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COMMUTATORS, ANTI-COMMUTATORS AND EULERIAN CALCULUS

PHILIP FEINSILVER

ABSTRACT. Consider operators obeying the commutation rule $Cy = 1 + qyC$ generalizing the rule $Dx = 1 + xD$, D denoting d/dx. The case $q = 1$ corresponds to boson creation-annihilation operators, $q = -1$ to fermion operators. We derive Leibniz' rule for general q. We find that the canonical representation of C such that $C1 = 0$ is given by the Eulerian derivative. The basics of Eulerian calculus are discussed and an indication of a discrete Hamiltonian theory analogous to the Heisenberg representation in quantum mechanics is given.

Introduction. In quantum theory one encounters creation and annihilation operators of two basic types: "boson" operators a, b such that $ab - ba = 1$ and "fermion" operators A, B such that $AB + BA = 1$. It is fairly easy to check, as will be seen later, that a standard representation of the pair (a, b) is $a = d/dx$, $b = x$ acting on a space of functions $f(x)$. Our approach is to find a general explicit formula for calculating commutators of the form $[h(a), g(b)]$ for fairly general h and g, e.g., any two polynomials. This is a generalization of Leibniz' rule for differentiating a product of functions. In finding a general Leibniz rule and, hence, an operator calculus for fermion operators it is natural to consider the general case of operators α, β such that $\alpha\beta = 1 + q\beta\alpha$ where q is a fixed parameter, $q = 1$ for bosons, $q = -1$ for fermions. Then the operator calculus turns out to be the "q-calculus" or, following G.C. Rota, "Eulerian" calculus. This calculus arises quite naturally in the study of elliptic functions and generalized hypergeometric functions. This paper provides another natural setting for the Eulerian calculus.

We first review the approach for the boson or Heisenberg case and then proceed to the general, Eulerian, case. We conclude with an Eulerian, discrete, analog of Hamiltonian theory corresponding to the Heisenberg representation in quantum mechanics.

I. **Preliminary discussion.** The main result of this paper is Leibniz' rule for the functional calculus of the "Eulerian derivative"

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$$\delta_q f(x) = \frac{f(x) - f(qx)}{x - qx}$$

which reduces to $f'(x)$ for $q \to 1$. Our approach indicates very clearly the role of the "separation parameter" q. It is essentially the variable $|q| \leqq 1$ that provides a homotopy from the commutator rule "$C \circ y = 1 + yC$" to the anti-commutator "$C \circ y = 1 - yC$", i.e., $C \circ y = 1 + qyC$, where C is an operator on functions of y and y is a multiplication operator, the $\circ$ indicating composition. When $q = 1$ one recovers ordinary calculus ("boson operators"); when $q = -1$ we obtain the functional calculus for anti-commuting operators ("fermion" operators). So the q provides a link between these basic cases.

NOTATION. x, y are the basic variables, i.e., operators act on functions of these variables.

$\circ$ denotes operator composition, e.g., $B \circ f(x)$; B applied to $f(x)$ we denote by $Bf(x)$, i.e., $Bf(x) = B \circ f(x)1$.

D—a linear operator such that $D \circ x = 1 + xD$.

C—a linear operator such that $C \circ y = 1 + qyC$.

∂_a—differentiation with respect to (any variable) a.

δ_q—"Eulerian differentiation" with respect to y; $\delta_q f(y) = (f(y) - f(qy))/(y - qy)$

D_q—Eulerian differentiation with respect to C.

We set $p = 1 - q$, $q_n = (1 - q^n)/p$, $Q = q^{-1}$, $Q_n = (1 - Q^n)/(1 - Q)$. $q_n! = \Pi_1^n q_j$. Note that

$$Q_n! = Q^{\binom{n}{2}} q_n!$$

$\binom{n}{k}_q = q_n!/q_{n-k}!\, q_k!$, the Gaussian binomial coefficient.

The standard notation for $\Pi_{j=0}^{n-1}(1 - aq^j)$ is $(a; q)_n$.

In this notation $q_n! = p^{-n}(q; q)_n$.

$\int$ denotes $\int$ over $\mathbf{R}$ unless otherwise indicated.

Other notations will be explained as they arise.

