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# CONVERGENCE QUESTIONS FOR LIMIT PERIODIC CONTINUED FRACTIONS

W. J. THRON AND HAAKON WAADELAND

**1. Introduction.** Given two sequences of complex numbers $\{a_n\}$, $\{b_n\}$, $n \geqq 1$, we define for complex $w$

$$s_n(w) = \frac{a_n}{b_n + w},\ n \geqq 1,$$

and

$$S_N^{(n)}(w) = s_{n+1}(S_N^{(n+1)}(w)),\ 0 \leqq n \leqq N-1,$$
$$S_N^{(N)}(w) = w.$$

Using one of the standard notations we then have

$$S_N^{(n)}(w) = \frac{a_{n+1}}{b_{n+1}} + \frac{a_{n+2}}{b_{n+2}} + \cdots + \frac{a_N}{b_N + w},\ 0 \leqq n \leqq N-1.$$

Instead of $S_N^{(0)}(w)$ we shall usually write $S_N(w)$.

The *continued fraction*

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{a_n}{b_n}\right) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots \tag{1.1}$$

then is the ordered pair $\langle\langle\{a_n\}, \{b_n\}\rangle, \{S_n(0)\}\rangle$. Here it is understood that $\{a_n\}$ and $\{b_n\}$ be such that $S_n(0)$ is defined as an extended complex number for all $n$ (or at least from a certain $n = n_0$ on). This is in particular the case if $a_n \neq 0$ for all $n$ or if $a_n = 0$ for all $n$ and simultaneously $b_n \neq 0$. The sequences $\{a_n\}$ and $\{b_n\}$ are called the *sequences of elements*, and $\{S_n(0)\}$ is the *sequence of approximants*. Convergence of a continued fraction means convergence of the sequence of approximants (possibly to $\infty$). In case of convergence the notation $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ is also used for $\lim_{n\to\infty} S_n(0)$.

The approximants of a continued fraction can also be represented as $S_n(0) = A_n/B_n$, where

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$$A_n = b_n A_{n-1} + a_n A_{n-2}$$

$$(1.2) \qquad\qquad n \geqq 1,$$

$$B_n = b_n B_{n-1} + a_n B_{n-2}$$

$$A_{-1} = 1,\ A_0 = 0,\ B_{-1} = 0,\ B_0 = 1.$$

See for instance [**5**].

It is easy to prove that for $n \geqq 1$

$$(1.3) \qquad F_n(w) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n + w} = \frac{A_n + A_{n-1}w}{B_n + B_{n-1}w}.$$

A continued fraction for which $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ exist is called a *limit periodic continued fraction*. Limit periodic continued fractions play an important role in the analytic theory of continued fractions, see for instance [**6**, Introduction]. The simplest example of a limit periodic continued fraction is the *periodic* continued fraction (of period length 1)

$$(1.4) \qquad \sum_{n=1}^{\infty}\left(\frac{a}{b}\right) = \frac{a}{b} + \frac{a}{b} + \cdots + \frac{a}{b} + \cdots.$$

This is meaningful if and only if $|a| + |b| > 0$ (and trivial if exactly one of the numbers $a$, $b$ is 0). It is well known (see for instance [**4**, p. 87]) that (1.4) converges if and only if the fixed points $x_1$ and $x_2$ of the linear fractional transformation

$$(1.5) \qquad s(w) = \frac{a}{b + w}$$

have different absolute values ($|x_1| < |x_2|$) or coincide ($x_1 = x_2$). In case of convergence we have $\lim_{n\to\infty} S_n(0) = x_1$. We recall that for $|x_1| < |x_2|$, $x_1$ is the *attractive* fixed point and $x_2$ the *repulsive* fixed point. For $|x_1| = |x_2|$, $x_1 \neq x_2$ the transformation (1.5) is elliptic, and the fixed points are neither attractive nor repulsive. We shall illustrate convergence of periodic continued fractions by two familiar examples.

EXAMPLE 1.1. Let $z$ be a complex number, and let in (1.4) $a = z$, $b = 1 - z$. The fixed points of $s$ are in this case $z$ and $-1$. $z$ is attractive in the disk $|z| < 1$ and repulsive for $|z| > 1$, $-1$ is repulsive in the disk $|z| < 1$ and attractive for $|z| < 1$. The fixed points coincide for $z = -1$. On the rest of the circle $|z| = 1$ they are distinct, but of the same absolute value. Hence the continued fraction

$$\frac{z}{1 - z} + \frac{z}{1 - z} + \cdots + \frac{z}{1 - z} + \cdots$$

converges to $z$ if $|z| < 1$, converges to $-1$ if $|z| > 1$ or if $z = -1$, and diverges on the rest of the circle $|z| = 1$.

EXAMPLE 1.2. Let $\alpha \neq 0$ be a complex number, kept fixed throughout the example, and let $z$ be a complex number. Let, furthermore, in (1.4) $a = \alpha z$, $b = 1$. The fixed points of $s$ are in this case

$$\frac{1}{2}(-1 \pm (1 + 4\alpha z)^{1/2}).$$

They differ in absolute value if and only if $z$ is not on the ray defined by $z = -t/(4\alpha)$, $t \geqq 1$, and coincide (with value $-1/2$) if and only if $z = -1/(4\alpha)$. For $z$ off the ray let $(1 + 4\alpha z)^{1/2}$ denote the branch with positive real part. Then the continued fraction

$$\frac{\alpha z}{1} + \frac{\alpha z}{1} + \cdots + \frac{\alpha z}{1} + \cdots$$

converges to $(1/2)((1 + 4\alpha z)^{1/2} - 1)$ for all $z$ not on the ray defined by $z = -t/(4\alpha)$, $t > 1$, and diverges on the ray.

