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MULTI-VALUED MAPPINGS OF CLOSED AND BOUNDED SUBSETS OF A NORMED LINEAR SPACE—A MAPPING DEGREE

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ABSTRACT. Previous extensions of degree theory to multi-valued mappings have required convexity or acyclicity conditions on the domains or point images of the mapping being considered. By using a straightforward combination of the results of D. G. Bourgin with results of K. Geba and A. Granas, a degree is defined in this paper which removes the acyclicity conditions, provided that the point images are acyclic in high enough dimensions. Using the degree, some fixed point theorems are developed.

0. Introduction. The study of non-acyclic upper semi-continuous multi-valued mappings was initiated by the work of D.G. Bourgin in [**2**]. Using a generalization of the Vietoris-Begle Theorem due to E. G. Skljarenko, [**14**], Bourgin defined the notion of a degree for a certain class of multi-valued mappings whose domain is the closed unit ball in a Banach space. In this paper, using a cohomology functor introduced by K. Geba and A. Granas, [**8**], the Bourgin degree is extended to include the class of compact non-acyclic upper semi-continuous multi-valued mappings with domains closed and bounded subsets of a normed linear space. Using this degree we obtain some fixed point theorems. Results similar to ours have been obtained using different methods by J. Bryszewski in the setting of Banach spaces where the closed and bounded subset is assumed to be either a closed ball or the boundary of a closed ball, [**5**].

We would like to recall that multi-valeud mappings have also been considered by: J. Bryszewski and L. Gorniewicz, [**4**], L. Gorniewicz, [**10**], A. Granas and J. W. Jaworowski, [**11**], R. Connelly, [**6**], for subsets of Euclidean space; F. E. Browder, [**3**], for subsets in Banach spaces; and L. Gorniewicz and A. Granas, [**9**], for acyclic multi-valued mappings.

1. Preliminaries. Let H^* denote the Čech cohomology functor with integer coefficients, $\mathbf{Z}$, from the category of metric spaces and continuous

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maps to the category of graded abelian groups and homomorphisms of degree zero. The homology theory will be the reduced singular homology.

In what follows we shall refer to a metric space as simply a space. A continuous mapping between spaces will be called a map.

Let A be a subset of a space Y. Denote by $\mathrm{rd}_Y A$ the relative dimension of A in Y. From the definition given in **[13]**, we have $\mathrm{rd}_Y A = \sup \dim C$, where C is a closed subset of Y contained in A, and $\dim C$ is the topological dimension of C. We will also assume that $\mathrm{rd}_Y A < 0$ if and only if A is empty and in this case we put $\mathrm{rd}_Y A = -\infty$.

We recall that a mapping $f\colon X \to Y$ between two spaces is called proper provided the counter image, $f^{-1}(A)$, of a compact subset A of Y is a compact subset of X. Also f is closed, if for each closed subset B of X, the image of B under f, $f(B)$, is a closed subset in Y. The fact that a proper map is closed is evident.

Definition 1.1. The singular set for a surjective map $f\colon X \to Y$, with respect to the integer k, is defined by

$$\sigma_k(f) = \{y \in Y \mid H^k(f^{-1}(y)) \neq 0\}.$$

The singular set for f is

$$\sigma(f) = \bigcup_k \sigma_k(f).$$

Definition 1.2. (See Bourgin, **[2]**). The effective bound for non-acyclicity of a surjective mapping $f\colon X \to Y$ is the integer $P(f) = 1 + \sup(\mathrm{rd}_Y \sigma_k(f) + k)$. the supremum taken over all non-negative integers k.

We note, if $\sigma_k(f)$ is the empty set for each k, then $P(f) = -\infty$. If n is a positive integer and f is a mapping so that $\mathrm{rd}_Y \sigma_k(f) \leqq n - 2 - k$, for each k, then $P(f) \leqq n - 1$.

Our results depend heavily on the following generalization of the Vietoris-Begle Theorem **[14]**.

Theorem 1.3. (Skljarenko). *Let $f\colon X \to Y$ be a proper surjective map. If* $P(f) < \infty$, *then for each $i > P(f)$, the induced homomorphism $(f)^i$: $H^i(Y) \to H^i(X)$ is an isomorphism.*

Recall the notion of an n-Vietoris map as defined in **[4]**.

Definition 1.4. A map $q\colon X \to Y$ is called an n-Vietoris map if the following two conditions are satisfied:

(i) q is a proper surjective map,
(ii) $\mathrm{rd}_Y \sigma_k(q) \leqq n - 2 - k$, for each k.