II. **Heisenberg operators.** We illustrate our approach by establishing the basic results for the operators $D \circ x = 1 + xD$. The canonical realization of these operators is of course multiplication by x and $Df(x) = D \circ f(x)1 = (d/dx)f(x)$, where 1 is a function (a "vacuum function") such that $D1 = 0$. For an explicit functional calculus it is convenient to restrict to functions of the following types:

(a) polynomials

(b) functions analytic in a neighborhood of, say, $0 \in \mathbf{C}$.

(c) functions in $\mathscr{S}$, Schwartz space,

$$= \{f: f \in C^\infty(\mathbf{R}),\ \lim_{|x| \to 0} |x^m f^{(n)}(x)| = 0,\ \forall m, n \geqq 0\}$$

or in $\mathscr{S}^* = \{\text{tempered distributions}\}$.
The reason is that these can all be derived from exponentials. Specifically, if we can define e^{aB} on some domain, then we have:

For (a) a polynomial or (b) power series $p(x)$, $p(B) = p(d/da)\, e^{aB}|_{a=0}$.

For (c) $f(x) \in \mathscr{S}$ or $\mathscr{S}^*$, $f(x) = \int e^{iax} \hat{f}(a)da$, where we normalize by $\hat{f}(a) = (1/2\pi) \int e^{-iax} f(x)dx$.

Then $f(B) = \int e^{iaB} \hat{f}(a)da$.

We can also compute inverses, when the following integrals exist, via

$$B^{-1} = \int_0^1 \lambda^B \frac{d\lambda}{\lambda} = \int_0^\infty e^{-aB}\, da.$$

Thus we have the following Lemma.

LEMMA 1. *From e^{aB} we can compute $f(B)$ for f in any of the classes* (*a*), (*b*), (*c*) *above* (*and also* [*formal*] *inverses*).

We now proceed with three fundamental theorems. (See pages 3–6 of [11]).

GENERALIZED LEIBNIZ LEMMA (GLM).

$$g(D) \circ f(x) = \sum_0^\infty \frac{f^{(n)}(x)\, g^{(n)}(D)}{n!} = e^{\partial_x \partial_D} f(x) g(D) = f(x + \partial_D) g(D).$$

PROOF.

1. $D^n \circ x = xD^n + nD^{n-1}, n > 0$.
   $n = 1$: By definition $[D, x] = 1$.
   $n = m + 1$: Multiply $D^m \circ x = xD^m + mD^{m-1}$ on the left by D.
2. Multiply $D^n \circ x = xD^n + nD^{n-1}$ by $t^n/n!$ and sum to yield $e^{tD} \circ x = xe^{tD} + te^{tD} = (x + t)e^{tD}$. Induction immediately yields $e^{tD} \circ x^n = (x + t)^n e^{tD}$ and so $e^{tD} \circ e^{sx} = e^{sx} e^{st} e^{tD}$.
3. Therefore

$$e^{tD} \circ e^{sx} = \sum_0^\infty (1/n!)\, s^n e^{sx} t^n e^{tD} = e^{\partial_x \partial_D} e^{sx} e^{tD}$$

and Lemma 1 completes the proof.

COROLLARY 1. $[g(D), x] = g'(D)$. $[D, f(x)] = f'(x)$.

COROLLARY 2. *Leibniz' Rule.*

$$\begin{aligned} D^n f(x) g(x) &= (D^n \circ f(x)) g(x) \\ &= \sum \frac{1}{k!} f^{(k)}(x)\, \partial_D^k D^n g(x) = \sum \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x). \end{aligned}$$

EXPONENTIAL LEMMA (EL).

$$e^{tD} f(x) = f(x + t), \quad g(D) e^{sx} = g(s) e^{sx}.$$

REMARK. If V and ξ are operator functions of, say, (x, D) and $[V, \xi] = 1$, $V1 = 0$, we would write explicitly $e^{tV}f(\xi)1 = f(\xi + t)1$.

PROOF. In the proof of GLM we have $e^{tD} \circ e^{sx} = e^{sx}e^{st}e^{tD}$. Apply this to the function 1 to get $e^{tD}e^{sx} = e^{sx}e^{st}$ since $D1 = 0$. Apply Lemma 1 to first $e^{tD}e^{sx} = e^{s(x+t)}$, then to $e^{tD}e^{sx} = e^{ts}e^{sx}$ to complete the proof.

