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A NONSTANDARD PROOF OF THE MARTINGALE CONVERGENCE THEOREM

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In this note we use A. Robinson's [5] nonstandard analysis to give an elementary proof of the almost everywhere convergence of an L^1-bounded submartingale. Here, the index set $\mathscr{I}$ is a countable subset of the real numbers $\mathbf{R}$; we assume that $\mathscr{I}$ contains the natural numbers $\mathbf{N}$, but any cofinal subset of $\mathbf{R}$ will do. The continuous parameter martingale convergence theorem usually reduces to the case considered here. Our proof does not use the notion of a stopping time. It does employ a nonstandard criterion for almost everywhere convergence and demonstrates the usefulness of that criterion. It also produces the limit function.

We shall use the notation from [4] to which we refer the reader for further details about nonstandard analysis in general. We assume that we are working with a fixed $\aleph_1$-saturated, nonstandard extension of a standard structure. Of course, $^*\mathbf{R}$ and $^*\mathbf{N}$ denote the nonstandard extensions of $\mathbf{R}$ and $\mathbf{N}$, and $a \cong b$ means that $a - b$ is infinitesimal in $^*\mathbf{R}$. If $(X, \mathscr{F}, \mu)$ is an internal measure space and $g: X \to {}^*\mathbf{R} \cup \{-\infty, +\infty\}$ is internal and $\mathscr{F}$-measurable, then (following K. Stroyan) we shall say that $g \cong 0$ nearly surely (n.s.) when the following holds: For some infinitesimal $\varepsilon > 0$, $\mu(|g| > \varepsilon) \cong 0$. Clearly, $g \cong 0$ n.s. if and only if for each $\varepsilon > 0$ in $\mathbf{R}$, $\mu(|g| > \varepsilon) < \varepsilon$.

We now establish a nonstandard criterion for almost everywhere convergence. Here, as later, $\mathscr{I}$ denotes a countable subset of $\mathbf{R}$ with $\mathbf{N} \subset \mathscr{I}$. The ordering on $\mathscr{I}$ is the ordering inherited from $\mathbf{R}$. We shall use n, m, and k to denote natural numbers, while i and j will denote elements of $\mathscr{I}$ or $^*\mathscr{I}$. Moreover, $\{i: n \leqq i \leqq m\}$ will denote the set of indices in just $\mathscr{I}$ with $n \leqq i \leqq m$, while if γ and η are in $^*\mathbf{N} - \mathbf{N}$, then $\{i: \gamma \leqq i \leqq \eta\}$ will denote the set of indices in $^*\mathscr{I}$ with $\gamma \leqq i \leqq \eta$. Given $n \in \mathbf{N}$, $\bigcup_{i \geqq n} A_i$ will denote $\bigcup \{A_i: i \in \mathscr{I}, i \geqq n\}$.

THEOREM 1. *Let $(X, \mathscr{F}, \mu)$ be a standard measure space with $\mu(X) < +\infty$, and for each $i \in \mathscr{I}$, let g_i be an extended real-valued, $\mathscr{F}$-measurable function on X.*

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i) *If* $g_i \to 0$ *a.e., then for each* γ *and* η *in* $^*\mathbf{N} - \mathbf{N}$ *with* $\gamma \leqq \eta$,

$$\sup_{\gamma \leq i \leq \eta} |g_i| \cong 0 \text{ n.s.}$$

ii) *Conversely,* $g_i \to 0$ *a.e. if there is an* $\eta \in {}^*\mathbf{N} - \mathbf{N}$ *such that for all* $\gamma \leqq \eta$ *in* $^*\mathbf{N} - \mathbf{N}$,

$$\sup_{\gamma \leq i \leq \eta} |g_i| \cong 0 \text{ n.s.}$$

iii) *Assume* $\mathscr{I} = \mathbf{N}$. *Then there is a null set* $A \subset X$ *such that* $g_i(x)$ *is a Cauchy sequence in* $\mathbf{R}$ *for each* $x \in X - A$ *if and only if the following holds: For some* $\eta \in {}^*\mathbf{N} - \mathbf{N}$ *and all* $\gamma \leqq \eta$ *in* $^*\mathbf{N} - \mathbf{N}$,