It is to be expected that convergence properties of limit periodic continued fractions are similar to those of periodic continued fractions and that convergence results for limit periodic continued fractions may be obtained by using the "nearness' of the "tails" $\mathrm{K}_{N+1}^{\infty}(a_n/b_n)$ to the continued fraction (1.4). This is in fact the case, as may be seen in [**4**, p. 93]. Slightly rephrased the first theorem there (Satz 2.41) states that if a continued fraction (1.1) is such that $a_n \neq 0$, $a_n \to a$, $b_n \to b$, $a, b \in \mathbf{C}$, and the fixed points of (1.5) are of different absolute values, then any tail $\mathrm{K}_{N+1}^{\infty}(a_n/b_n)$ from a certain $N = N_0$ on converges to a complex number $f^{(N)}$, where $f^{(N)} \to x_1$ (the attractive fixed point). The convergence of the tails implies convergence of the continued fraction itself, possibly to $\infty$.

Continued fractions are similarly defined if $\{a_n\}$, $\{b_n\}$ are sequences of complex valued functions rather than of complex numbers. Pointwise considerations will contain nothing new, but often one needs more (for instance uniform convergence) in order to conclude from properties of the functions $a_n$, $b_n$ properties of the possible $\lim_{n\to\infty} S_n(0)$. In the two examples it is easy to prove that the approximants (which of course all are rational) are holomorphic in the domains in question (i.e., $|z| < 1$ and $|z| > 1$ in Ex. 1.1 and $z \neq -t/(4\alpha)$ in Ex. 1.2), and that the sequences of approximants converge to holomorphic functions in the domains in question, in accordance with what we already know. A result on uniform convergence of limit periodic continued fractions is given in [**4**, p. 93] (Satz 2.42), where it is proved that if $a_n$ and $b_n$ are complex valued functions on some set, $a_n \to a$ and $b_n \to b$ uniformly on that set, and $x_1$ and $x_2$ (now functions) are subject to boundedness conditions $c \leqq |x_2| \leqq C$, $|x_1/x_2| \leqq \mathfrak{D} < 1$ on the same set, then from a certain $N = N_0$ on, any tail $K_{N+1}^{\infty}(a_n/b_n)$ converges uniformly. We shall illustrate this by showing two examples with the previous ones as special cases.

EXAMPLE 1.3. Let $z$ be a complex variable and $\{F_n\}$, $\{G_n\}$ sequences of complex numbers such that $F_n \to F$, $G_n \to -F$, $F \in \mathbf{C}\backslash\{0\}$. Then for any $r \in (0, 1)$ there is an $N_0$, such that for all $N \geqq N_0$ the continued fraction (general $T$-fraction)

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{F_n z}{1 + G_n z}\right)$$

converges in $|z| < r/|F|$ uniformly to a holomorphic function $f^{(N)}$, where $f^{(N)}(z) \to z$ as $N \to \infty$, and in $|z| > 1/r|F|$ to a holomorphic function $g^{(N)}$, where $g^{(N)}(z) \to -1$ as $N \to \infty$. Note that the $N$ in $f^{(N)}$ and $g^{(N)}$ is merely a superscript and does note signify a derivative.

EXAMPLE 1.4. Let $z$ be a complex variable and $\{\alpha_n\}$ a sequence of complex numbers such that $\alpha_n \to \alpha \in \mathbf{C}\backslash\{0\}$. Let $\{D_k\}$ be a sequence of bounded sub-domains of the plane cut along the ray $z = -t/(4\alpha)$, $t \geqq 1$, such that for all $k \geqq 1$, $\mathrm{cl}(D_k) \subset D_{k+1}$ and $\bigcup_{k=2}^{\infty} D_k =$ the cut plane. Then for any $k$ there is an $N_k$, such that for all $N \geqq N_k$ the continued fraction (regular $C$-fraction)

$$\underset{n=N+1}{\overset{\infty}{\mathrm{K}}}\left(\frac{\alpha_n z}{1}\right)$$

converges in $D_k$ uniformly to a holomorphic function $f^{(N)}$, where

$$f^{(N)}(z) \to \frac{1}{2}((1 + 4\alpha z)^{1/2} - 1)$$

as $N \to \infty$. See [**4**, p. 95].

We shall now look at two other convergence properties of periodic continued fractions, which turn out to have parallels for limit periodic continued fractions. Assume for simplicity that $a$ and $b$ in (1.4) are both $\neq 0$. If the fixed points are $x_1$ and $x_2$ (not necessarily of distinct absolute values, not even necessarily distinct), then the continued fraction (1.4) can be written

$$\frac{-x_1x_2}{-(x_1 + x_2)} + \frac{-x_1x_2}{-(x_1 + x_2)} + \cdots + \frac{-x_1x_2}{-(x_1 + x_2)} + \cdots$$

It is readily seen that $S_n(x_1) = x_1$ and $S_n(x_2) = x_2$ for all $n$. This suggests the following two ideas: 1. For a limit-periodic continued fraction the sequence $\{S_n(x_1)\}$, where $x_1$ is the attractive fixed point of (1.5), may converge to the value of the continued fraction more rapidly than the sequence $\{S_n(0)\}$ (To replace $\{S_n(0)\}$ by $\{S_n(x_1)\}$ in this case shall be called "right modification".); and 2. For a limit-periodic continued fraction the sequence $\{S_n(x_2)\}$, where $x_2$ is the repulsive fixed point, may converge, and if so, to a "wrong value", i.e., a value $\neq \lim_{n\to\infty} S_n(0)$ ("Wrong modification".). The observations on periodic continued fractions even suggest the possibility of convergence of $\{S_n(x_1)\}$ or $\{S_n(x_2)\}$ in the elliptic case.

The idea 1 is studied in the paper **[6]** (where also further references to other papers on the same idea or related ideas are made) and it turned out, that the tails of any limit periodic continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/1)$, no matter how slowly $a_n \to a \neq 0$, will be subject to substantial improvement of the convergence by the "right modification" in the case $|x_1| < |x_2|$. In the case $x_1 = x_2$ the same holds under additional conditions on the rate at which $a_n \to a$. (To put $b_n = 1$ for all $n$ as in the paper **[6]** is a rather inessential restriction since any $\mathrm{K}_{n=1}^{\infty}(\alpha_n/\beta_n)$ with all $\beta_n \neq 0$ is equivalent to some continued fraction of the form $\mathrm{K}_{n=1}^{\infty}(a_n/1)$.)

