ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Voluem 12, Number 2, Spring 1982

CUTTING AND PASTING INVOLUTIONS

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ABSTRACT. In a recent paper by the author it was shown that the only invariants for Z_p-equivariant cutting and pasting (where p is an odd prime and Z_p is the cyclic group of order p) are oriented Z_p-stratified cobordism and certain Euler characteristic criteria. This paper shows that for certain classes of Z_2-manifolds, i.e., manifolds with involution, there are Z_2-equivariant cutting and pasting results analogous to the odd prime case. An example is given to show that the results for Z_p-manifolds do not strictly carry over to Z_2-manifolds.

1. Introduction. Let M^n and N^n be non-null n-dimensional closed smooth (oriented) Z_2-manifolds, i.e., (oriented) manifolds with (orientation preserving) involutions. This paper gives necessary and sufficient conditions under which two Z_2-manifolds M^n and N^n are Z_2-equivariant cut and paste equivalent for certain classes of Z_2-manifolds. The results are analogous to the odd prime case [7], but the results do not strictly carry over for involutions.

It should be noted that the cutting and pasting relation used in this paper is not the same as SK_* as found for example in K.K.N.O. [4]. Our cutting and pasting relation has equivariant bordism as an invariant, which is not the case for $\mathrm{SK}_*^{Z_2}$. In Kosniowski's book [5], it is shown that $\mathrm{SK}_*^{Z_2}$ is determined precisely by the Euler characteristics of the manifolds in question, and the Euler characteristics of the fixed sets in each dimension.

We will aboid Kosniowski's language of slice types in equivariant bordism [5]. Instead, we will use a similar notion called "Z_2-stratified cobordism" which was originally defined in [8]. The motivation for the definition of a stratified bordism in the above sense is that it geometrically suggests how one would perform equivarient surgery used in detecting Z_2 cutting and pasting invariants.

DEFINITION 1.1. If M^n and N^n are n-dimensional closed smooth

AMS MOS subject classifications (1970); Primary 57D65, 57D85, 57D90.
Key words and phrases: Euler characteristic, Z_2-stratified cobordism, Z_2-equivariant cut and paste equivalence, involution, antipodal map, signature.
This paper constitutes part of the author's Ph.D. thesis written at M.I.T.
Received by the editors on September 5, 1979.

(oriented) Z_2-manifolds, then M^n *is* (*oriented*) Z_2-*equivariant cut and paste equivalent to* N^n *in one step* if there exists a compact smooth (oriented) n-dimensional Z_2-manifold P^n and four disjoint equivariant imbeddings i, $\hat{\imath}$, j, and $\hat{\jmath}$ of a closed smooth $(n-1)$-dimensional Z_2-manifold T^{n-1} into ∂P^n such that

a) $\partial P^n = i(T^{n-1}) + \hat{\imath}(T^{n-1}) + j(T^{n-1}) + \hat{\jmath}(T^{n-1})$, where $+$ denotes disjoint union, and

b) there are (orientation preserving) Z_2-equivariant diffeomorphisms

$$M^n \xrightarrow[\sim]{\phi} \begin{cases} P^n \text{ with identifications} \\ i(t) \sim \hat{\imath}(t), \text{ for every } t \in T^{n-1} \\ j(t) \sim \hat{\jmath}(t), \text{ for every } t \in T^{n-1} \end{cases}$$

$$N^n \xrightarrow[\sim]{\psi} \begin{cases} P^n \text{ with identifications} \\ i(t) \sim j(t), \text{ for every } t \in T^{n-1} \\ \hat{\imath}(t) \sim \hat{\jmath}(t), \text{ for every } t \in T^{n-1} \end{cases}$$

DEFINITION 1.2. If M^n and N^n are n-dimensional closed smooth (oriented) Z_2-manifolds, then M^n *is* (*oriented*) Z_2-*equivariant cut and paste equivalent to* N^n if there exist n-dimensional closed smooth (oriented) Z_2-manifolds $V_1^n, V_2^n, \ldots, V_k^n$ with $M^n = V_1^n$, $N^n = V_k^n$, and V_i^n (oriented) Z_2-equivariant cut and paste equivalent to V_{i+1}^n in one step, for $i = 1, 2, \ldots, (K-1)$.