$$\max_{\gamma \leq i \leq j \leq \eta} |g_i - g_j| \cong 0 \text{ n.s.}$$

Proof. (i) If $g_i \to 0$ a.e., then given $\varepsilon > 0$ in $\mathbf{R}$ and setting $A_i^\varepsilon = \{|g_i| > \varepsilon\}$ for each $i \in \mathscr{I}$, we have

$$\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{i \geqq k} A_i^\varepsilon\right) = 0,$$

whence $\lim_{k\to\infty} \mu(\bigcup_{i\geqq k} A_i^\varepsilon) = 0$. It follows that for $\gamma \leqq \eta$ in $^*\mathbf{N} - \mathbf{N}$, $^*\mu(\bigcup_{\gamma \leq i \leq \eta} A_i^\varepsilon) \cong 0$, and so $^*\mu(\sup_{\gamma \leq i \leq \eta} |g_i| > \varepsilon) < \varepsilon$. Since this is true for any $\varepsilon > 0$ in $\mathbf{R}$, $\sup_{\gamma \leq i \leq \eta} |g_i| \cong 0$ n.s.

(ii) If there is an $\eta \in {}^*\mathbf{N} - \mathbf{N}$ such that for each $\gamma \leqq \eta$ in $^*\mathbf{N} - \mathbf{N}$ and each $\varepsilon > 0$ in $\mathbf{R}$ we have $^*\mu(\sup_{\gamma \leq i \leq \eta} |g_i| > \varepsilon) \cong 0$ whence $^*\mu(\bigcup_{\gamma \leq i \leq \eta} A_i^\varepsilon) \cong 0$, then it follows that for any $\delta > 0$ in $\mathbf{R}$ there is a $k \in \mathbf{N}$ with $\mu(\bigcup_{i \geqq k} A_i^\varepsilon) \leqq \delta$. Therefore, $\mu(\bigcap_{k=1}^\infty \bigcup_{i\geqq k} A_i^\varepsilon) = 0$. Let

$$B = \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{i \geqq k} A_i^{1/m}.$$

Then $\mu(B) = 0$ and $g_i(x) \to 0$ for $x \in X - B$.

(iii) The proof is left to the reader; we shall not need this result.

The above criterion for almost everywhere convergence was suggested by Egorov's Theorem. A sufficient condition based on the Borel-Cantelli Lemma was used by Hersh and Greenwood [3] to consider the convergence of L^2-bounded martingales. A "maximal" condition similar to the one used here does appear in the body of their proof; further use of this maximal condition seems to be needed to carry out their proof.

We now fix an increasing family of σ-algebras $\{\mathscr{F}_i: i \in \mathscr{I}\}$ in a standard set X. We let $\mathscr{F}$ denote the smallest σ-algebra containing each $\mathscr{F}_i$, i.e., $\mathscr{F} = \sigma(\bigcup_{i\geqq 1} \mathscr{F}_i) = \sigma(\bigcup_{n=1}^\infty \mathscr{F}_n)$. Let P be a fixed probability measure on $(X, \mathscr{F})$. Let $\mathscr{J}$ be a finite or infinite subset of $\mathscr{I}$. Recall that a family $\{Y_j: j \in \mathscr{J}\}$ of extended real-valued functions on X is said to be adapted to $\{\mathscr{F}_j: j \in \mathscr{J}\}$ if Y_j is $\mathscr{F}_j$-measurable for each $j \in \mathscr{J}$. If, moreover,

$\int_A Y_{i_0} dP \leqq \int_A Y_{j_0} dP$ (or, respectively, $\int_A Y_{i_0} dP = \int_A Y_{j_0} dP$) for each pair $i_0 < j_0$ in $\mathscr{J}$ and each $A \in \mathscr{F}_{i_0}$, then $\{Y_j : j \in \mathscr{J}\}$ is called a submartingale (or, respectively, a martingale). For completeness, we prove the following inequalities of Doob.