COROLLARY. $g(D)1 = g(0)$ (set $s = 0$).

DUALITY LEMMA.

$$g(D)f(x) = f(\partial_a)e^{ax}g(a)|_{a=0}.$$

PROOF. Apply Lemma 1, noting that

$$e^{tD}f(x) = f(x + t) = f(\partial_a)e^{a(x+t)}|_0 = f(\partial_a)e^{ax}e^{ta}|_0.$$

GLM allows one to express any product so that the derivatives are always on the right; EL shows that D is the infinitesimal generator for the translation group; and the Duality Lemma illustrates the duality $D \leftrightarrow x$ inherent in the rule $D \circ x = 1 + xD$.

We will proceed to find the results corresponding to GLM for the commutation rules $C \circ y = 1 + qyC$ and $C \circ y = h + qyC$. There are some results roughly analogous to EL but we will note that there are difficulties.

III. **Commutator: anti-commutator homotopy.** We now assume the relation $C \circ y = 1 + qyC$.

PROPOSITION 1. $C^n \circ y = q^n yC^n + q_n C^{n-1}$.

PROOF. For $n = 1$, this is by definition. For $n = m + 1$,

$$\begin{aligned} CC^m \circ y &= q^m CyC^m + q_m C^m \\ &= q^m(1 + qyC)C^m + q_m C^m \\ &= q^{m+1}yC^{m+1} + q_{m+1}C^m \end{aligned}$$

as required. We now observe that $D_q C^n = q_n C^{n-1}$. Thus, $C^n \circ y = q^n yC^n + D_q C^n$. Let's assume a rule of the form

$$C^n \circ y^m = \sum p_n(m, k)y^k D_q^{m-k}C^n.$$

Then

$$\begin{aligned} C^n \circ y^{m+1} &= \sum p_n(m, k)y^k D_q^{m-k}C^n y \\ &= \sum P_n(m, k)[y^{k+1}q^{n-m+k}D_q^{m-k}C^n + y^k D_q^{m-k+1}C^n]. \end{aligned}$$

We see that

$$p_n(m + 1, k) = p_n(m, k - 1)q^{n-m+k-1} + p_n(m, k).$$

Since $p_n(0, 0) = 1$, $p_n(0, k) = 0$, $k > 0$, we have, for $k \geqq 0$,

$$p_n(m+1, k+1) - p_n(m, k+1) = p_n(m, k)q^{n-m+k}$$

and

$$p_n(m, k+1) = \sum_{j=0}^{m-1} q^{n+k-j} p_n(j, k).$$

Thus

$$p_n(m, k) = q^{nk+1+2+\cdots+(k-1)} g_n(m, k) = q^{nk+\binom{k}{2}} g_n(m, k)$$

where

$$g_n(m, k+1) = \sum_0^{m-1} q^{-j} g_n(j, k).$$

PROPOSITION 2. *Let* $I_0(m) \equiv 1$. $I_{k+1}(m) = \sum_{j=0}^{m-1} s^j I_k(j)$, *for* $k \geqq 0$, *with* $I_k(m) = 0$ *for* $k > m$. *Then*

$$\text{(1)} \qquad \prod_{j=0}^{m-1} (1 + vs^j) = (-v; s)_m = \sum_0^{m-1} v^k I_k(m),$$

$$\text{(2)} \qquad I_k(m) = \binom{m}{k}_s s^{\binom{k}{2}}.$$

PROOF:

1. Put $E_m = \sum_0^{m-1} v^k I_k(m)$. Then

$$\begin{aligned} E_m - E_{m-1} &= \sum v^k (I_k(m) - I_k(m-1)) \\ &= \sum v^k s^{m-1} I_{k-1}(m-1) = vs^{m-1} E_{m-1}, \end{aligned}$$

i.e., $E_m = (1 + vs^{m-1})E_{m-1}$. And we have initially $E_1 = 1$.