THEOREM 1.5. *If $q: X \to Y$ is an n-Vietoris map, then $q^i: H^i(Y) \to H^i(X)$ is an isomorphism for each $i > n - 1$.*

PROOF. Since q is an n-Vietoris map, $P(q) \leqq n - 1$ and q is a proper surjective map, hence the conclusion follows from (1.3).

2. Multi-valued maps. Let X and Y be two spaces and assume that for every point $x \in X$, a non-empty subset $F(x)$ of Y is given: in this case we say that F is a multi-valued map from X to Y and write $F: X \to Y$. Let F be a multi-valued map, we can associate with F the following diagram of maps

$$\begin{tikzcd} X \arrow[rr, "F"] & & Y \\ & \Gamma(F) \arrow[ul, "q"] \arrow[ur, "p"'] & \end{tikzcd}$$

in which $\Gamma(F) = \{(x, y) \in X \times Y | y \in F(x)\}$ is the graph of F and the natural projections p and q are given by $q(x, y) = x$ and $p(x, y) = y$.

DEFINITION 2.1. The multi-valued mapping F: $X \to Y$ is called upper semi-continuous (usc), if the graph, $\Gamma(F)$, of F is closed in the product $X \times Y$.

We note, if F is usc then $F(x) \subset Y$, is a closed subset of Y for each $x \in X$.

DEFINITION 2.2. The upper semi-continuous mapping F: $X \to Y$ is said to be compact provided there is a compact subset $K \subset Y$, so that the closure of the image of X under F is contained in K, i.e., Cl $F(X) \subset$ K.

We observe that if F: $X \to Y$ is a compact usc mapping, then for each point $x \in X$, $q^{-1}(x)$ is mapped homeomorphically to $F(x)$ under p. It follows that $H^k(F(x))$ is isomorphic to $H^k(q^{-1}(x))$ for each $x \in X$. We shall denote the singular set of F, $\sigma(F)$, to be the singular set of q, i.e., $\sigma_k(F) = \sigma_k(q)$ and $\sigma(F) = \sigma(q)$. The effective bound for non-acyclicity of an usc compact mapping F: $X \to Y$, denoted P(F), is given by P(q), i.e., P(F) = P(q).

Our analysis is restricted to the case $Y = E$ is a normed linear space and X is a closed and bounded subset of E which supports the structure of a C^0-manifold with boundary. Following D. G. Bourgin, [**2**], we consider multi-valued mappings which satisfy the following definition.

DEFINITION 2.3. The multi-valued mapping $F: X \to E$, defined on the closed and bounded subset X of E, is called an admissible transformation provided:

(i) F is upper semi-continuous and compact,

(ii) F is fixed point free on the boundary of X,
(iii) $\mathrm{P}(F) < \infty$,
(iv) if $x \in \sigma(F)$, then $H^*(F(x))$ is finitely generated.

DEFINITION 2.4. Let $F\colon X \to Y$ be a multi-valued mapping, a point $x \in X$ is called a fixed point for F provided $x \in F(x)$.

Denote by $\mathscr{L}_E = \{L_\alpha \subset E \mid L_\alpha$ is a finite dimensional linear subset of $E\}$ and if $L_\alpha \in \mathscr{L}_E$, define $\mathrm{d}(\alpha) = \dim L_\alpha$. If $X \subset E$, let $\mathscr{L}_X = \{X_\alpha \mid X_\alpha = X \cap L_\alpha \neq \varnothing\}$. We note, if X is a closed and bounded subset of E, then X_α is a compact sunset of L_α for each α. We will also denote by $\mathrm{Bd}\ X_\alpha = \mathrm{Bd}\ X \cap L_\alpha$, provided $L_\alpha \cap \mathrm{Bd}\ X \neq \varnothing$, where $\mathrm{Bd}\ X$ is the boundary of X. The symbols, F_α and $\dot{F}_\alpha$, will denote the restrictions of an admissible transformation F to X_α and $\mathrm{Bd}\ X_\alpha$, respectively. Similarly $\Gamma(F_\alpha)$ and $\Gamma(\dot{F}_\alpha)$ are the corresponding graphs of F_α and $\dot{F}_\alpha$.

