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A SIMPLE PROOF AND GENERALIZATION OF WEGLORZ' CHARACTERIZATION OF NORMALITY FOR IDEALS

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ABSTRACT. A condition equivalent to normality for κ-complete ideals on a regular uncountable cardinal κ has been established by B. Weglorz as a corollary to his study of Ramsey and pseudonormal ideals. By isolating a critical combinatorial property (see Lemma 3) we are able to provide a direct, elementary proof of this equivalence and to generalize the result to arbitrary non-principal ideals.

1. Notation and definitions. Our notation is that used in Baumgartner, Taylor, Wagon [1]. If κ is a regular uncountable cardinal, an *ideal on* κ is a collection I of subsets of κ such that whenever $X, Y \in I$ and $Z \subseteqq X \cup Y$, then $Z \in I$. I is called *non-principal* if I contains all the singleton subsets. I is called *proper* if $\kappa \notin I$. I is *κ-complete* if whenever $\beta < \kappa$ and $\{X_\alpha | \alpha < \beta\} \subseteqq I$, then $\bigcup_{\alpha<\beta} X_\alpha \in I$. An important ideal on κ is the generalized Fréchet ideal, $I_\kappa = \{X \subseteqq \kappa | \, |X| < \kappa\}$. Note that if I is a non-principal, κ-complete ideal on κ, then $I_\kappa \subseteqq I$. However, we do not wish to restrict our attention in this paper to κ-complete ideals; the phrase "I is an (arbitrary) ideal on κ" will simply mean "I is a proper, non-principal ideal on κ".

If I is an ideal on κ, then $I^+ = \{X \subseteqq \kappa \mid X \notin I\}$ and $I^* = \{X \subseteqq \kappa \mid \kappa - X \in I\}$. Sets in I are said to be of "I-measure zero", sets in I^+ are said to be of "positive I-measure", and sets in I^* are said to be of "I-measure one."

If I is an ideal on κ and $A \in I^+$, then the *restriction of I to A*, denoted by $I|A$, is the ideal on κ given by $I|A = \{X \subseteqq \kappa \mid X \cap A \in I\}$.

If I is an ideal on κ and $A \subseteqq \kappa$ and $f: A \to \kappa$ is a function, f is called *I-small* if and only if for every $\alpha < \kappa, f^{-1}(\{\alpha\}) \in I$; f is called *regressive on* A if and only if for every $\alpha \in A - \{0\}, f(\alpha) < \alpha$.

If $\{X_\alpha | \alpha < \kappa\}$ is a sequence of κ-many subsets of κ, then the *diagonal union* of the sequence, denoted by $\nabla\{X_\alpha | \alpha < \kappa\}$ or by $\nabla_{\alpha<\kappa} X_\alpha$, is defined to be $\{\beta < \kappa \mid \exists \alpha < \beta, \beta \in X_\alpha\} = \bigcup\{X_\alpha - (\alpha + 1) \mid \alpha < \kappa\}$.

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2. Statement of results. The following three properties of an arbitrary ideal I on κ will be considered:

(i) if $\{X_\alpha | \alpha < \kappa\} \subseteq I$, then $\nabla\{X_\alpha | \alpha < \kappa\} \in I$. This condition asserts that I is closed under diagonal unions. An ideal with this property will be called *normal*.

(ii) if $f: \kappa \to \kappa$ is an I-small function on κ, then $\{\inf(f^{-1}(\{\alpha\})) | \alpha < \kappa\} \in I^*$. We will refer to this property as *Weglorz' condition.*

(iii) if $A \in I^+$ and $f: A \to \kappa$ is I-small, then $\{\inf(f^{-1}(\{\alpha\})) | \alpha < \kappa\} \in (I|A)^*$.

THEOREM 1. (Weglorz [2]). *Property* (ii) *implies property* (i) *if I is a κ-complete ideal on κ.*

Weglorz' proof of this consists of Theorem 3.11 together with the relevant parts of Theorems 2.5 and 3.5 from [2]. In fact, the actual result proved in [2] is a bit stronger.

Below, we will give proofs of the following theorems.