REMARK 1.1. The above definitions were stated for manifolds without orientation and for oriented manifolds with orientation preserving involutions. Our cutting and pasting theorems will involve both of the above cases. Note that cutting and pasting in the sense of Definition 1.2 is an equivalence relation. If M^n is a Z_2-manifold as above, denote by $[M^n]$ or $\{M^n\}$ the class of M^n under Z_2-equivariant cut and paste equivalence for M^n unoriented, or M^n oriented with orientation preserving Z_2-action, respectively.

NOTATION 1.1. If M^n is an n-dimensional compact smooth (oriented) Z_2-manifold, let

a) M_m = union of the m-dimensional components of the fixed point set of M^n,

b) $(M^n - M_n)$ = union of all n-dimensional components of M^n which are not fixed by Z_2. Here—denotes set complement,

c) χ denote the Euler characteristic.

DEFINITION 1.3. Let M^n be an n-dimensional closed smooth (oriented) Z_2-manifold. Then M^n *bounds an* (*oriented*) Z_2-*stratified bordism* if there exists an $(n+1)$-dimensional compact smooth (oriented) Z_2-manifold

W^{n+1} with a Z_2-equivariant (orientation preserving) diffeomorphism ψ: $M^n \to \partial W^{n+1}$, and W_{m+1} being empty if M_m is empty, for every $m = -1, 0, 1, \ldots, n$. By convention, M_{-1} is empty.

DEFINITION 1.4. Let M^n and N^n be n-dimensional closed smooth (oriented) Z_2-manifolds. Then M^n is *(oriented) Z_2-stratified cobordant to N^n* if

a) M_m is empty if and only if N_m is empty for each $m = 0, 1, \ldots, n$,
b) $(M^n - M_n)$ is empty if and only if $(N^n - N_n)$ is empty,
c) $(M^n + N^n)$ bounds an (oriented) Z_2-stratified bordism,
d) $\chi(M_0) = \chi(N_0)$.

REMARK 1.2. It is clear that (oriented) Z_2-stratified cobordism is an equivalence relation on n-dimensional closed smooth (oriented) Z_2-manifolds.

2. Results analogous to the Odd Prime Case. According to [2], we see that the normal representation of the fixed point sets of an involution is given fiberwise by the antipodal map A. This shows, incidentally, that for orientation preserving involutions, the fixed point manifolds occur in even codimensions.

First there is a lemma necessary.

LEMMA 2.1. *Let M^n and N^n be non-null n-dimensional closed smooth free Z_2-manifolds. Then M^n is Z_2-equivariant cut and paste equivalent to N^n as free* (requiring that the manifold P^n in Definition 1.1 be free) *Z_2-manifolds if and only if*

a) $\chi(M^n) = \chi(N^n)$, *and*
b) $[M^n] = [N^n] \in N_n(Z_2)$, *where $N_n(Z_2)$ is the n-dimensional cobordism group of free Z_2-manifolds in the sense of* [2]

PROOF. See the proof of Lemma 2.1 in [7].

There is also an oriented version.

COROLLARY 2.1. *Let M^n and N^n be non-null closed smooth oriented free Z_2-manifolds. Then M^n is oriented Z_2-equivariant cut and paste equivalent to N^n as free Z_2-manifolds if and only if*

a) $\chi(M^n) = \chi(N^n)$, *and*
b) $[M^n] = [N^n] \in \Omega_n(Z_2)$, *where $\Omega_n(Z_2)$ is the n-dimensional cobordism group of free Z_2-manifolds in the sense of* [2].

Our cutting and pasting results begin with the following theorem.

THEOREM 2.1. *Let M^n and N^n be even dimensional closed smooth oriented Z_2-manifolds. Then M^n is oriented Z_2-equivariant cut and paste equivalent to N^n if and only if*

a) $\chi(M^n) = \chi(N^n)$,
b) $\chi(M_m) = \chi(N_m)$ for $m = 0, \ldots, n$, and
c) *M^n is oriented Z_2-stratified cobordant to N^n.*

PROOF. See the proof of Theorem 1.1 in [7]. The above theorem follows immediately from the proof of the odd prime case in [7], since the manifolds and their fixed point sets occur in even dimensions.

If the even dimensional Z_2-manifolds M^n and N^n are not endowed with orientations, there is the following result.