THEOREM 2.5. *Let q_α denote the projection of $\Gamma(F_\alpha)$ to X_α. If* $\mathrm{d}(\alpha) - 1 = k > \mathrm{P}(F)$, *then* $(\dot{q}_\alpha)^k\colon H^k(\mathrm{Bd}\, X_\alpha) \to H^k(\Gamma(\dot{F}_\alpha))$ *and* $(q_\alpha)^k\colon H^k(X_\alpha) \to H^k(\Gamma(F_\alpha))$, *are isomorphisms.*

PROOF. Since both X_α and $\mathrm{Bd}\ X_\alpha$ are compact and q_α is an n-Vietoris mapping, the conclusions follow from (1.5).

DEFINITION 2.6. Two admissible transformations F and G are called admissibly homotopic, if there exists an upper semi-continuous multi-valued mapping, $H\colon X \times I \to E$, $I = [0, 1]$ the closed unit interval, such that $H(x, t)$ is an admissible transformation for each $t \in I$, $H(x, 0) = F(x)$, and $H(x, 1) = G(x)$, for each $x \in X$.

COROLLARY 2.7. *Assume H is an admissible homotopy between the admissible transformations F and G, then the conclusion of* (2.5) *is valid for the projections $q(x, t)$, generated by $H(x, t)$, for each $t \in I$, and each of the induced isomorphisms are equal.*

3. **General position.** For a multi-valued mapping $F\colon X \to E$. define the multi-valued mapping $f\colon X \to E$ by $f(x) = x - F(x)$.

DEFINITION 3.1. The multi-valued mapping f is called an admissible compact vector field provided F is an admissible transformation.

THEOREM 3.2. *Let $f\colon X \to E$ be a compact vector field, then the image of X under f, $f(X)$, is closed bounded and $f(\mathrm{Bd}\ X)$ is disjoint from the origin, θ, of E.*

PROOF. If $f(x) = x - F(x)$, then F is usc, compact, and fixed point free on the boundary of X, hence the conclusion follows.

From (3.2) there is a bounded, symmetric, open neighborhood U about θ for which $f(\mathrm{Bd}\,X) \cap U = \varnothing$.

THEOREM 3.3. *Let C be a compact subset in E. If U is any symmetric open subset of E at θ, then there is an element $L_S \in \mathscr{L}_E$ and a continuous function $Q_S\colon C \to L_S$ so that $x - Q_S(x) \in U$, for all $x \in C$.*

PROOF. A complete proof of (3.3) can be found in [**12**, p. 500]. We will only indicate the construction of Q_S. Since C is compact there exist a finite number of points $\{x_i\}_{i=1}^k$ in C so that $C \subset \bigcup_{i=1}^k (U + x_i)$. Let L_S be the linear span of $\{x_i\}_{i=1}^k$, $\{\rho_i\}_{i=1}^k$ a partition of unity associated with $\{U + x_i\}_{i=1}^k$ and define $Q_S\colon C \to L_S$ by $Q_S(x) = \sum_{i=1}^k \rho_i(x)x_i$.

If $F\colon X \to E$ is an admissible transformation, then $\mathrm{Cl}\,F(X)$ is compact and there exists an open, bounded, symmetric, convex neighborhood of θ so that $f(\mathrm{Bd}\,X) \cap U = \varnothing$. From (3.3) there is a finite dimensional linear subspace $L_S \subset E$ and a continuous function Q_S: $\mathrm{Cl}\,F(X) \to L_S$, so that $x - Q_S(x) \in U$, for $x \in \mathrm{Cl}\,F(X)$

DEFINITION 3.4. If $f\colon X \to E$ is a compact vector field and U is an open, bounded, symmetric, convex neighborhood of θ in E, so that $f(\mathrm{Bd}\,X) \cap U = \varnothing$, then an element $L_\alpha \in \mathscr{L}_E$ is called admissible with respect to F and U, where $f(x) = x - F(x)$, provided $L_S \subset L_\alpha$ and $\mathrm{d}(\alpha) > \max(\mathrm{P}(F) + 2, 1)$.