THEOREM 2 (DOOB [2, p. 314]). *Given $n < m$ in $\mathbf{N}$, let $\{Y_i : n \leqq i \leqq m\}$ be a submartingale adapted to $\{\mathscr{F}_i : n \leqq i \leqq m\}$. Fix $\lambda \in \mathbf{R}$ and $A \in \mathscr{F}_n$. Let $\bar{M} = \{\sup_{n \leqq i \leqq m} Y_i > \lambda\} - A$ and let $\underline{M} = \{\inf_{n \leqq i \leqq m} Y_i < \lambda\} - A$. Then*

$$\text{(i) } \lambda P(\bar{M}) \leqq \int_{\bar{M}} Y_m dP,$$

$$\text{(ii) } \int_{X-A} Y_n \, dP - \lambda P(\underline{M}) \leqq \int_{(X-A)-\underline{M}} Y_m dP.$$

PROOF. We assume that $\mathscr{J} = \mathbf{N}$. The general case follows by taking appropriate limits with respect to increasing finite subsets of $\mathscr{J}$; the order on the family of finite subsets of $\mathscr{J}$ is given by containment.

(i) Define B_i by induction so that $B_n = \{Y_n > \lambda\} - A$ and for $n < i \leqq m$, $B_i = [\{Y_i > \lambda\} - \bigcup_{j=n}^{i-1} B_j] - A$. Then, using the submartingale property on $B_i \in \mathscr{F}_i$, we have

$$\int_{\bar{M}} Y_m \, dP = \int_{\bigcup_{i=n}^{m} B_i} Y_m \, dP \geqq \sum_{i=n}^{m} \int_{B_i} Y_i \, dP$$

$$\geqq \sum_{i=n}^{m} \lambda \, P(B_i) = \lambda \, P(\bar{M}).$$

(ii) Define C_i and D_i by induction so that $C_n = \{Y_n < \lambda\} - A$, $D_n = \{Y_n \geqq \lambda\} - A$, and for $n < i \leqq m$, $C_i = \{x \in D_{i-1} : Y_i < \lambda\}$ and $D_i = D_{i-1} - C_i$. Then

$$\begin{aligned}
\int_{X-A} Y_n dP &\leqq \lambda P(C_n) + \int_{D_n} Y_n dP \\
&\leqq \lambda P(C_n) + \int_{D_n} Y_{n+1} \, dP \\
&\leqq \lambda P(C_n) + \lambda P(C_{n+1}) + \int_{D_{n+1}} Y_{n+1} dP \\
&\;\;\vdots \\
&\leqq \lambda P(\underline{M}) + \int_{D_m} Y_m dP.
\end{aligned}$$

Recall that for $Y \in L^1(X, \mathscr{F}, P)$, $E[Y \mid \mathscr{F}_i] \in L^1(X, \mathscr{F}_i, P)$ is the Radon-Nikodym derivative of the measure obtained by integrating Y over sets in $\mathscr{F}_i$. The following result is proved here using a simplification of a proof by Alda [1].

THEOREM 3. *Let Y be $\mathscr{F}$-measurable and integrable on X. Then $E[Y|\mathscr{F}_i] \to Y$ a.e.*

PROOF. Fix γ and η in $^*\mathbf{N} - \mathbf{N}$ with $\gamma < \eta$. Since $\mathscr{F} = \sigma(\bigcup_{n=1}^{\infty} \mathscr{F}_n)$, there is for any standard set $A \in \mathscr{F}$ and any $\varepsilon > 0$ an $n \in \mathbf{N}$ and an $E \in \mathscr{F}_n$ such that $P(A \triangle E) < \varepsilon$, and so there is an $E_A \in \mathscr{F}_\gamma$ with $^*P(^*A \triangle E_A) \cong 0$. It follows that there is an internal $\mathscr{F}_\gamma$-measurable function Y_0 on $^*X$ such that $\int_{^*X} |{}^*Y - Y_0|\, d^*P \cong 0$, whence $|{}^*Y - Y_0| \cong 0$ n.s. Let $Y_i = E[^*Y|\mathscr{F}_i]$ for $\gamma \leqq i \leqq \eta$. Then $\{|Y_i - Y_0|: \gamma \leqq i \leqq \eta\}$ is an internal submartingale and this is also true if $|{}^*Y - Y_0|$ is adjoined as a last element. Given $\varepsilon > 0$ in $\mathbf{R}$ and setting $B_\varepsilon = \{\sup_{\gamma \leqq i \leqq \eta} |Y_i - Y_0| > \varepsilon\}$, we have by (i) of Theorem 2,