2. We check that

$$I_k(m) = \binom{m}{k}_s s^{\binom{k}{2}}$$

satisfies $I_k(m) - I_k(m-1) = s^{m-1} I_{k-1}(m-1)$. We have, the dot indicating s to the power $\binom{k}{2}$ as a common factor,

$$\begin{aligned} s^{\binom{k}{2}} \cdot \frac{s_m s_{m-1} \cdots s_{m-k+1}}{s_k s_{k-1} \cdots s_1} - \frac{s_{m-1} \cdots s_{m-k+1} s_{m-k}}{s_k s_{k-1} \cdots s_1} &= s^{\binom{k}{2}} s^{m-k} \binom{m-1}{k-1}_s \\ &= s^{m-1} s^{\binom{k-1}{2}} \binom{m-1}{k-1}_s \end{aligned}$$

accordingly.

REMARK. 2 follows from 1 and the "q-binomial theorem." See Theorems 2.1 and 3.3 of [3]. These results are quite classical; references are given in [3, pp. 30, 51]. Also see [7].

Thus $g_n(m, k) = I_k(m)$ with $s = Q$. Consequently

$$g_n(m, k) = Q^{\binom{k}{2}}\binom{m}{k}_Q = Q^{km}q^{(k^2+k)/2}\binom{m}{k}_q,$$

using $Q_m! = Q^{\binom{m}{2}}q_m!$. And so

$$p_n(m, k) = q^{nk+k^2-mk}\binom{m}{k}_q.$$

And we have finally

GENERAL LEIBNIZ RULE (GLR).

$$C^n \circ y^m = q^{mn} \sum_k \frac{(q^{k-m-n}\delta_q D_q)^k}{q_k!} y^m C^n.$$

PROOF. Replacing $k \to m - k$,

$$\begin{aligned} C^n \circ y^m &= \sum q^{nk+k^2-mk} \frac{q_m!}{q_{m-k}!q_k!} y^k D_q^{m-k} C^n \\ &= \sum q^{mn-nk+k^2-mk} \frac{1}{q_k!} \delta_q^k y^m D_q^k C^n. \end{aligned}$$

REMARKS. We observe that this agrees with GLM for $q \to 1$. However, for $q \neq 1$, we cannot in general find a rule for $g(C) \circ f(y)$ because the powers are linked via q. We do have

COROLLARY 1. $g(C) \circ y = yg(qC) + D_q g(C)$.

COROLLARY 2. $C \circ f(y) = f(qy)C + \delta_q f(y)$.

COROLLARY 3.

$$C^n \circ f(y) = \sum_k \frac{1}{q_k!} \delta_q^k f(yq^{n-k}) D_q^k C^n = \sum_k \binom{n}{k}_q \delta_q^k f(yq^{n-k}) C^{n-k}$$

Anti-commutators. In case $q = -1$ we have $C \circ y + yC = 1$. GLR states that

$$C^n \circ y^m = (-1)^{mn} \sum \frac{((-1)^{k-n-m}\delta_{-1}D_{-1})^k}{q_k!} y^m C^n.$$

The Eulerian derivative satisfies

$$\delta_{-1}f(y) = \frac{f(y) - f(-y)}{2y} = \frac{f^*(y)}{y}$$

where $f^*(y)$ = odd part of f.

It follows immediately that $\delta_{-1}^2 = 0$. Furthermore, we have

PROPOSITION 3. For $C \circ y + yC = 1$,

(1) *An even power of y or C commutes with all functions of either variable.*

(2) *For m, n odd, $C^n \circ y^m = -y^m C^n + y^{m-1} C^{n-1}$.*

PROOF. $C^n \circ y^m = (-1)^{mn}[y^m C^n + (-1)^{m+n-1}\delta_{-1} y^m D_{-1} C^n]$.
(1) For m or n even, the 2nd term $= 0$.
(2) For m, n odd, we have $-y^m C^n + y^{m-1}C^{n-1}$.

LEIBNIZ' RULE FOR ANTI-COMMUTATORS.

$$g(C) \circ f(y) = f(y)g(C) - 2f^*(y)g^*(C) + \frac{f^*(y)}{y}\,\frac{g^*(C)}{C}.$$

PROOF: From Corollary 1 to GLR, $g(C) \circ y = yg(-C) + g^*(C)/C$, while even powers of y commute with $g(C)$. Thus,

$$g(C) \circ y^{2k} = y^{2k} g(C) \text{ and}$$
$$g(C) \circ y^{2k+1} = y^{2k+1} g(-C) + y^{2k} g^*(C)/C.$$

Denote by $f^e(y)$ the even part of y, $f^e(y) = f(y) - f^*(y)$.
Then

$$g(C)f(y) = f^e(y)g(C) + f^*(y)g(-C) + \frac{f^*(y)}{y}\,\frac{g^*(C)}{C},$$

which yields the result.