Choose L_α admissible with respect to F and U. If $x \in L_\alpha$, define $f_\alpha(x) = x - Q_\alpha(F_\alpha(x))$, where Q_α is the restriction of Q_S to $F(X_\alpha)$ composed with the inclusion of L_S into L_α. Also denote by $\dot f_\alpha$ the restriction of f_α to $\mathrm{Bd}\,X_\alpha$. Consider the following diagram for L_α admissible for F and U:

$$
\begin{tikzcd}
I_\alpha & & \\
L_S \arrow[u, "\mathrm{inc}"] & F(\mathrm{Bd}\,X_\alpha) \arrow[l, "Q_S"'] \arrow[ul, "Q_\alpha"'] & \Gamma(\dot F_\alpha) \arrow[l, "\dot p_\alpha"'] \arrow[d, "\dot q_\alpha"] \\
& & \mathrm{Bd}\,X_\alpha \arrow[ul, "\dot F_\alpha"]
\end{tikzcd}
$$

Formally we have $\dot f_\alpha(x) = (\dot q_\alpha - Q_\alpha \dot p_\alpha)\dot q_\alpha^{-1}(x) \subset L_\alpha$, for each $x \in \mathrm{Bd}\,X_\alpha$. We designate the map in parenthesis by T_α, hence $f_\alpha = T_\alpha \circ q_\alpha^{-1}$. The restriction of T_α to $\mathrm{Bd}\,X_\alpha$ will be denoted by $\dot T_\alpha$.

The graph of $\dot F_S$, $\Gamma(\dot F_S)$, is a compact subset of $L_S \times \mathrm{Cl}\,F(x)$, hence the continuity of $\dot T_S$ implies that the image of $\dot T_S$ is contained in a ball centered at $\theta \in L_S$ of some finite radius, R. If $\theta \in \mathrm{Im}\,\dot T_S$, then $Q_S(y) = x$, for some $x \in \mathrm{Bd}\,X_S$ and $y \in F(x)$, therefore $x - y \in f(\mathrm{Bd}\,X_S)$ and $(x - y) = Q_S(y) - y \in U$, which contradicts the construction of U. It follows that the image of $\dot{T}_S$ is contained in an annulus, $A^S(r, R) \subset L_S$, centered at θ. We therefore have the theorem.

THEOREM 3.5. *Assume F, f, U and L_S are chosen as above. If L_α is chosen admissible for F and U, then*

$$(\dot{f}_\alpha)^k = [(\dot{q}_\alpha)^k]^{-1} \circ (\dot{T}_\alpha)^k \colon H^k(A^{k+1}(r, R)) \to H^k(\mathrm{Bd}\, X_\alpha),$$

$k = \mathrm{d}(\alpha) - 1$, *is a homomorphism and the annulus $A^{k+1}(r, R)$ depends only on L_S.*

PROOF. Since L_α is admissible, $k = \mathrm{d}(\alpha) - 1 > \mathrm{P}(F)$ and (2.5) implies $(\dot{q}_\alpha)^k$ is an isomorphism, we therefore have the indicated decomposition. Since $L_S \subset L_\alpha$, we have $A^S(r, R) \subset L_\alpha$ and can construct $A^{k+1}(r, R)$.

4. $\mathbf{H}^{\infty-n}$ Cohomology theory. In [8] K. Geba and A. Granas constructed a cohomology functor, $H^{\infty-*}$, with integer coefficients, $\mathbf{Z}$, from the category of closed and bounded subsets of a normed linear space and compact vector fields, to the category of graded abelian groups and homomorphisms of degree zero. For completeness we shall indicate the construction of this functor, for more details refer to [7] or [8].

Let L_α, $L_\beta \in \mathscr{L}_E$, so that $L_\alpha \subset L_\beta$ and $\mathrm{d}(\alpha) + 1 = d(\beta)$. Denote by L_β^+ and L_β^- the two closed half-spaces of L_β determined by L_α. For $X \subset E$, a closed and bounded subset of E, denote by $X_\beta^+ = X \cap L_\beta^+$ and $X_\beta^- = X \cap L_\beta^-$. The triples $(X_\beta, X_\beta^+, X_\beta^-)$ are refered to as a triad.

THEOREM 4.1. (MAYER-VIETORIS, [1, p. 94]). $(X_\beta, X_\beta^+, X_\beta^-)$ *be the above triad. Then the sequence,*

$$\cdots \to H^k(X_\beta^+ \cap X_\beta^-) \to H^{k+1}(X_\beta) \to H^{k+1}(X_\beta^+) \oplus H^{k+1}(X_\beta^-) \to \cdots$$

is exact.

We have $X_\alpha = X_\beta^+ \cap X_\beta^-$, hence (4.1) yields a homomorphism from $H^k(X_\alpha)$ into $H^{k+1}(X_\beta)$. Denote this homomorhpism by $\Delta_{\alpha,\beta}^k$.