$$\varepsilon\, {}^*P(B_\varepsilon) \leqq \int_{B_\varepsilon} |Y_\eta - Y_0| \, d^*P \leqq \int_{B_\varepsilon} |{}^*Y - Y_0|\, d^*P \cong 0.$$

Therefore, $P(B_\varepsilon) \cong 0$ for any $\varepsilon > 0$ in $\mathbf{R}$, and so $\sup_{\gamma \leqq i \leqq \eta} |Y_i - Y_0| \cong 0$ n.s. Since $|Y_0 - {}^*Y| \cong 0$ n.s., $\sup_{\gamma \leqq i \leqq \eta} |Y_i - {}^*Y| \cong 0$ n.s., whence $|Y_i - Y| \to 0$ a.e.

We now consider a fixed $\{\mathscr{F}_i\}$-adapted submartingale $\{Y_i: i \in \mathscr{I}\}$ such that $Y_i \geqq 0$ for each $i \in \mathscr{I}$ and $L = \sup_i \int_X Y_i dP < +\infty$. Since $\int_X Y_i dP \leqq \int_X Y_j dP$ when $i < j$ in $\mathscr{I}$, $L = \lim_{i \to \infty} \int_X Y_i dP$. Thus $L = {}^\circ\!\int_{^*X} Y_i\, d^*P$ for each infinite $i \in {}^*\mathscr{I}$. Given any infinite $i \in {}^*\mathscr{I}$, we let

$$S_i = L - \lim_{\substack{m \to \infty \\ m \in N}} {}^\circ\!\int_{^*X} (Y_i \wedge m)\, d^*P.$$

We call S_i the singular part of the integral of Y_i. One can find an $\alpha \in {}^*\mathbf{N} - \mathbf{N}$ such that $S_i = {}^\circ\!\int_{\{Y_i \geqq \alpha\}} Y_i\, d^*P$.

PROPOSITION 1. *There is an $\eta \in {}^*\mathbf{N} - \mathbf{N}$ such that for each infinite $i \leqq \eta$ in $^*\mathscr{I}$, $S_i = S_\eta$.*

PROOF. Given an infinite $i \in {}^*\mathscr{I}$, choose $\alpha \in {}^*\mathbf{N} - \mathbf{N}$ so that $S_i = {}^\circ\!\int_{\{Y_i \geqq \alpha\}} Y_i\, d^*P$. The set $D_i = \{Y_i \geqq \alpha\}$ has infinitesimal measure, so for $j \geqq i$ in $^*\mathscr{I}$ and $m \in \mathbf{N}$, ${}^\circ\!\int_{D_i} (Y_j \wedge m)\, d^*P = 0$, whence

$$S_i = {}^\circ\!\int_{D_i} Y_i\, d^*P \leqq {}^\circ\!\int_{D_i} Y_j\, d^*P \leqq S_j.$$

Let $\{\gamma_n: n \in \mathbf{N}\}$ be a decreasing sequence in $^*\mathbf{N} - \mathbf{N}$ such that $\lim_{n \to \infty} S_{\gamma_n} = \inf\{S_i: i \text{ infinite in } {}^*\mathscr{I}\}$. By $\aleph_1$-saturation, there exists an $\eta \in {}^*\mathbf{N} - \mathbf{N}$ so that $\eta \leqq \gamma_n$ for all $n \in \mathbf{N}$. Clearly $S_i = S_\eta$ for each infinite $i \leqq \eta$ in $^*\mathscr{I}$.