We thus have "complete solutions" for $q^2 = 1$. A similar approach would work for q any integer root of unity.

q-commuting operators. We consider now a homotopy from the non-commuting to the commuting case. Assume, then, $C \circ y = h + qyC$. Observe that then $C \circ Y = 1 + qYC$, where $Y = yh^{-1}$. By GLR, then,

$$C^n \circ y^m = C^n \circ Y^m h^m = h^m q^{mn} \sum (1/q_k!)(q^{k-m-n}\,\delta_q D_q)^k Y^m C^n$$
$$= q^{mn} \sum (1/q_k!)(q^{k-m-n} h \delta_q D_q)^k y^m C^n,$$
$$C \circ f(y) = C \circ f(hY) = f(qhY)C + h\delta_q f(hY) = f(qy)C + h\delta_q f(y).$$
$$g(C) \circ y = yg(qC) + hD_q g(C).$$

In the limit $h \to 0$ we get that $C^n \circ y^m = q^{mn} y^m C^n$, $g(C) \circ y = yg(qC)$, $C \circ f(y) = f(qy)C$. These are, of course, easy to check directly.

BINOMIAL THEOREM FOR q-OPERATORS. *If $ab = qba$, then*

$$(a + b)^n = \sum_k \binom{n}{k}_q b^k a^{n-k}.$$

PROOF. We can write $a + b = (1 + ab^{-1})b = b(1 + qab^{-1})$. Similarly,

$$(a + b)^n = b^n(-ab^{-1}; q)_n = b^n \sum_k \binom{n}{k}_q (ab^{-1})^k q^{\binom{k}{2}}$$

by Prop. 2. By induction on k it follows that $b^n(ab^{-1})^k = b^{n-k}a^k Q^{\binom{k}{2}}$ and hence the result.

In a quantum field theory with operators C_j, y_j such that $C_j \circ y_k = \delta_{jk} - y_k C_j$ and D_j, x_j such that $[D_j, x_k] = \delta_{jk}$ we have, e.g., for a functional calculus,

$$\begin{aligned} C_j \circ f(y_k) &= f(-y_k)C_j, \quad j \neq k, \\ C_j \circ f(y_j) &= f(-y_j)C_j + f^*(y_j)/y_j, \\ D_j \circ f(x_k) &= f(x_k)D_j, \quad j \neq k, \\ D_j \circ f(x_j) &= f(x_j)D_j + f'(x_j), \end{aligned}$$

and similarly for functions $f(y_1, y_2, \ldots)$, $g(C_1, C_2, \ldots)$, $\ldots$.

IV. **Eulerian calculus.** To recover ordinary calculus from the operators D, x we apply operator equations to a particular "vacuum" function $1 \in \ker D$. Similarly, Eulerian calculus results by applying the operator calculus for C and y to a function $1 \in \ker C$.

PROPOSITION 4. *The canonical representation of Eulerian calculus is given by: multiplication by y and $Cf(y) = \delta_q f(y)$.*

PROOF. Apply $C \circ f(y) = f(y)C + \delta_q f(y)$ to 1.

REMARK. For example, applying Corollary 3 to $g(y)1$ yields Hahn's q-Leibniz rule for functions ([**14**], (2.5)):

$$\delta_q^n(f(x)g(x)) = \sum_k \binom{n}{k}_q \delta_q^k f(xq^{n-k})\delta_q^{n-k} g(x).$$

In the following, unless noted otherwise, we assume $C1 = 0$.

PROPOSITION 5. *If $q = -1$, then $C^2 = 0$.*

PROOF. $C^2 = 0$ is the same as $\delta_{-1}^2 = 0$.

REMARK. If we interpret C as an "annihilation" operator and 1 as a vacuum state, so that $C1 = 0$, then Prop. 5 is essentially the Pauli exclusion principle, namely that no state could have more than one electron (or fermion), i.e., $C^2 = 0$. In this case, y^n would correspond to y^{n_2}, $n_2 =$ congruence class of n mod 2.

