLLARY. *For each complex number x such that $|x| < 1$,*

$$\prod_{n=1}^{\infty} (1 - x^n)^6 (1 - x^{2n-1})^2$$

$$= -2P(x) \sum_{n=1}^{\infty} (-1)^n (4n)^2 x^{2n^2} + Q(x) \sum_{=-\infty}^{\infty} (-1)^n (4n+1)^2 x^{2n^2+n}. \tag{6}$$

PROOF. Let a, x be given and rewrite the right side of (5) as:

$$2P(x) + 2P(x) \sum_{n=1}^{\infty} (-1)^n x^{2n^2}(a^{4n} + a^{-4n})$$

$$- Q(x)(a + a^{-1}) - Q(x) \sum_{n=1}^{\infty} (-1)^n x^{2n^2+n}(a^{4n+1} + a^{-4n-1})$$

$$- Q(x) \sum_{n=1}^{\infty} (-1)^n x^{2n^2-n}(a^{4n-1} + a^{-4n+1}).$$

Further, let $G(a, x)$ be defined by

$$G(a, x) = \prod_{n=1}^{\infty} (1 - ax^n)^2 (1 - a^{-1}x^n)^2 (1 - a^2x^{2n-1})(1 - a^{-2}x^{2n-1}),$$

so that the left side of (5) become $(1 - a)(1 - a^{-1}) \prod (1 - x^n)^2 G(a, x)$. Now, put $a = e^{2it}$, and for brevity $f(t) = \prod_{n=1}^{\infty}(1 - x^n)^2 G(e^{2it}, x)$. Multiplying both sides of (5) by 4^{-1}, we have

$$
\begin{aligned}
f(t)\sin^2 t = \frac{1}{2}P(x) - \frac{1}{2}Q(x)\cos 2t \\
+ P(x)\sum_{n=1}^{\infty}(-1)^n x^{2n^2}\cos 8nt \\
- \frac{1}{2}Q(x)\sum_{n=1}^{\infty}(-1)^n x^{2n^2+n}\cos(8n+2)t \\
- \frac{1}{2}Q(x)\sum_{n=1}^{\infty}(-1)^n x^{2n^2-n}\cos(8n-2)t
\end{aligned}
$$

We now differentiate the foregoing identity twice with respect to t to get

$$
\begin{aligned}
&2f(t)\cos^2 t + 2D_t[f(t)\cos t]\sin t + D_t[\sin^2 t \cdot f'(t)] \\
&= 2Q(x)\cos 2t - 4P(x)\sum_{n=1}^{\infty}(-1)^n x^{2n^2}(4n)^2\cos 8nt \\
&\quad + 2Q(x)\sum_{n=1}^{\infty}(-1)^n x^{2n^2+n}(4n+1)^2\cos(8n+2)t \\
&\quad + 2Q(x)\sum_{n=1}^{\infty}(-1)^n x^{2n^2-n}(4n-1)^2\cos(8n-2)t.
\end{aligned}
$$

In the foreoging we put $t = 0$, cancel a factor of 2 from both sides of the resulting identity and effect a trivial transformation to obtain the desired conclusion.

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