COROLLARY 4.2. (see [8]). *The system* $\{H^{\mathrm{d}(\alpha)-n}(X_\alpha);\ \Delta_{\alpha,\beta}^{\mathrm{d}(\alpha)-n}\}$ *forms a direct system of Abelian groups.*

Following [8] we define the cohomology functor $H^{\infty-n}$ by $H^{\infty-n}(X) =$ dir limit $\{H^{\mathrm{d}(\alpha)-n}(X_\alpha);\ \Delta_{\alpha,\beta}\}$. In the following we will be interested only in the case $n = 1$. In this case, [7, p. 105], $H^{\infty-1}(X)$ is isomorphic to the 0th singular homology group of $V = E\backslash X$, the complement of X in E. It follows that $H^{\infty-1}(A(r, R))$ is isomorphic to the group of integers, $\mathbf{Z}$, where $A(r, R) = \{x \in E \mid r \leqq \|x\| \leqq R\}$ is the annulus determined by r and R.

Let $1 \in \mathbf{Z}$ be a generator of $\mathbf{Z}$. In the following assume X is a closed and bounded subset of E and $\{U_i\}_{i \in J}$, is the family of all bounded components of $E \backslash X$. Given $u_i \in U_i$. define g_i: $X \to A(r_i, R_i)$, $0 < r_i \leqq R_i$, by $g_i(x) = x - u_i$.

LEMMA 4.3. (See [7, p. 105]). *The Abelian group $H^{\infty-1}(X)$ is a free Abelian group with generators $\omega_i = g_i^*(1)$. If f: $X \to A(r, R)$ induces a homomorphism from $H^{\infty-1}(A(r, R))$ into $H^{\infty-1}(X)$, then*

$$f^{\infty-1}(1) = \sum_{s=1}^{t} n_s \omega_{i_s}$$

where $n_s \in \mathbf{Z}$.

From (3.5) the multi-valued compact vector field $\dot{f}$, induces a homomorphism $(\dot{f})^{\mathrm{d}(\alpha)-1}$ from the group $H^{\mathrm{d}(\alpha)-1}(A^{\mathrm{d}(\alpha)}(r, R))$ into the group $H^{\mathrm{d}(\alpha)-1}(\mathrm{Bd}\ X_\alpha)$.

THEOREM 4.4. *The family $\{(\dot{f}_\alpha)^{\mathrm{d}(\alpha)-1}\}$ induces a homomorphism*

$$(\dot{f})^{\infty-1} : H^{\infty-1}(A(r, R)) \to H^{\infty-1}(\mathrm{Bd}\ X).$$

PROOF. Assume L_α is chosen in the admissible range for F and U. For $\mathrm{d}(\alpha) + 1 = \mathrm{d}(\beta)$, let $\Delta_{\alpha, \beta}$, $\bar{\Delta}_{\alpha, \beta}$, and $\delta_{\alpha, \beta}$ be the Mayer-Vietoris homomorphisms for the triads $(\mathrm{Bd}\ X_\beta, \mathrm{Bd}\ X_\beta^+, \mathrm{Bd}\ X_\beta^-)$, $(A^{\mathrm{d}(\beta)}(r, R), A_+^{\mathrm{d}(\beta)}(r, R), A_-^{\mathrm{d}(\beta)}(r, R))$, and $(\Gamma(\dot{F}_\beta), \Gamma_+(\dot{F}_\beta), \Gamma_-(\dot{F}_\beta))$, respectively. We have that $\dot{q}_\beta$ is a mapping of the first and third triad. Let $e_{\mathrm{d}(\alpha)}^+$ and $e_{\mathrm{d}(\alpha)}^-$ be the upper and lower hemispherical cells of the unit sphere $S^{\mathrm{d}(\alpha)}$, in L_β, determined by the linear subspace L_α. By construction we have $\mathrm{Im}\ \dot{T}_\beta = \mathrm{Im}\ \dot{T}_S \subset A^S(r, R)$ for all admissible β. Since $A^S(r, R) \subset (e_{\mathrm{d}(\alpha)}^+ \times [r, R]) \cap (e_{\mathrm{d}(\alpha)}^- \times [r, R])$, $\dot{T}_\beta$ is a mapping of the second and third triads, hence from (3.5) the theorem follows.

5. The degree. Given an admissible transformation F: $X \to E$, from the closed and bounded subset X of E, we can construct the multi-valued compact vector field f: $X \to E$, by $f(x) = x - F(x)$. Let U be an open, symmetric, bounded, convex neighborhood of θ in E so that $U \cap f(\mathrm{Bd}X) = \emptyset$. Pick $L_S \in \mathscr{L}_E$ satisfying (3.3), then applying (4.3) and (4.4), we have,

$$(\dot{f})^{\infty-1}(1) = \sum_{s=1}^{t} n_s \omega_{i_s}$$

$n_s \in Z$.