We now fix $\eta \in {}^*\mathbf{N} - \mathbf{N}$ so that $S_i = S_\eta$ for any infinite $i \leqq \eta$ in $^*\mathscr{I}$. Let $S = S_\eta$. If $S = 0$, we set $A_i = \varnothing$ for all $i \in {}^*\mathscr{I}$. If $S > 0$ and $i \leqq \eta$ in $^*\mathscr{I}$, we let α_i be the largest element $\rho \in {}^*\mathbf{N}$ such that

$$\int_{\{Y_i \geqq \rho\}} Y_i \, d^*P \geqq S - S/\rho,$$

and we let $A_i = \{Y_i \geqq \alpha_i\}$. The proof of Proposition 1 shows that for each infinite $i \leqq \eta$ in $^*\mathscr{I}$ we have $\alpha_i \in {}^*\mathbf{N} - \mathbf{N}$, $\int_{A_i} Y_i \, d^*P \cong \int_{A_i} Y_\eta \, d^*P \cong S$, $^*P(A_i) \cong 0$, and $\int_{A_\eta - A_i} Y_\eta \, d^*P \cong 0$.

Given any set $B \in \mathscr{F}$, there is a set $E_B \in \mathscr{F}_\eta$ such that $^*P(^*B \triangle E_B) \cong 0$. Let $\nu(B) = {}^\circ\int_{E_B - A_\eta} Y_\eta \, d^*P$. Since

$$\lim_{m \to \infty} {}^\circ\int_{\{Y_\eta \geqq m\} - A_\eta} Y_\eta \, d^*P = 0,$$

there exists for each $\varepsilon > 0$ in $\mathbf{R}$ a $\delta > 0$ in $\mathbf{R}$ such that $\nu(B) < \varepsilon$ when $P(B) < \delta$. Thus ν is σ-additive on $\mathscr{F}$ and absolutely continuous with respect to P. Let Z be the Radon-Nikodym derivative $d\nu/dP$.

PROPOSITION 2. *The nonnegative submartingale* $Y_i \to Z$ a.e.

PROOF. For each $i \in \mathscr{I}$, let $Z_i = E[Z|\mathscr{F}_i]$. By Theorem 3, $Z_i \to Z$ a.e.; we will show that $Y_i - Z_i \to 0$ a.e. Given $\varepsilon > 0$ in $\mathbf{R}$, it follows from the properties of the A_i's that there exists an $n \in \mathbf{N}$ such that $P(A_n) < \varepsilon/2$ and ${}^\circ\int_{A_\eta - {}^*A_n} Y_\eta \, d^*P < \varepsilon^2/2$. For each $m \geqq n$ in $\mathbf{N}$, set $B_m = \{\sup_{n \leqq i \leqq m}(Y_i - Z_i) > \varepsilon\} - A_n$. Since $Y_i - Z_i$ is a submartingale and $^*B_m \in {}^*\mathscr{F}_m \subset \mathscr{F}_\eta$, it follows from the definition of ν that

$$\varepsilon P(B_m) \leqq \int_{B_m} (Y_m - Z_m) \, dP \leqq {}^\circ\int_{^*B_m} Y_\eta \, d^*P - \int_{B_m} Z \, dP$$
$$\leqq {}^\circ\int_{A_\eta - {}^*A_n} Y_\eta \, d^*P + \nu(B_m) - \int_{B_m} Z \, dP < \frac{\varepsilon^2}{2}.$$

Therefore, for any $m \geqq n$ in $\mathbf{N}$, $P(\sup_{n \leqq i \leqq m}(Y_i - Z_i) > \varepsilon) \leqq P(B_m) + P(A_n) < \varepsilon$. It follows that for $\gamma \leqq \eta$ in $^*\mathbf{N} - \mathbf{N}$, $^*P(\sup_{\gamma \leqq i \leqq \eta}(Y_i - Z_i) > \varepsilon) < \varepsilon$ for each $\varepsilon > 0$ in $\mathbf{R}$ and thus for some positive $\varepsilon \cong 0$ in $^*\mathbf{R}$. On the other hand, if for each $\varepsilon > 0$ in $\mathbf{R}$ we set $M_\varepsilon = \{\inf_{\gamma \leqq i \leqq \eta}(Y_i - Z_i) < -\varepsilon\} - A_\gamma$, then

$$\int_{^*X - A_\gamma} (Y_\gamma - Z_\gamma) \, d^*P + \varepsilon {}^*P(M_\varepsilon) \leqq \int_{(^*X - A_\gamma) - M_\varepsilon} (Y_\eta - Z_\eta) \, d^*P.$$