To see that it makes sense to follow idea 2 even if it leads to convergence to a "wrong" value let us see what the "wrong modification" does to the continued fractions in Ex.1.1 and Ex.1.2. In Ex.1.1. we have for all $n$, $S_n(z) = z$ in the entire plane, also for $|z| > 1$, where $z$ is the repulsive fixed point and for $|z| = 1$, $z \neq -1$, where the transformation (1.5) is elliptic and where the continued fraction (1.4) does not enverge. Furthermore we have for all $n$, $S_n(-1) = -1$ for all $z \neq 0$ in the plane. In both cases the modification leads to an analytic continuation of the function defined by the continued fraction and provides a process converging to the extension. Quite similarly in Ex.1.2 the use of the repulsive fixed point leads to analytic continuation across the ray $z = -t/(4\alpha)$, $t \geqq 0$, to a function, defined and analytic on a two-sheeted Riemann surface with a branch point of order one at $z = -1/(4\alpha)$. It is to be expected that similar things hold for the continued fractions in Ex.1.3 and Ex.1.4. That this indeed is the case, although under rather strong conditions on the rate at which $a_n \to a$, $b_n \to b$, is proved in the paper [7]. The use of the "wrong modification" (or rather a combination of "right" and "wrong modifications") in order to increase the domain of convergence and hence to obtain analytic continuation was introduced for ordinary $T$-fractions in **[8]**, but the full extent of the consequences of this modification was not established until somewhat later **[2]**. In a more general setting the method was later suggested by Gill in, for instance, the paper **[1]**.

The results in [7] for general $T$-fractions and regular $C$-fractions are established by means of a lemma on boundedness of modified sequences in combination with a result of Jones and Thron on correspondence and convergence **[3]**. Although the paper [7] does not contain a direct proof of the convergence of the wrong modification more generally, the method most certainly can be applied to other types of continued fractions.

The purpose of the rest of the present paper is to establish directly the convergence of the wrong modification of a limit periodic continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$, where $a_n \to a$ and $b_n \to b$ sufficiently fast.

**2. The modified approximants.** Following the notations in [7] we shall write, in case of a limit periodic continued fraction (1.1) with $a_n \to a$, $b_n \to b$,

$$a_n = a + \delta_n, \qquad b_n = b + \eta_n \tag{2.1}$$

Here $\delta_n \to 0$ and $\eta_n \to 0$ as $n \to \infty$ (at a rate to be discussed later). Let $x$ be one of the fixed points of the transformation (1.5). Then $-(x + b)$ is the other fixed point, since they are the roots of the quadratic equation $x^2 + bx - a = 0$. (In the applications $a_n$, $b_n$, $a$, $b$, $x$ are complex functions of a complex variable, but we use for simplicity $a_n$, $b_n$, ... instead of $a_n(z)$, $b_n(z)$, ... .)

A slight rearrangement of the three term recursion formula (1.2) for the $A_k$'s gives for $n \geqq 2$

$$A_n + xA_{n-1} = (b + x)A_{n-1} + aA_{n-2} + \eta_n A_{n-1} + \delta_n A_{n-2}.$$

Keeping in mind that $a = x(b + x)$ we have

$$\begin{aligned}
A_n + xA_{n-1} &= (b + x)[A_{n-1} + xA_{n-2}] + \eta_n A_{n-1} + \delta_n A_{n-2}\\
A_{n-1} + xA_{n-2} &= (b + x)[A_{n-2} + xA_{n-3}] + \eta_{n-1} A_{n-2} + \delta_{n-1} A_{n-3}\\
&\vdots\\
A_2 + xA_1 &= (b + x)[A_1 + xA_0] + \eta_2 A_1 + \delta_2 A_0\\
A_1 + xA_0 &= (b + x)[A_0 + xA_{-1}] + \eta_1 A_0 + \delta_1 A_{-1}
\end{aligned}$$

since $A_0 = 0$ and $A_{-1} = 1$. Multiplication by $1, (b + x), \ldots, (b + x)^{n-1}$ followed by addition of all equalities gives

$$\begin{aligned}
A_n + xA_{n-1} = (b + x)^n x &+ \sum_{k=0}^{n-1} (b + x)^{n-1-k} \eta_{k+1} A_k\\
&+ \sum_{k=0}^{n-1} (b + x)^{n-1-k}\, \delta_{k+1} A_{k-1}.
\end{aligned} \tag{2.2}$$

The same method for the $B_k$'s gives, since $B_0 = 1$ and $B_{-1} = 0$,

$$\begin{aligned}
B_n + xB_{n-1} &= (b + x)[B_{n-1} + xB_{n-2}] + \eta_n B_{n-1} + \delta_n B_{n-2}\\
B_{n-1} + xA_{n-2} &= (b + x)[B_{n-2} + xB_{n-3}] + \eta_{n-1} B_{n-2} + \delta_{n-1} B_{n-3}\\
&\vdots\\
B_2 + xB_1 &= (b + x)[B_1 + xB_0] + \eta_2 B_1 + \delta_2 B_0\\
B_1 + xB_0 &= (b + x)[B_0 + xB_{-1}] + \eta_1 B_0 + \delta_1 B_{-1}.
\end{aligned}$$

Multiplication by 1, $(b + x)$, ..., $(b + x)^{n-1}$ followed by addition of all equalities gives

$$(2.3)\quad \begin{aligned} B_n + xB_{n-1} = (b + x)^n &+ \sum_{k=0}^{n-1} (b + x)^{n-1-k}\, \eta_{k+1}\, B_k \\ &+ \sum_{k=0}^{n-1} (b + x)^{n-1-k}\, \delta_{k+1}\, B_{k-1}. \end{aligned}$$

Under the condition $(b + x) \neq 0$ we get from (2.2), (2.3) and (1.3), for $n \geqq 1$

$$(2.4)\quad S_n(x) = \frac{x + \sum_{k=0}^{n-1} \frac{A_k}{(b + x)^{k+1}} \eta_{k+1} + \sum_{k=0}^{n-1} \frac{A_{k-1}}{(b + x)^{k+1}} \delta_{k+1}}{1 + \sum_{k=0}^{n-1} \frac{B_k}{(b + x)^{k+1}} \eta_{k+1} + \sum_{k=0}^{n-1} \frac{B_{k-1}}{(b + x)^{k+1}} \delta_{k+1}}$$