DEFINITION 5.1. Assume F, U, and L_S are chosen as above, then define the relative degree of F, denoted $\mathrm{d}[F, U, Q_S]$, by

$$\mathrm{d}[F, U, Q_S] = \sum_{s=1}^{t} n_s.$$

The relative degree of F as defined depends on the choice of U and Q_S, we now show these dependents can be removed.

LEMMA 5.2. *Assume L_α and L_β are chosen admissible for F and U, Q_α: $\mathrm{Cl}\ F(x) \to L_\alpha$ and Q_β: $\mathrm{Cl}\ F(x) \to L_\beta$, satisfy* (3.3) *relative to U, then* $\mathrm{d}[F, U, Q_\alpha] = \mathrm{d}[F, U, Q_\beta]$.

PROOF. Pick L_ν so that $L_\alpha \cup L_\beta \subset L_\nu$ and define a homotopy Q: $\mathrm{Cl}F(x) \times I \to L_\nu$ by $Q(x, t) = tQ_\alpha(x) + (1 - t)Q_\beta(x)$. Since U is convex, then for fixed $t \in I$, $x - Q(x, t) \in U$, for all $x \in \mathrm{Cl}\ F(x)$. There is therefore the induced homotopy $\dot{T}(x, t) = t\dot{T}_\alpha(x) + (1 - t)\dot{T}_\beta(x)$. The image of $\dot{T}$ is contained in $A(r, R)$, hence $\dot{T}_\alpha$ and $\dot{T}_\beta$ induce the same homomorphism from $H^{\infty-1}(A(r, R))$ into $H^{\infty-1}(\Gamma(\dot{F}))$ and the lemma follows.

LEMMA 5.3. *The relative degree of F is independent of U satisfying* $U \cap f(\mathrm{Bd}\ X) = \emptyset$.

PROOF. Assume U_1 and U_2 are open, bounded, symmetric, convex neighborhoods of θ in E so that $U_1 \cap f(\mathrm{Bd}\ X) = U_2 \cap f(\mathrm{Bd}\ X) = \emptyset$. We first assume $U_1 \subset U_2$, then from (5.2) we may assume the function Q_1 associated with U_1 is the restriction of the function Q_2 associated with U_2. In this case the family $\{\dot{T}_\alpha\}$ for Q_1 and Q_2 agree and the independence of U follows. In the general case construct the neighborhood $U = U_1 \cap U_2$; then the representation is independent of the two inclusions $U \subset U_1$ and $U \subset U_2$ and the Lemma follows.

DEFINITION 5.4. Given an admissible transformation F: $X \to E$, define the degree of F, $\mathrm{d}[F]$, to be the relative degree of F with respect to some U and Q.

THEOREM 5.5. *The degree of an admissible transformation F: $X \to E$ is well defined.*

PROOF. From (5.2) and (5.3), $\mathrm{d}[F, U, Q_S]$ is independent of U and Q_S.

6. Properties of the degree. In [**2**, p. 1115] D. G. Bourgin introduced the notion of degree for admissible transformations defined on the unit ball, K, of a Banach space, E. From the fact that $\mathrm{Bd}\ K = S$, the unit sphere of E, we have $H^{\infty-1}(\mathrm{Bd}\ K) \cong \mathbf{Z}$, and $f^{\infty-1}(1) = n$, for some $n \in \mathbf{Z}$. Hence our notion of degree is an extension of Bourgin's degree.

Assume F: $X \to E$ is an admissible transformation which is fixed point free on X. By defining $f(x) = x - F(x)$ as the multivalued compact vector field associated with F, we have that the image of f does not cover θ. It follows that there is a bounded, convex, symmetric, open neighborhood of θ in E, so that $U \cap \mathrm{Im}\ F = \emptyset$. Following (3.3) we can construct $L_S \in \mathscr{L}_E$ and a mapping Q_S: $\mathrm{Cl}\ F(x) \to L_S$, so that the following commutative diagram exist,

$$\begin{tikzcd} H^{\infty-1}(X) \arrow[rr, "(\mathrm{inc})^{\infty-1}"] & & H^{\infty-1}(\mathrm{Bd}X) \\ & H^{\infty-1}(A(r,R)). \arrow[ul, "(f)^{\infty-1}"] \arrow[ur, "(\dot{f})^{\infty-1}"'] & \end{tikzcd}$$

Hence the image of the homomorphism $(\mathrm{inc})^{\infty-1} \circ (f)^{\infty-1}$ is equal to the image of the homomorphism $(\dot{f})^{\infty-1}$.