Exponential functions. The "exponentials" for Eulerian calculus are well known. We discuss them now. We look for eigenfunctions $E(ay)$ satisfying

$$\delta_q E(ay) = aE(ay) \text{ analogous to } De^{ax} = ae^{ax}.$$

PROPOSITION 6. *For $|q| < 1$,*

$$E(y) = \prod_{j=0}^{\infty} (1 - pyq^j)^{-1} = (py; q)_\infty^{-1} = \sum_{j=0}^{\infty} \frac{y^j}{q_j!}. \tag{1}$$

$$e(y) = E(y)^{-1} = (py; q)_\infty = \sum_{j=0}^{\infty} \frac{(-y)^j}{Q_j!}. \tag{2}$$

(3) $$\delta_q e(y) = -e(qy).$$

PROOF. Note that $\sum_0^\infty y^j/q_j!$ satisfies $\delta_q E(y) = E(y)$, since $\delta_q y^j = q_j y^{j-1}$.
(1) First we find by Prop. 2 that

$$(py;q)_m = \sum_0^{m-1} (-y)^k \binom{m}{k}_q q^{\binom{k}{2}} p^k$$

noting that

$$\binom{m}{k}_q = \frac{(q^{m-k+1};q)_k}{(q;q)_k} \xrightarrow[m\to\infty]{} (q;q)_k^{-1},$$

we thus should have

$$\sum_0^\infty \frac{(-1)^k y^k q^{\binom{k}{2}}}{q_k!} = \sum_0^\infty \frac{(-1)^k y^k}{Q_k!} = (py;q)_\infty.$$

This is easily checked since for $|q| < 1$, the infinite product converges absolutely and is bounded uniformly for y in any compact subset of **C**, e.g., for $|y| < K$, by e^{RK}, $R = |p|/(1 - |q|)$; while the right-hand sides are holomorphic in y, being polynomials.

2. Now consider the equation $\delta_q E(y) = E(y)$, with $E(0) = 1$. That is, $E(y) - E(qy) = pyE(y)$ or $E(qy) = (1 - py)E(y)$. Assuming $E(y)$ is continuous, in fact it will then be holomorphic, at 0 we have

$$1 = E(0) = \lim_{m\to\infty} E(q^m y) = \prod_{j=0}^\infty (1 - pyq^j)E(y),$$

i.e., $E(y) = (py;q)_\infty^{-1}$. $E(y)$ and the expansion $\sum_0^\infty y^j/q_j!$ both satisfy $\delta_q E(y) = E(y)$, $E(0) = 1$, and are holomorphic around 0. Number 1 now follows from the fact that $\delta_q F(y) = F(y)$, $F(0) = 0$, F continuous at 0, has the unique solution $F = 0$ by the same argument as above.

3. It is easy to check that $\delta_q e(y) = -e(qy)$ directly. We will check using the commutation rule $C \circ f(y) = f(qy)C + \delta_q f(y)$: $e(y)E(y) = 1$ implies $0 = Ce(y)E(y) = e(qy)E(y) + E(y)\delta_q e(y)$.

REMARKS.
(1) It is immediate from (1) that $\delta_q E(ay) = aE(ay)$ as desired. Also, we have $\delta_q e(ay) = -ae(qay)$.
(2) $E(y)$ does not have any apparent group properties for $q \neq 1$, e.g.,

$$E(a+b)E(a)^{-1}E(b)^{-1} = \prod_0^\infty \left(1 + \frac{p^2 q^{2j} ab}{1 - pq^j(a+b)}\right).$$

The following transformation is interesting in this context. Put $A_j = -pq^j a$, $B_j = -pq^j b$. Choose $T_j = A_j + B_j = \tanh \alpha_j \tanh \beta_j$ and $U_j = A_j B_j = \tanh \alpha_j + \tanh \beta_j$. So, e.g., $A_j = (1/2)(T_j + (T_j^2 - 4U_j)^{1/2})$,

$B_j = (1/2)(T_j - (T_j^2 - 4U_j)^{1/2})$, $\tanh \alpha_j = (1/2)(U_j + (U_j^2 - 4T_j)^{1/2})$, $\tanh \beta_j = (1/2)(U_j - (U_j^2 - 4T_j)^{1/2})$. That is, interchange sums and products. Then

$$E(a + b) = \prod_j (1 + T_j)^{-1} = \prod_j (1 + \tanh \alpha_j \tanh \beta_j)^{-1},$$
$$E(a + b)E(a)^{-1}E(b)^{-1} = \prod_j (1 + \tanh (\alpha_j + \beta_j)),$$
$$e(a)e(b) = \prod_j (1 + \tanh \alpha_j \tanh \beta_j)(1 + \tanh (\alpha_j + \beta_j)).$$

These follow directly from the expressions in terms of a and b.