Obviously the formula remains valid if $x$ is replaced by $-(b + x)$ and simultaneously $b + x$ by $-x$, provided that $x \neq 0$; hence

$$(2.4')\quad S_n(-(b + x)) = \frac{-(x + b) + \sum_{k=0}^{n-1} \frac{A_k}{(-x)^{k+1}} \eta_{k+1} + \sum_{k=0}^{n-1} \frac{A_{k-1}}{(-x)^{k+1}} \delta_{k+1}}{1 + \sum_{k=0}^{n-1} \frac{B_k}{(-x)^{k+1}} \eta_{k+1} + \sum_{k=0}^{n-1} \frac{B_{k-1}}{(-x)^{k+1}} \delta_{k+1}}$$

Observe that in the periodic case, where $\delta_k = \eta_k = 0$ for all $k$ the formulas (2.4) and (2.4′) reduce to the earlier made observation $S_n(x) = x$, $S_n(-(b + x)) = -(b + x)$. It is also worth noting that if the continued fraction is of the form $\mathrm{K}(a_n/1)$ or $\mathrm{K}(1/b_n)$, then the formulas are simplified, since in these cases they only involve two sums instead of four.

**3. Convergence of $\{S_n(x)\}$.** In the case $|x| < |x + b|$, $\{S_n(x)\}$ is the sequence of modified approximants with the right modification. From **[6]** we know that this, under very mild conditions, converges to the value $\lim_{n\to\infty} S_n(0)$ substantially faster than the sequence $\{S_n(0)\}$. One might therefore think that $|x| < |x + b|$ is the only interesting case, but this is not true. For the purpose of analytic continuation we need a result on convergence of $\{S_n(x)\}$ covering simultaneously the cases where $x$ is attractive, repulsive or neither. In the applications to the case of complex valued functions of a complex variable the result can be used to conclude uniform convergence on sets in $\mathbf{C}$ where all three possibilities $|x(z)| < |x(z) + b(z)|$, $|x(z)| = |x(z) + b(z)|$, and $|x(z)| > |x(z) + b(z)|$ occur. In the following we shall therefore assume that the fixed points satisfy a condition of the form

$$(3.1)\quad r < \left| \frac{x}{x + b} \right| < 1/r$$

for some $r$, $0 < r < 1$, kept fixed in the argument.

We want control over the sums involved, and in order to gain such control we shall have to impose strong conditions upon the $\delta_k$'s and the $\eta_k$'s. Convergence of the four series in (2.4) is sufficient for convergence of $\{S_n(x)\}$ if the two sums in the denominator do not add up to $-1$. It is not a priori necessary, but it seems to be hard to obtain results without assuming that the four series converge, even absolutely.

Before stating the lemma we shall look at the periodic case again in order to see why we need very strong conditions on the $\delta_k$'s and the $\eta_k$'s to be sure of convergence of the series. In the periodic case all $\delta_k$'s and $\eta_k$'s are zero. If the fixed points of (1.5) are $x_1$ and $x_2$, then

$$B_n = (-1)^n(x_1^n + x_1^{n-1}x_2 + \cdots + x_1x_2^{n-1} + x_2^n)$$
$$A_n = (-1)^n x_1x_2(x_1^{n-1} + x_1^{n-2}x_2 + \cdots + x_1x_2^{n-2} + x_2^{n-1}).$$

If (3.1) holds, i.e., if $r < |x_2/x_1)| < 1/r$, then

$$\left|\frac{B_n}{x_i^n}\right| \leqq 1 + \frac{1}{r} + \cdots + \frac{1}{r^n} < \frac{\frac{1}{1-r}}{r^n},$$

$$\left|\frac{A_p}{x_i^n}\right| \leqq |x_{3-i}|\left(1 + \frac{1}{r} + \cdots + \frac{1}{r^{n-1}}\right) < \frac{\frac{|x_{3-i}|r}{1-r}}{r^n}, \quad i = 1, 2.$$

This shows us, that conditions on $|\delta_k|$ and $|\eta_k|$, no matter how strong they are, if they also include the case $\delta_k = \eta_k = 0$ for all $k$, then the best we can hope for in (2.4) is that for some $C > 0$

$$\left|\frac{A_n}{(b+x)^n}\right| < \frac{C}{r^n}, \quad \left|\frac{B_n}{(b+x)^n}\right| < \frac{C}{r^n}. \tag{3.2}$$

This indicates that conditions of the type

$$|\delta_n| < k_n r^n, \quad |\eta_k| < k_n r^n, \quad \sum_{n=1}^{\infty} k_n < \infty \tag{3.3}$$

are the weakest worth trying if we want the series to converge and if we stick to conditions on the absolute values of $\delta_n$ and $\eta_n$. The considerations made here or the subsequent more precise treatment do not exclude the possibility of convergence of the series under other types of conditions on the $\delta_n$'s and $\eta_n$'s, conditions where also the arguments of $\delta_n$ and $\eta_n$ are included. Nor do they exclude the possibility of convergence of the sequence $\{S_n(x_i)\}$ in cases where the series do not converge. Discussions on such questions are outside of the scope of the present paper.

So far we only know that the inequalities (3.2) hold in the periodic case. The question whether they still hold if $\delta_n$ and $\eta_n$ are not too large is answered in the following lemma.