THEOREM 6.1. *If* $F: X \to E$ *is an admissible transformation which satisfies*

(i) $F(\mathrm{Bd}\ X) \subset X$, *and*

(ii) *there is an admissible homotopy* H: $\mathrm{Bd}\ X \times I \to X$, *such that for each* $x \in \mathrm{Bd}\ X$, $H(x, 0) = \dot{F}(x)$, $H(x, 1) = \theta$, *and* $H(x, t)$ *is fixed point free, for each* $t \in I$,

then $\mathrm{d}[F] = 1$.

PROOF. By using (ii) and the assumption that $\theta \notin \mathrm{Bd}\ X$, we may construct an annulus, $A(r, R)$, so that the image of $\dot{f}(x) = x - \dot{F}(x)$ is contained in $A(r, R)$, $\mathrm{Bd}\ X \subset A(r, R)$, and for each $(x, t) \in \mathrm{Bd}\ X \times I$, $x - H(x, t) \subset A(r, R)$. By defining G: $\mathrm{Bd}\ X \times I \to A(r, R)$, as $G(x, t) = x - H(x, t)$, we can construct a homotopy connecting $\dot{f}$ to the inclusion of Bd X into $A(r, R)$. It follows that $(\dot{f})^{\infty-1}(1)$ is a generator of $H^{\infty-1}(\mathrm{Bd}\ X)$ and $\mathrm{d}[F] = 1$.

COROLLARY 6.2. *Assume* V *is an open bounded subset of* E *so that* $\theta \in V$ *and* $\mathrm{Cl}\ V = X$ *is simply connected. Let* F *and* H *be as in* (6.1), *then* F *has a fixed point.*

PROOF. Assume F has no fixed point in X. By using the construction prior to (6.1) we have $(\mathrm{inc})^{\infty-1} \circ (f)^{\infty-1} = (\dot{f})^{\infty-1}$. From the argument in (6.1) we may assume $\dot{f}$ is the generator of $H^{\infty-1}(\mathrm{Bd}\ X)$ representing the component of $E \backslash \mathrm{Bd}\ X$ containing θ. Since X is simply connected and $\theta \notin \mathrm{Bd}\ X$, then the image of $(\mathrm{inc})^{\infty-1}$ intersects the image of $(\dot{f})^{\infty-1}$ in the identity of $H^{\infty-1}(\mathrm{Bd}\ X)$. It follows that $\mathrm{d}[F] = 0$, however this contradicts (6.1).

THEOREM 6.3. *Assume* V *is an open bounded subset of* E *so that* $X = \mathrm{Cl}\ V$ *is simply connected. If* $F: X \to E$ *is an admissible transformation such that* $\mathrm{d}[F] \neq 0$, *then either*

(i) F *admits a fixed point, or*

(ii) *there is a point* $x \in \mathrm{Bd}\ X$ *and* $0 < \lambda < 1$, *so that* $x \in \lambda F(x)$.

PROOF. By translation we may assume $\theta \in V$ and denote by C the bounded component of $E \backslash \mathrm{Bd}\ X$ containing θ. Suppose to the contrary that:

(1) $x \notin F(x)$, for all $x \in X$, and

(2) $x \notin \lambda F(x)$, for all $x \in \mathrm{Bd}\ X$ and $0 < \lambda < 1$.

From (1) we can conclude that the image of $(\mathrm{inc})^{\infty-1} \circ (f)^{\infty-1}$ is contained in the image of $(\dot{f})^{\infty-1}$. From (1) and (2) we can construct an annulus $A(r, R)$ and a homotopy H: Bd $X \times I \to A(r, R)$ defined by $H(x, t) = x - tF(x)$. Clearly H is an admissible homotopy joining $\dot{f}$ to the inclusion of Bd X into $A(r, R)$, hence $(\dot{f})^{\infty-1}$ is a generator of $H^{\infty-1}(\mathrm{Bd}\, X)$ representing C. However X is assumed simply connected, hence the image of $(\mathrm{inc})^{\infty-1}$ intersects the image of $(\dot{f})^{\infty-1}$ only at the identity. It now follows that $d[F] = 0$, a contradiction.