THEOREM 2.2. *Let M^n and N^n be even dimensional closed smooth Z_2-manifolds, and each with odd dimensional fixed point sets occuring in at most one odd dimension. Then M^n is Z_2-equivariant cut and paste equivalent to N^n if and only if*
a) $\chi(M^n) = \chi(N^n)$,
b) $\chi(M_m) = \chi(N_m)$ for $m = 0, \ldots, n$, *and*
c) *M^n is Z_2-stratified cobordant to N^n.*

PROOF. We know that conditions a), b), and c) are necessary for Z_2-equivariant cut and paste equivalence. See the proof of Theorem 1.1 in [7].

To show the sufficiency of conditions a), b), and c), examine the proof of Theorem 1.1 in [7] and see what must be changed. Let W^{n+1} be a Z_2-stratified cobordism between M^n and N^n, and let W_{m+1} be the embedded fixed point cobordism between M_m and N_m, from $m = 0, \ldots, n$.

In examining the surgery in obtaining W^{n+1} from $(M^n \times [0, 1])$, one finds that there are non-negative even integers λ_m and a non-negative even integer λ such that

$$\begin{aligned}\Big[M^n &+ \sum_m (\lambda_m + \chi(W_{m+1}) - \chi(M_m))(S^n, S^m) \\ &+ ((\lambda + (\chi(W^{n+1}) - \chi(M^n))) + (\sum_m (\chi(M_m) - \chi(W_{m+1}))))S^n\Big] \\ &= \Big[N^n + \sum_m \lambda_m(S^n, S^m) + \lambda S^n\Big].\end{aligned}$$

Some remarks on the above equality are in order. Remark 1.2 indicates that square brackets [ ] denote an unoriented Z_2-equivariant cutting and pasting class. Also, (S^n, S^m) denotes the Z_2-manifold which is the boundary of Z_2-manifold (D^{n+1}, D^{m+1}). Here (D^{n+1}, D^{m+1}) is the $(n+1)$-disk with Z_2-action $D^{n-m} \times D^{m+1} \to D^{n-m} \times D^{m+1}$ given by $(x, y) \to (A(x), y)$ where A is the antipodal map, and the action is equivariantly smoothed. Note that S^m is then the fixed point set of (S^n, S^m).

Additionally, it is easy to check that

$$((\lambda + (\chi(W^{n+1}) - \chi(M^n))) + (\sum_m (\chi(M_m) - \chi(W_{m+1}))))$$

is even, and that the action on the above even number of copies of S^n is gotten simply by interchanging the S^n's pairwise.

Notice that if M^n and N^n have no odd dimensional fixed point dimensions, then $(\chi(W_{m+1}) - \chi(M_m)) = 0$ for each even m and $(\chi(W^{n+1}) - \chi(M^n)) = 0$, by the proof of Lemma 2.15 in [7].

So, $[M^n + \lambda_m(S^n, S^m) + \lambda S^n] = [N^n + \lambda_m(S^n, S^m) + \lambda S^n]$.

A Z_2-equivariant analogue of Lemma 2.16 in [7] lets us deduce that $[M^n] = [N^n]$, if there are no odd dimensional fixed point dimensions.

On the other hand if M^n and N^n have fixed point manifolds in at most one odd dimension m, then we may deduce

$$\begin{aligned}[M^n &+ (\lambda_m + \chi(W_{m+1}) - \chi(M_m))(S^n, S^m)\\ &+ (\lambda + \chi(M_m) - \chi(W_{m+1}))S^n]\\ &= [N^n + \lambda_m(S^n, S^m) + \lambda S^n],\end{aligned}$$

by applying Z_2-analogues of Lemmas 2.15 and 2.16 in [7].

Also, m being odd implies $\chi(M_m) = 0$, so that

$$\begin{aligned}&[M^n + (\lambda_m + \chi(W_{m+1}))(S^n, S^m) + (\lambda - \chi(W_{m+1}))S^n]\\ &= [N^n + \lambda_m(S^n, S^m) + \lambda S^n].\end{aligned}$$

Previous remarks let us deduce that $\chi(W_{m+1}) = 0 \pmod 2$. To complete the proof, it suffices to show that

$$[(\lambda_m + \chi(W_{m+1}))(S^n, S^m) + (\lambda - \chi(W_{m+1}))S^n] = [\lambda_m(S^n, S^m) + \lambda S^n]$$

by the Z_2-analogue of Lemma 2.16 in [7].