LEMMA 1. *Let* $K_{n=1}^{\infty}(a_n/b_n)$ *be a limit periodic continued fraction with* $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, *where* $a$ *and* $b$ *are such that the fixed points* $x$ *and* $-(b + x)$ *of the linear fractional transformation* $s(w) = a/(b + w)$ *satisfy a condition*

$$r < \left|\frac{x}{b + x}\right| < 1/r \tag{3.1}$$

*for some positive* $r < 1$. *Let* $A_n$ *and* $B_n$ *be the normalized numerator and denominator of the n-th approximant, and let* $a_n = a + \delta_n$, $b_n = b + \eta_n$. *Then there is a* $\gamma_0 > 0$, *depending only on* $|x/(b + x)|$, $|a|$ *and* $r$, *such that the following holds for all* $\gamma$, $0 \leqq \gamma \leqq \gamma_0$: *If*

$$|\delta_n| \leqq \gamma r^n, \quad |\eta_n| \leqq \gamma r^n \tag{3.4}$$

*for all* $n$, *then there is a* $C > 0$, *depending only on* $|a|$, $r$ *and* $\gamma$, *such that for all* $n \geqq 0$

$$\left|\frac{A_n}{(x + b)^n}\right| < \frac{C}{r^n}, \quad \left|\frac{B_n}{(x + b)^n}\right| < \frac{C}{r^n}. \tag{3.5}$$

PROOF. We shall restrict ourselves to the $B_n$-case. The proof of the $A_n$-inequalities is similar. In the recursion formulas (1.2) for $B_n$

$$B_n = b_n B_{n-1} + a_n B_{n-2}, \; n \geqq 1$$
$$B_{-1} = 0 \qquad B_0 = 1$$

we introduce

$$D_k = \frac{B_k}{(b + x)^k}, \quad k \geqq -1. \tag{3.6}$$

Then

$$D_{-1} = 0, D_0 = 1, D_1 = \frac{b_1}{b + x} = \frac{b + \eta_1}{b + x}$$

and, since $x(b + x) = a$, we have

$$D_n = \frac{b}{b + x} D_{n-1} + \frac{x}{b + x} D_{n-2} + \frac{\eta_n}{b + x} D_{n-1} + \frac{\delta_n}{(b + x)^2} D_{n-2}$$

for $n \geqq 1$. With

$$\Delta_k = D_k - D_{k-1}, \quad k \geqq 0 \tag{3.7}$$

we have $\Delta_0 = 1$, $\Delta_1 = -x/(b + x) + \eta_1/(b + x)$ and

$$\Delta_n = -\frac{x}{b + x} \Delta_{n-1} + \frac{\eta_n}{b + x} \sum_{\nu=0}^{n-1} \Delta_\nu + \frac{\delta_n}{(b + x)^2} \sum_{\nu=0}^{n-2} \Delta_\nu \tag{3.8}$$

for $n \geqq 2$ (it also holds for $n = 1$).

Assume now that the inequalities (3.4) hold for some fixed $\gamma$. Then we have in particular

$$|\Delta_1| \leqq \left|\frac{x}{b+x}\right| + \frac{\gamma r}{|x|}\left|\frac{x}{b+x}\right| = \left|\frac{x}{b+x}\right|\left(1+\frac{\gamma r}{|x|}\right).$$

Since $|(b+x)/x| < 1/r$ and $|1/(x(b+x))| = 1/|a|$ we have $1/|x| < 1/(r|a|)^{1/2}$ (and similarly $1/|b+x| < 1/(r|a|)^{1/2}$), and hence

$$|\Delta_1| < \left|\frac{x}{b+x}\right|\left(1+\gamma\left(\frac{r}{|a|}\right)^{1/2}\right) < \frac{K}{r},$$

with

$$K = 1 + \gamma\left(\frac{r}{|a|}\right)^{1/2} \tag{3.9}$$

Since $K > 1$, we also have $|\Delta_0| < K/r^0$.

Let $n \geqq 2$ be such that $|\Delta_k| < K/r^k$ for $k = 0, 1, \ldots, n-1$. From (3.8) then follows

$$\begin{aligned}|\Delta_n| &< \left|\frac{x}{b+x}\right|\frac{K}{r^{n-1}} + \gamma r^n\left(\frac{1}{|b+x|}+\frac{1}{|b+x|^2}\right)\left(K+\frac{K}{r}+\cdots+\frac{K}{r^{n-1}}\right)\\ &\leqq \frac{K}{r^n}\left[r\left|\frac{x}{b+x}\right| + \gamma\frac{r^{n+1}}{1-r}\left(\frac{1}{r|a|}+\frac{1}{(r|a|)^{1/2}}\right)\right].\end{aligned}$$

Since $r|x/(b+x)| < 1$, there is a $\gamma$, such that the factor in brackets is $< 1$. (Any positive $\gamma$ with

$$\gamma < \left(1 - r\left|\frac{x}{b+x}\right|\right)\left[\frac{r^3}{1-r}\left(\frac{1}{r|a|}+\frac{1}{(r|a|)^{1/2}}\right)\right]^{-1} \tag{3.10}$$

works.) By induction we thus have for all $n \geqq 0$, $|\Delta_n| < K/r^n$, provided that $\gamma$ is sufficiently small (i.e., satisfies (3.10)).

From this follows, that for all $n$

$$\begin{aligned}|D_n| &\leqq |\Delta_0| + |\Delta_1| + \cdots + |\Delta_n| < K\left(1+\frac{1}{r}+\cdots+\frac{1}{r^n}\right)\\ &= \frac{K}{r^n}(1+r+\cdots+r^n) < \frac{C_1}{r^n},\end{aligned}$$

where

$$C_1 = \frac{K}{1-r} = \frac{1+\gamma(r|a|)^{1/2}}{1-r}.$$

(Observe that $\gamma = 0$ gives back the bound we found in the periodic case.) This concludes the proof of the $B_n$-part of the lemma. The proof of the $A_n$-part is almost identical, except for a few slight differences of merely technical character. The $\gamma$-condition (3.10) is the same in the $A_n$-case, but $C_1$ (with similar meaning) is replaced by

$$C_2 = \frac{(|a|r)^{1/2}(1 + r/\gamma|a|)}{1 - r}.$$

With $C = \text{Max}\,(C_1, C_2)$ the lemma is thus completely established.