THEOREM 6.4. *Assume V is an open bounded subset of E so that $X =$ Cl V is simply connected. If F: $X \to E$ is an admissible transformation so that*

(i) $F(\mathrm{Bd}\, X) \subset X$,

(ii) *F admits a unique fixed point $x_0 \in X$, and*

(iii) $F(\mathrm{Bd}\, X)$ *is contractible in X to x_0,*

then $\mathrm{d}[F] = 1$.

PROOF. We have x_0 is a fixed point for F and F is admissible, hence $x_0 \in V$. From (iii) and the assumption that X is simply connected, we can construct a mapping τ: $F(\mathrm{Bd}\, X) \times I \to X$ so that,

(1) if $y \in F(\mathrm{Bd}\, X)$, then $\tau(y, t)$ is a path from y to x_0, i.e., $\tau(y, 0) = y$ and $\tau(y, 1) = x_0$,

(2) $\tau(y, t) \notin \mathrm{Bd}\, X$, for $t \in (0, 1)$.

Using τ, construct a homotopy H: Bd $X \times I \to X$ by $H(x, t) = \{\tau(y,t) \mid y \in \dot{F}(x)\}$. We have that H connects $\dot{F}$ to the constant map taking X to $\{x_0\}$. If $x \in$ Bd X so that $x \in H(x, t)$, for some t, then $x = \tau(y, t)$, for some $y \in \dot{F}(x)$. By using (2) we have $t = 0$ and therefore $x \in \dot{F}(x)$. However this contradicts F being admissible. It follows that H is fixed point free on Bd X and by using (6.1) we have $\mathrm{d}[F] = 1$.

COROLLARY 6.5. *Assume V is an open bounded subset of E so that* Cl$V = X$ *is contractible. If F: $X \to E$ is an admissible transformation so that $F(\mathrm{Bd}\, X) \subset X$ and F admits a unique fixed point, then* $\mathrm{d}[F] = 1$.

PROOF. If X is contractible, then (iii) in (6.4) is clearly satisfied and the result follows.

We recall the notion of codimension as presented in [**8**, p. 265].

DEFINITION 6.6. If $X \subset E$ is a closed and bounded subset of E, then the codimension of X in E, denoted by codim X, is defined by codim$X = \inf\{n \in \mathbf{Z} \mid H^{\infty-n}(X, A) \neq 0$, for some $A \subset X\}$.

LEMMA 6.7. [**8**, p. 265]. *If* codim $X = n$, *then* $H^{\infty-k}(X) = 0$, *for all* $0 \leqq k < n$.

THEOREM 6.8. *Let X be a closed and bounded subset of E so that* codim$X > 1$. *If F: $X \to E$ is an admissible transformation so that* $\mathrm{d}[F] \neq 0$, *then F has a fixed point.*

PROOF. Assume F does not have a fixed point. Again we have the commutative diagram constructed prior to (6.1) and can conclude that $(\mathrm{inc})^{\infty-1} \circ (f)^{\infty-1} = (\dot{f})^{\infty-1}$. Since codim $X > 1$, we have from (6.7) $H^{\infty-1}(X) = 0$, hence $\mathrm{d}[F] = 0$, a contradiction.

Let H: $X \times I \to E$ be an admissible homotopy with no fixed points on Bd $X \times I$ and denote by $H_1(x) = H(x, 1)$ and $H_0(x) = H(x, 0)$. Define the homotopy h: $X \times I \to E$ by, $h(x, t) = x - H(x, t)$ and let $\dot{h}$ denote the restriction of h to Bd $X \times I$. Since H is fixed point free on Bd $X \times I$ and I is compact, there is an annulus $A(r, R)$, so that the image of $\dot{h}$ is contained in $A(r, R)$. It follows that the transformations $\dot{h}_0$ and $\dot{h}_1$ induce the same homomorphism from $H^{\infty-1}(A(r, R))$ into $H^{\infty-1}(\mathrm{Bd}X)$. Hence we have $\mathrm{d}[H_1] = \mathrm{d}[H_0]$ and the following result.

THEOREM 6.8. The degree is a homotopy invariant.

THEOREM 6.9. Let $x_0 \in X$ and assume F: $X \to E$ is an admissible constant transformation taking X onto $\{x_0\}$, then $\mathrm{d}[F] = 1$.

PROOF. We may assume $\theta = x_0$. Since F is admissible, $\theta \notin$ Bd X and is therefore contained in one of the components of $E\backslash$Bd X. Hence $(\dot{f})^{\infty-1}$ can be taken as a generator of $H^{\infty-1}$ (Bd X) and the theorem follows.

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