We may as well assume λ_m is even by throwing in additional copies of (S^n, S^m) if needed.

Case 1) $\chi(W_{m+1}) = 0$:
There is nothing to show.

Case 2) $\chi(W_{m+1}) > 0$:
Then

$$\begin{aligned}&\left[(\lambda + \chi(W_{m+1}))(S^n, S^m) + (\lambda - \chi(W_{m+1}))S^n\right]\\ &= \left[\left(\frac{\lambda_m + \chi(W_{m+1})}{2}\right)(S^m \times (S^{n-m}, S^0))\right.\\ &\left. + \left(\frac{\lambda_m + \chi(W_{m+1})}{2}\right)(S^{m+1} \times S^{n-m-1}) + (\lambda - \chi(W_{m+1}))S^n\right],\end{aligned}$$

where the actions on S^m and S^{m+1} are trivial, the action on S^{n-m-1} is given by the antipodal map, and the free Z_2-action of the $(\lambda - \chi(W_{m+1}))S^n$ is gotten by interchanging the S^n pairwise.

Also, cutting along S^m (m is odd) and applying the theorem of D. Sullivan on W. Neumann in [6], gives

$$\Big[\Big(\frac{\lambda_m + \chi(W_{m+1})}{2}\Big)(S^m \times (S^{n-m}, S^0))$$

$$+ \Big(\frac{\lambda_m + \chi(W_{m+1})}{2}\Big)(S^{m+1} \times S^{n-m-1}) + (\lambda - \chi(W_{m+1}))S^n\Big]$$

$$= \Big[\frac{\lambda_m}{2}(S^m \times (S^{n-m}, S^0)) + \frac{\lambda_m}{2}(S^{m+1} \times S^{n-m-1})$$

$$+ \frac{\chi(W_{m+1})}{2}(S^{m+1} \times S^{n-m-1}) + (\lambda - \chi(W_{m+1}))S^n\Big]$$

$$+ \Big[\lambda_m(S^n, S^m) + \frac{\chi(W_{m+1})}{2}(S^{m+1} \times S^{n-m-1}) + (\lambda - \chi(W_{m+1}))S^n\Big]$$

$$= \Big[\lambda_m(S^n, S^m) + \lambda S^n\Big],$$

by Lemma 2.1.

Case 3) $\chi(W_{m+1}) < 0$:

Then

$$(\lambda_m + \chi(W_{m+1}))(S^n, S^m) + (\lambda - \chi(W_{m+1}))S^n$$
$$= (\lambda_m + \chi(W_{m+1}))(S^n, S^m) + (-\chi(W_{m+1}))S^n + \lambda S^n.$$

So that,

$$\Big[(\lambda_m + \chi(W_{m+1}))(S^n, S^m) + (-\chi(W_{m+1}))S^n + \lambda S^n\Big]$$

$$\overset{*}{=} \Big[\Big(\frac{\lambda_m + \chi(W_{m+1}))}{2}\Big)(S^m \times (S^{n-m}, S^0))$$

$$+ \frac{\lambda_m + \chi(W_{m+1}))}{2}(S^{m+1} \times S^{n-m-1})$$

$$+ (-\chi(W_{m+1}))S^n + \lambda S^n\Big], \text{ with } Z_2\text{-actions as in Case 2)}$$

$$\overset{*}{=} \Big[\frac{\lambda_m}{2}(S^m \times (S^{n-m}, S^0) + \frac{(\lambda_m + \chi(W_{m+1}))}{2}(S^{m+1} \times S^{n-m-1})$$

$$+ (-\chi(W_{m+1}))S^n + \lambda S^n\Big]$$

but cutting along S^m,

$$\overset{*}{=} \Big[\frac{\lambda_m}{2}(S^m \times (S^{n-m}, S^0)) + \frac{(\lambda_m + \chi(W_{m+1})}{2}(S^{m+1} \times S^{n-m-1})$$

$$+ \left(-\frac{\chi(W_{m+1})}{2}\right)(S^{m+1} \times (S^{n-m-1})) + \lambda S^n\Big]$$

$$= \left[\frac{\lambda_m}{2}(S^m \times (S^{n-m}, S^0)) + \frac{\lambda_m}{2}(S^{m+1} \times S^{n-m-1}) + \lambda S^n\right]$$

$$= \left[\lambda_m(S^n, S^m) + \lambda S^n\right].$$

This completes Case 3).