LEMMA 2. *Under the conditions of Lemma* 1, *except that* (3.4) *is replaced by*

$$(3.4) \qquad |\delta_n| \leqq \gamma r'^n, \quad |\eta_n| \leqq \gamma r'^n,$$

*where $r'$ is a positive number $< r$, the four series*

$$\sum_{k=0}^{\infty} \frac{A_k}{(b + x)^{k+1}}\eta_{k+1}, \quad \sum_{k=0}^{\infty} \frac{B_k}{(b + k)^{k+1}}\eta_{k+1},$$

$$\sum_{k=0}^{\infty} \frac{A_{k-1}}{(b + x)^{k+1}}\delta_{k+1}, \quad \sum_{k=0}^{\infty} \frac{B_{k-1}}{(b + x)^{k+1}}\delta_{k+1}$$

*all converge absolutely to sums, bounded in absolute value by a function $B(r, r', |a|, \gamma)$ of only $r$, $r'$, $|a|$ and $\gamma$ and such that $B(r, r', |a|, \gamma) \to 0$ as $\gamma \to 0$.*

PROOF. From lemma 1 we have the following inequalities for the terms of the series:

$$\left|\frac{A_k}{(b + x)^{k+1}}\eta_{k+1}\right| \leqq \frac{1}{|b + x|}\frac{C}{r^k}\gamma r'^{k+1}$$

$$\leqq \frac{\gamma r'}{(r|a|)^{1/2}} C\left(\frac{r'}{r}\right)^k < \gamma C\left(\frac{r}{|a|}\right)^{1/2}\left(\frac{r'}{r}\right)^k, \; k \geqq 1,$$

and the same inequality for $k \geqq 0$ if $A_k$ is replace by $B_k$,

$$\left|\frac{A_{k-1}}{(b + x)^{k+1}}\delta_{k+1}\right| \leqq \frac{1}{|b + x|^2}\frac{C}{r^{k-1}}\gamma r'^{k+1}$$

$$\leqq \frac{r\gamma'^2}{r|a|} C\left(\frac{r'}{r}\right)^{k-1} < \gamma C\frac{r}{|a|}\left(\frac{r'}{r}\right)^{k-1}, \; k \geqq 2,$$

and the same inequality for $k \geqq 1$ if $A_k$ is replaced by $B_k$.

From this follows that all four series converge absolutely, and

$$\left|\sum_{k=0}^{\infty} \frac{A_k}{(b + x)^{k+1}}\eta_{k+1}\right| < \gamma C\left(\frac{r}{|a|}\right)^{1/2}\sum_{k=1}^{\infty}\left(\frac{r'}{r}\right)^k$$

$$= \gamma C\left(\frac{r}{|a|}\right)^{1/2}\frac{r'}{r - r'}$$

$$< \gamma C\left(\frac{r}{|a|}\right)^{1/2}\frac{r}{r - r'},$$

$$\left|\sum_{k=0}^{\infty} \frac{A_{k-1}}{(b+x)^{k+1}} \delta_{k+1}\right| < \frac{\gamma r'}{(|a|r)^{1/2}} + \gamma C \frac{r}{|a|} \sum_{k=2}^{\infty} \left(\frac{r'}{r}\right)^{k-1}$$
$$< \gamma \left(\frac{r}{|a|}\right)^{1/2} + \gamma C \frac{r}{|a|} \frac{r'}{r - r'}$$
$$< \gamma \left(\frac{r}{|a|}\right)^{1/2} + \gamma C \frac{r}{|a|} \frac{r}{r - r'},$$

$$\left|\sum_{k=0}^{\infty} \frac{B_k}{(b+x)^{k+1}} \eta_{k+1}\right| < \gamma C \left(\frac{r}{|a|}\right)^{1/2} \sum_{k=0}^{\infty} \left(\frac{r'}{r}\right)^{k} = \gamma C \left(\frac{r}{|a|}\right)^{1/2} \frac{r}{r - r'},$$

$$\left|\sum_{k=0}^{\infty} \frac{B_{k-1}}{(b+x)^{k+1}} \delta_{k+1}\right| < \gamma C \frac{r}{|a|} \sum_{k=1}^{\infty} \left(\frac{r'}{r}\right)^{k-1} = \gamma C \frac{r}{|a|} \frac{r}{r - r'}.$$

Since $C\, r/(r - r') > 1$, the following expression is a common upper bound for all four sums:

$$B(r, r', |a|, \gamma) = \gamma C \left(\frac{r}{|a|}\right)^{1/2} + \left(\frac{r}{|a|}\right) \frac{r}{r - r'}.$$

Since $\lim_{\gamma \to 0} C = \text{Max}(1, (|a|r)^{1/2}/(1 - r)$, we have $B(r, r', |a|, \gamma) \to 0$ as $\gamma \to 0$. Lemma 2 is thus proved.

From formula (2.4) and lemma 2 follows the next proposition.

PROPOSITION 1. *Let* $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ *be a limit periodic continued fraction with* $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, *where* $a$ *and* $b$ *are such that the fixed points* $x$ *and* $-(b + x)$ *of the linear fractional transformation* $s(w) = a/(b + w)$ *satisfy a condition*

$$r < \left|\frac{x}{b + x}\right| < 1/r \tag{3.1}$$

*for some positive* $r < 1$. *Then there is a* $\gamma_0 > 0$, *depending only on* $|a|$, $|x/(b + x)|$ *and* $r$, *such that the following holds for all positive* $\gamma < \gamma_0$ *and all positive* $r' < r$: *If*

$$|a - a_n| \leqq \gamma r'^n, \quad |b - b_n| \leqq \gamma r'^n \tag{3.4'}$$

*for all* $n$, *then the sequence* $\{S_n(x)\}$ *converges, possibly to* $\infty$. *For fixed values of* $a, b, r, r'$ *there is a positive* $H(\gamma)$, *with* $H(\gamma) \to 0$ *as* $\gamma \to 0$, *such that*

$$\left|\lim_{n\to\infty} S_n(x) - x\right| < H(\gamma).$$

REMARK 1. In proposition 1, as well as in Lemma 2, the $a_n$, $b_n$-conditions may, without any essential change in the proof, be weakened by replacing them by conditions of the type $|a - a_n| \leqq \gamma_n r^n$, $|b - b_n| \leqq \gamma_n r^n$, where $\sum_{n=1}^{\infty} \gamma_n$ converges and Max $\gamma_n$ is sufficiently small.