Integration. Let's consider the solution to the simplest type of Eulerian differential equation $\delta_q f(y) = g(y)$. Thus $f(y)$ is the Eulerian integral of g. This is easy to solve as follows:

$$f(y) - f(qy) = pyg(y),$$
$$f(qy) - f(q^2y) = pqyg(y),$$
$$\ldots$$
$$f(q^jy) - f(q^{j+1}y) = pq^jyg(q^jy).$$

We thus have the following result.

PROPOSITION 7. *Let f be continuous at 0 and $|q| < 1$. Then if $\delta_q f(y) = g(y)$,*

$$f(y) = f(0) + py \sum_{j=0}^{\infty} q^j g(q^j y).$$

One could develop integration theory and study discrete differential equations in this context. We will, however, conclude with a discrete version of Hamiltonian flows.

Hamiltonian formalism. We first review the Heisenberg case. We are given an operator $H(x, D)$, for convenience assumed to be analytic with an expansion such that all D's are on the right—this can be adjusted by GLM. We then consider the operator flow $x(t) = e^{tH}xe^{-tH}$, $z(t) = e^{tH}ze^{-tH}$, where we use "z" for the momentum variable, $z = z(0) = D$. Then it is easy to see by GLM that:

$$\dot{x}(t) = [H, x(t)] = \frac{\partial H}{\partial z}(x(t), z(t)), \; x(0) = x, \tag{1}$$

$$\dot{z}(t) = [H, z(t)] = -\frac{\partial H}{\partial x}(x(t), z(t)), \; z(0) = z, \tag{2}$$

$$[z(t), x(t)] = [z, x] = 1. \tag{3}$$

This is essentially the Heisenberg formulation of quantum mechanics. We will present an "Eulerian" version which is a discretized analog. Assume we have $H(y, C)$ and consider $y(t) = E(tH)ye(tH)$. Denoting Eulerian differentiation with respect to t by τ_q we have, using Corollary 2 to GLR,

$$\begin{aligned}\tau_q y(t) &= E(qtH)y\tau_q e(tH) + Hy(t)\\ &= Hy(t) - E(qtH)ye(qtH)H\\ &= Hy(t) - y(qt)H.\end{aligned}$$

We thus have

PROPOSITION 8. *Given $H(y, C)$ a "q-Hamiltonian," denote Eulerian differentiation with respect to t by τ_q. Then*
(1) $y(t) = E(tH)ye(tH)$ *satisfies*

$$\begin{aligned}\tau_q y(t) &= Hy(t) - y(qt)H\\ &= D_q H(y(t), C(t)) + y(t)H(y(t), qC(t)) - y(qt)H(y, C).\end{aligned}$$

(2) $C(t) = E(tH)Ce(tH)$ *satisfies*

$$\begin{aligned}\tau_q C(t) &= HC(t) - C(qt)H\\ &= -Q\delta_Q H(y(t), C(t)) + C(t)H(Qy(t), C(t)) - C(qt)H(y, C).\end{aligned}$$

(3) $C(t) \circ y(t) = 1 + qy(t)C(t)$.
(4) $H(y(t), C(t)) = H(y, C)$ *(conservation of energy).*

PROOF. From above,

$$\begin{aligned}\tau_q y(t) &= Hy(t) - y(qt)H\\ &= y(t)H(y(t), qC(t)) - y(qt)H + D_q H(y(t), C(t));\end{aligned}$$

D_q denoting Eulerian derivative with respect to $C(t)$, using Corollary 1 to GLR. Similarly,

$$\tau_q C(t) = HC(t) - C(qt)H.$$

From $Cf(y) = f(qy)C + \delta_q f(y)$ we have, replacing $f(y)$ by $F(Qy)$,

$$F(y)C = CF(Qy) - \delta_q f(y),$$

and

$$\delta_q f(y) = \frac{F(Qy) - F(y)}{py} = Q\frac{F(Qy) - F(y)}{-(1-Q)y} = Q\delta_Q F(y).$$

Number 3 follows directly from $E(y)e(y) = 1$.