Here, $\int_{^*X - A_\gamma}(Y_\gamma - Z_\gamma) \, d^*P \cong \nu(X) - \int_X Z \, dP = 0$. Moreover, we have just shown that $Y_\eta - Z_\eta$ is either negative or infinitesimal except on a set E of infinitesimal measure. Since $\int_{E - A_\gamma} Y_\eta \, d^*P \cong 0$,

$$^\circ\int_{(^*X - A_\gamma - M_\varepsilon)} (Y_\eta - Z_\eta) \, d^*P \leqq 0.$$

Therefore, $^*P(M_\varepsilon) \cong 0$ for each $\varepsilon > 0$ in $\mathbf{R}$. It now follows that for any $\gamma \leqq \eta$ in $^*\mathbf{N} - \mathbf{N}$, $\sup_{\gamma \leqq i \leqq \eta} |Y_i - Z_i| \cong 0$ n.s., whence $Y_i - Z_i \to 0$ a.e.

It now follows from Proposition 2 that $S_i = S$ for every infinite $i \in {}^*\mathscr{I}$. Moreover, when $S = 0$, the above proof shows that $P(\exists i \in \mathscr{I}$ with $Y_i > Z_i) = 0$. The nonnegative submartingale $\{Y_i\}$ is called uniformly integrable if $S = 0$ (see [**4**, Page 131]). We now prove the convergence theorem for L^1-bounded submartingales.

THEOREM 4. *Let* $\{Y_i: i \in \mathscr{I}\}$ *be an* $\{\mathscr{F}_i\}$*-adapted submartingale with* $\sup_i \int_X |Y_i|\, dP < +\infty$. Then Y_i converges a.e.

PROOF. For each $m \in \mathbf{N}$, $\{Y_i \vee -m\}$ is a submartingale as is $\{(Y_i \vee -m) + m\}$. By Proposition 2, $Y_i \vee -m$ converges a.e. for each $m \in \mathbf{N}$. Given $m \in \mathbf{N}$, let $D_m = \{\inf_{i \in \mathscr{I}} Y_i < -m\}$. For almost all $x \in X - \bigcap_{m=1}^{\infty} D_m$, $Y_i(x)$ converges. We will show that $P(\bigcap_{m=1}^{\infty} D_m) = 0$. Assume not. Then for some $\varepsilon > 0$ in $\mathbf{R}$, $P(D_m) \geqq \varepsilon$ for all $m \in \mathbf{N}$ and thus for all $m \in {}^*\mathbf{N}$. Fix $m_0 \in {}^*\mathbf{N} - \mathbf{N}$. For each $\eta \in {}^*\mathbf{N} - \mathbf{N}$, let $M_\eta = \{\inf_{1 \leqq i \leqq \eta} Y_i < -m_0\}$. Then

$$\int_{{}^*X} {}^*Y_1\, d{}^*P + m_0\, {}^*P(M_\eta) \leqq \int_{{}^*X - M_\eta} Y_\eta d{}^*P.$$

Since both integrals are finite and m_0 is infinite, ${}^*P(M_\eta) \cong 0$. Since η is arbitrary in ${}^*\mathbf{N} - \mathbf{N}$, ${}^*P(\inf_{i \in {}^*\mathscr{I}} Y_i < -m_0) \cong 0$, a contradiction.

Assume $\{Y_i\}$ is an L^1-bounded martingale, i.e., both $\{Y_i\}$ and $\{-Y_i\}$ are submartingales, and assume the submartingale $\{|Y_i|\}$ is uniformly integrable. Then it is a well-known and now easily obtained fact that for $Z = \lim Y_i$, we have $Y_i = E[Z \mid \mathscr{F}_i]$ a.e. for each i.

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