REMARK 2. The condition that $\gamma$ be sufficiently small is inessential as far as the tail of the continued fraction is concerned. If (3.1) holds for some positive $r < 1$ and (3.4′) holds for all $n$ for some positive $\gamma$, then $\gamma r'^m$ is "sufficiently small" from a certain $m = N$ on, even if $\gamma$ is not, and hence $\lim_{n\to\infty} S_n^{(m)}(x)$ converges. Since

$$\lim_{m\to\infty} (\lim_{n\to\infty} S_n^{(m)}(x)) = x,$$

it follows that for all sufficiently large $m$ the sequence $\{S_n^{(m)}\}_{n=m+1}^{\infty}$ converges to a finite number.

**4. Some words on analytic continuation.** If $a_n$ and $b_n$ are complex valued functions of certain variables such that the continued fraction is limit periodic when the variables are in certain given sets the question of uniform convergence of $\{S_n(x)\}$ arises. Since the argument above is based upon absolute convergence of series by comparison with geometric series, the question of uniform convergence will essentially be settled by establishing uniform bounds for the quantities occurring in the terms of the series. Rather than going into details in a general setting we shall indicate the flavor of the argument by using two examples.

EXAMPLE 4.1. As in Ex.1.3 we shall study the limit periodic general $T$-fraction

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{F_n z}{1 + G_n z},$$

where $F_n \to 1$, $G_n \to -1$ as $n \to \infty$. (The choice $F = 1$ is merely a matter of normalization, as long as $F \neq 0, \infty$.) Let, furthermore, for some fixed $R > 1$ and $\varepsilon > 0$, $|F_n - 1| \leqq \varepsilon/R^n$, $|G_n + 1| \leqq \varepsilon/R^n$. In this case we know from Ex.1.3 that for any $\rho > 1$ there is an $N_0$ such that any tail

$$\mathop{\mathrm{K}}_{n=N+1}^{\infty} \frac{F_n z}{1 + G_n z}$$

with $N \geqq N_0$ converges uniformly on $|z| < 1/\rho$ to a holomorphic function $f^{(N)}$, and from **[9]** that this function has a meromorphic extension to $|z| < R$. Furthermore the same tail converges uniformly on $|z| > \rho$ to a homomorphic function $g^{(N)}$ with a meromorphic extension to $|z| > 1/R$. We even know that the continued fraction itself (not only the tail) converges uniformly on compact sets of $|z| < 1$ minus poles to a meromorphic function and uniformly on compact sets of $|z| > 1$ minus poles to a meromorphic function, neither identically $\infty$.

Here we have $a(z) = z$, $b(z) = 1 - z$, $x(z) = z$, $-(x(z) + b(z)) = -1$. For $\rho > 1$ let $D_\rho$ denote the annulus $D_\rho = \{z | 1/\rho < |z| < \rho\}$. Fix an $R_1$ such that $1 < R_1 < R$ and choose an $R_2$ such that $R_1 < R_2 < R$. In the following we assume that $z \in D_{R_1}$. Then

$$1/R_2 < 1/R_1 < \left|\frac{x(z)}{x(z) + b(z)}\right| < R_1 < R_2.$$

In the use of the lemmas and the proposition $1/R_2 = r$ and $1/R = r'$. Hence (3.1) is satisfied in the whole annulus $D_{R_1}$, even in a stronger version, since $r < 1/R_1$ and $R_1 < 1/r$. Furthermore (3.4′) holds for any $n$ with $\gamma = \varepsilon R_1$. If $\varepsilon$ is sufficiently small (simple computation from (3.10) shows that any $\varepsilon$ with

$$\varepsilon < \frac{(R_2 - R_1)(R_2 - 1)}{R_1R_2 + (R_1R_2)^{1/2}} \tag{3.11}$$

works), then the conclusion of Lemma 1 holds, and, as is easily seen, with a $C$ depending only upon $R_1$ and $R_2$. The series in Lemma 2 are easily seen to converge uniformly in the annulus $D_{R_1}$, and since the partial sums are all polynomials the sums are holomorphic in the annulus. It is also easily seen that the absolute value of the sums have a common bound $B$, depending only upon $R_1$, $R_2$ and $\varepsilon$, and such that $B \to 0$ as $\varepsilon \to 0$. This implies that for sufficiently small $\varepsilon$ (an explicit value that works is also here easy to give), the denominator in (2.4) is bounded away from zero, and hence $\{S_n(x(z))\}$ converges uniformly on $D_{R_1}$ to a holomorphic function. In the part of the annulus lying in the open unit disk this coincides with $\lim S_n(0)$, and hence we have an analytic continuation. If $\varepsilon$ is not small enough to permit a direct use of Lemma 1, we can study a tail of the continued fraction, in which case the condition (3.11) is replaced by

$$\frac{\varepsilon}{R^N} < \frac{(R_2 - R_1)(R_2 - 1)}{R_1R_2 + (R_1R_2)^{1/2}}, \tag{3.11′}$$

which, regardless of $\varepsilon$, is true from a certain $N$ on (depending on $\varepsilon$).

Switching the roles of $x(z)$ and $-(b(z) + x(z))$ leads under similar conditions to uniform convergence of $S_n(-b(z) - x(z)) = S_n(-1)$ in the annulus and to analytic continuation of the function to which the continued fraction converges in $|z| > 1$ to the domain $|z| > 1/R_1$.

EXAMPLE 4.2. As in Ex.1.4 we shall study the limit periodic regular $C$-fraction $\mathrm{K}_{n=1}^{\infty}(\alpha_n z/1)$, where $\alpha_n \to \alpha \neq 0, \infty$ as $n \to \infty$. Without loss of generality we shall assume $\alpha = 1$ in the following. We know from Ex.1.4 the following convergence property: Given a sequence $\{D_k\}$ of bounded sub-domains of the plane cut along the ray $z = -t/4$, $t \geqq 1$, such that for all $k \geqq 1$, $\mathrm{cl}(D_k) \subset D_{k+1}$ and $\bigcup_{n=1}^{\infty} D_k =$ the cut plane. Then to any $k \geqq 1$ there is an $N_k$ such that for any $N \geqq N_k$ the tail $\mathrm{K}_{n=N+1}^{\infty}(\alpha_n z/1)$ converges uniformly on $D_k$ to a holomorphic function.