There is also the following theorem.

THEOREM 2.3. *Let M^n and N^n be odd dimensional closed smooth Z_2-manifolds, with fixed point sets occuring only in even dimensions. Then $[M^n] = [N^n]$ if and only if*

a) $\chi(M_m) = \chi(N_m)$, $m = 1, \ldots, n$, *and*

b) M^n *is Z_2-stratified cobordant to* N^n.

PROOF. See the proof of Theorem 1.1 in [7].

3. Where the Z_p-analogue fails. This section shows that for involutions, Z_2-stratified cobordism and the "standard" Euler criteria are not always enough to equivariantly cut and paste between two Z_2-manifolds.

PROPOSITION 3.1. *Let $M^{4\ell+1}$ and $N^{4\ell+1}$ be closed smooth $(4\ell + 1)$-dimensional oriented Z_2-manifolds. Assume that the fixed point manifolds of $M^{4\ell+1}$ and $N^{4\ell+1}$ occur only in a single odd dimension $(2K + 1)$. Moreover, assume that there is an oriented Z_2-stratified cobordism $W^{4\ell+2}$ between $M^{4\ell+1}$ and $N^{4\ell+1}$ with $\chi(W^{4\ell+2}) = 1 \bmod(2)$. If $\{M^{4\ell+1}\} = \{N^{4\ell+1}\}$, then $\ell \leqq K$.*

PROOF. Since $\chi(W^{4\ell+2}) = 1 \bmod(2)$, then any oriented cobordism $\tilde{W}^{4\ell+2}$ between $M^{4\ell+1}$ and $N^{4\ell+1}$ is such that $\chi(\tilde{W}^{4\ell+2}) = 1 \bmod(2)$. This follows from the fact that the signature τ of a closed smooth oriented $(4\ell + 2)$-dimensional manifold $V^{4\ell+2}$ is zero. Moreover, [10] shows that $\tau(V^{4\ell+2}) = \chi(V^{4\ell+2}) \bmod(2)$. Thus if $W^{4\ell+2}$ is as in the statement of the proposition, then $\chi(W^{4\ell+2} \bigcup_\partial(-W^{4\ell+2})) = 0 \bmod(2)$. Note that "$-$" denotes the opposite orientation. But $\chi(W^{4\ell+2} \bigcup_\partial(-\tilde{W}^{4\ell+2}) = \chi(W^{4\ell+2}) + \chi(\tilde{W}^{4\ell+2})$. So that, $\chi(\tilde{W}^{4\ell+2}) \equiv \chi(W^{4\ell+2}) \bmod(2)$ and $\chi(\tilde{W}^{4\ell+2}) = 1 \pmod 2$.

Hence, if $\{M^{4\ell+1}\} = \{N^{4\ell+1}\}$, the explicit oriented Z_2-stratified cobordism $\tilde{W}^{4\ell+2}$ that may be constructed between $M^{4\ell+1}$ and $N^{4\ell+1}$ by a sequence of one-step oriented Z_2-equivariant cut and paste operations in analogy to that of the odd prime case in [7] is such that $\chi(\tilde{W}^{4\ell+2}) = 1 \pmod 2$. Closely examining the cobordism $\tilde{W}^{4\ell+2}$ at each step shows that there is some codimension one Z_2 equivariant "cut" $T^{4\ell}$ such that $\chi(T^{4\ell}) = 1 \bmod(2)$. Conner and Floyd [2] show that $2K \geqq 2\ell$. This completes the proof of the proposition.

There is an immediate corollary.

COROLLARY 3.1. *Let* $(S^{4\ell+1}, S^{2K+1})$ *be an oriented* $(4\ell+1)$*-sphere with* Z_2*-action as described previously. If* $\{S^{4\ell+1}, S^{2K+1})\} = \{2(S^{4\ell+1}, S^{2K+1})\}$, *then* $\ell \leqq K$.

The above corollary indicates that one cannot hope in general to "equalize" odd dimensional spheres with orientation preserving involutions. In other words, Lemma 2.17 in [7] is not always true for involutions.

There are other "stable results" for cutting and pasting involutions, i.e., results that do not completely allow "absorption" of the Z_2-equivariant spheres in the sense of [7]. A fuller understanding of involutions could possibly give a complete theory of Z_2-cutting and pasting in the above sense.

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