Notice that $H(y, qC)$ may not commute with $H(y, C)$, e.g.,

$$(y + C) \circ (y + qC) = y^2 + 1 + 2qyC + qC^2$$

while

$$(y + qC) \circ (y + C) = y^2 + q + (q^2 + 1)yC + qC^2.$$

A thorough investigation of this "mechanics" should prove quite fascinating.

Concluding remarks. Much work has been done on q-difference equations [9] [14] [15], combinatorial and number theoretical applications of q-theory [2] [3] [4] [12] [17] and systems of orthogonal and related polynomials satisfying q-difference relations [1] [4] [8]. For original work by Ramanujan, L.J. Rogers and the English school we refer to [7] [16] [18] [20]. Some of the most important recent work deals with discovering precise relationships between classical orthogonal polynomials and special functions and their q-analogs [7] [8] [17]. The operator methods have appeared in the physics literature [5] [6]; the algebra treated above generalizes the Heisenberg algebra used in [11] to study the classical polynomials.

Finally, note that limits besides $q \to 1$, for example, $q \to -1$ and $q \to 0$ have been considered and found to yield important results in various contexts [6] [8].

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BIBLIOGRAPHY

1. W. A. Al-Salam, *q-Appell polynomials* Annali di Matematica **77** (1967), 33–45.

2. G. E. Andrews, *On the foundations of combinatorial theory, V. Eulerian differential operators*, Studies in Applied Math. **50** (1971), 345–375.

3. ———, The Theory of Partitions, Addison-Wesley, Reading, Mass., 1976.

4. G. E. Andrews and R. Askey, *Enumeration of partitions: The role of eulerian series and q-orthogonal polynomials*, in Higher Combinatorics, Aigner, M., ed., Reidel Publ. Co., Dordrecht-Holland, 1977.

5. M. Arik, D.D., Coon Y. Lam, *Operator algebra of dual resonance models*, J. Math. Phys. **16** (1975), 1776–1779.

6. M. Arik, D. D. Coon, *Hilbert spaces of analytic functions and generalized coherent states*, J. Math. Phys. **17** (1976), 524–527.

7. R. Askey, *Ramanujan's extensions of the gamma and beta functions*, Am. Math. Monthly **87** (1980), 346–359.

8. R. Askey and M. Ismail, *A generalization of ultraspherical polynomials*, University of Wisconsin M.R.C. Technical Summary Report #1851, May 1978.

9. G. D. Birkhoff, Collected Mathematical Papers, A.M.S., New York, 1950, Vol. 1.

10. P. Feinsilver, Operator calculus, Pacific J. Math. **78** (1978), 95–116.

11. ———, Special Functions, Probability Semigroups and Hamiltonian Flows, Springer-Verlag Lecture Notes in Mathematics, Vol. 696, esp. Chpts. II and II.

12. J. Goldman, and G.-C. Rota, *On the foundations of combinatorial theory, IV.*

finite vector spaces and Eulerian generating functions, Studies in Applied Math. **49** (1970), 239–258.

13. C. Graves, *On the principles which regulate the interchange of symbols*, Proc. Royal Irish Academy, (1853–1857), 144–152.

14. W. Hahn, *Über Orthogonalpolynome, die q-Differenzen-gleichungen genügen*, Math. Nachr. **2** (1949), 4–34.

15. ———, *Beiträge zur Theorie der Heineschen Reihen*, Math. Nachr., **2**, (1949), 340–379.

16. F. H. Jackson, *Basic integration*, Quarterly J. of Math. **2** (1951), 1–6.

17. S. C. Milne, *A q-analog of restricted growth functions, Dobinski's equality and Charlier polynomials*, Trans. A.M.S. **245** (1978), 89–118.

18. L. J. Slater, Generalized Hypergeometric Functions, Cambridge Univ. Press, 1966.

19. W. V. Waldenfels, An algebraic central limit theorem in the anti-commuting case, Z. Wahr. **42** (1978), 135–140.

20. G. N. Watson, "The Mock Theta Functions," Proc. London Math. Soc. **42** (1937), 274.

Department of Mathematics, Southern Illinois University,
Carbondale, IL 62901