Here we have $a(z) = z$, $b(z) = 1$, and $x(z)$ and $-(x(z) + 1)$ are the two roots $(1/2)(-1 \pm (1 + 4z)^{1/2})$ of the quadratic equation $x^2 + x - z = 0$. We shall later choose which one we shall call $x(z)$ and which one we shall call $-(x(z) + 1)$.

Let, furthermore, for some fixed positive $\theta < 1$ and $\varepsilon > 0$, $|\alpha_n - 1| \leqq \varepsilon\theta^n$ for $n \geqq 1$. In the following we shall assume that $z$ is in the angular domain $D_\beta$, $\pi - \beta < \arg(z + 1/4) < \pi + \beta$, for some fixed positive $\beta < \pi$ to be decided later. Writer $z = -1/4 + (\rho^2/4)e^{i\phi}$, $\pi - \beta < \phi < \pi + \beta$, and choose the notation such that

$$x(z) = \frac{-1 + \rho^{i\phi/2}}{2}, \quad -(x(z) + 1) = -\frac{1 + \rho e^{i\phi/2}}{2}.$$

In $D_\beta$ we have

$$\tan\frac{\pi - \beta}{4} < \left|\frac{x(z)}{x(z) + 1}\right| < \tan\frac{\pi + \beta}{4}.$$

If $\beta$ is such that $\theta_0 = \tan(\pi - \beta/4) > \theta$, i.e., $0 < \beta < \pi - 4\arctan\theta$, then we have in $D_\beta$ that

$$\theta < \theta_0 < \left|\frac{x(z)}{x(z) + 1}\right| < 1/\theta_0 < 1/\theta.$$

Let, for arbitrary fixed $M > 0$, $D_\beta^{(M)}$ denote the intersection of $D_\beta$ and the disk $|z| < M$. Using the two lemmas and Proposition 1 it is easily proved that at least a tail of the continued fraction is such that the sequence of $x(z)$ – modified approximants converges uniformly on $D_\beta^{(M)}$ to a holomorphic function. In the lower part of the angular opening, $\pi < \arg(z + 1/4) < \pi + \beta$, the $x(z)$ – modification represents the "wrong" modification, and in the upper part, $\pi < \arg(z + 1/4) < \pi$, it represents the "right" modification and the limit coincides there with the value of the continued fraction (or the tail). Hence this method leads to analytic continuation "from above" across the slit on the negative real axis. By using the same angular domain, but describing it as

$$-\pi - \beta < \arg(z + 1/4) < -\pi + \beta,$$

we can quite similarly obtain an analytic continuation "from below". In this case the $x(z)$-modification is the "right" modification in the lower part of the angular opening $-\pi < \arg(z + 1/4) < -\pi + \beta$ and the "wrong" modification in the upper part $-\pi - \beta < \arg(z + 1/4) < -\pi$.

In the two examples above details are left out, since the results are contained in [7] and the purpose of the examples in the present paper is merely to indicate how those results can be obtained in a different way.

**5. Final remarks.** As mentioned in the introduction Gill suggested in [1] the use of the repulsive fixed point in order to obtain analytic continuation. The main result in the paper [1] is a theorem related to our Proposition 1, but of a much more general nature. First, it has to do with compositions $T_n$ of more general nonparabolic linear fractional transformations

$$t_n(w) = \frac{a_n + c_n w}{b_n + d_n w}, \; a_n d_n - b_n c_n \neq 0$$

with $a_n \to a$, $b_n \to b$, $c_n \to c$, $d_n \to d$ as $n \to \infty$, with finite fixed points $\alpha_n \to \alpha$ (attractive) and $\beta_n \to \beta$ (repulsive) rather than the more special ones $s_n(w)$ (with $c_n = 0$, $d_n = 1$ in $t_n(w)$) used for generating continued fractions. Second, he studies sequences $\{T_n(\mu_n)\}$, where $\mu_n - \beta_n \to 0$ (at a rate described in the theorem) rather than the sequence $\{T_n(\beta)\}$.

Since Gill's method differs from our methods in [7] as well as in the present paper, the two latter represent at least new presentations of results on the wrong modification. Whether or not they are more than that depends essentially upon whether or not our Proposition 1 follows from Gill's Theorem 2.

A complete comparison of Gill's result (in the case $c_n = 0$, $d_n = 1$, $\mu_n = \beta$) with our result is beyond the scope of the present paper. We shall restrict ourselves to a few remarks without going into details. Gill's result is expressed in terms of the fixed points, and in the case $|\alpha/\beta| < 1$ products $\Pi_{k=1}^{n}|\alpha_k/\beta_k|$ are contained in his conditions in a manner very related to the way the ratio $|\alpha/\beta|^n$ is part of our conditions. However, since in his result the strong condition on convergence (geometric convergence) only is required for the sequence $\{\beta_n\}$, whereas $\alpha_n$ may tend to $\alpha$ more slowly, there exist, by virtue of the equalities $\alpha_n + \beta_n = -b_n$, $\alpha_n\beta_n = -a_n$, continued fractions, where Gill's Theorem 2 can be used, but not our Proposition 1. Gill presents such an example in his paper, Example 2. On the other hand, in the case $|\alpha/\beta| = 1$ Gill's theorem does not permit too fast convergence of $a_n$ to $a$ and $b_n$ to $b$ simultaneously. For instance, neither the case of a periodic continued fraction nor the cases of geometric convergence of $a_n$ to $a$ and $b_n$ to $b$ are covered by Gill's theorem when $|\alpha/\beta| = 1$. Hence the results on analytic continuation of functions defined by $C$- or $T$-fractions as treated in [7] or in the Examples 4.1 and 4.2 of the present paper can not be proved by using merely Gill's theorem, since in these cases it can not "help us across" the border line where the attractive and repulsive fixed points switch roles.

In conclusion we, therefore, may say that there are cases covered by Gill's result and not by our results and vice versa.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, CO 80309

MATEMATISK INSTITUTT, UNIVERSITETET I TRONDHEIM, NLHT 7000 TRONDHEIM, NORWAY