THEOREM 2. *Property* (iii) *implies property* (i) *if I is an arbitrary ideal on κ.*

THEOREM 3. *Property* (ii) *implies property* (iii) *if I is an arbitrary ideal on κ.*

Note that Theorem 1 is then implied by the conjunction of Theorems 2 and 3.

To be thorough, we remind the reader of the following theorem.

THEOREM 4. *Property* (i) *implies property* (ii) *if I is an arbitrary ideal.*

Before proceeding with the proofs, we wish to stress once again that the conjunction of Theorems 2, 3, and 4 establishs the equivalence of the three properties for arbitrary ideals. This results in a more general notion of normality than the usual one in which κ-completeness is assumed as well. Of course, if I is normal and $I_\kappa \subseteq I$, then I is κ-complete. But there are some interesting normal ideals which do not extend the generalized Fréchet ideal.

EXAMPLE. Let NS_κ denote the ideal of non-stationary subsets of κ and consider the ideal on $\aleph_2$ defined by $I = \{X \subseteq \aleph_2 | X \in NS_{\aleph_2}$ and $X \cap \aleph_1 \in NS_{\aleph_1}\}$. I is a non-principal countably complete normal ideal on $\aleph_2$. To verify that I is closed under diagonal unions, one needs only to invoke Fodor's result that for regular, uncountable κ, NS_κ is normal, together with the simple fact that

$$\left(\nabla_{\alpha<\aleph_2} X_\alpha\right) \cap \aleph_1 = \nabla_{\alpha<\aleph_1} (X_\alpha \cap \aleph_1).$$

This example demands reconciliation with the familiar fact that "NS_κ is the minimal normal ideal on κ". This reconciliation is achieved by recalling the correct statement of the fact in quotes: NS_κ is the minimal normal ideal on κ extending I_κ. (See Proposition (iii), Chapter *I*, §2.1 of [1]).

We should also point out that reference [1], Chapter IV, Theorem 4.10(a) contains a different proof of Weglorz' Theorem above. This proof is a bit complex, however, since it simultaneously extablishes several related results.

3. Proofs. Our proof of Theorem 2 is a modification of Weglorz' proof. We begin by stating two easy facts in the form of lemmas so that they may be referred to subsequently.

LEMMA 1. *If I is an arbitrary ideal on κ, then I is normal if and only if for every $A \in I^+$ and for every f: $A \to \kappa$ which is regressive on A, there exists an $\alpha < \kappa$ such that $f^{-1}(\{\alpha\}) \in I^+$.*

LEMMA 2. *If $A \in I^+$, then $X \in (I|A)^*$ if and only if $X \cup (\kappa - A) \in I^*$.*

The heart of the matter is the following lemma.

LEMMA 3. *Let I be an arbitrary ideal, let $B \in I^+$ and let g: $B \to \kappa$ be regressive on B. Then there is a subset $C \subseteqq B$ with $C \in I^+$ such that $g|C$: $C \to \kappa - C$.*

In words, Lemma 3 asserts for arbitrary ideals I that if a function is regressive on a set B of positive I-measure, then some restriction of that function to a set C of positive I-measure maps into the complement of C.

PROOF OF LEMMA 3. Suppose g: $B \to \kappa$ is regressive and that $B \in I^+$. For each $\alpha \in B$, define n_α to be the least integer n such that $g^n(\alpha) \notin B$. [Here, g^n is the n-fold iterate of g.] Note that such an n must exist because g is regressive on B. Let $C_n = \{\alpha \in B | n_\alpha = n\}$, let $D = \bigcup_{n \in \omega} C_{2n+1}$, and let $E = \bigcup_{n \in \omega} C_{2n}$. Observe that $B = \bigcup_{n \in \omega} C_n$, that if $\alpha \in C_n$, then $g(\alpha) \in C_{n-1}$, and that B is the disjoint union of D and E; consequently, $g|D$: $D \to \kappa - D$ and $g|E$: $E \to \kappa - E$ and the proof is complete since at least one of D or E must have positive I-measure.

PROOF OF THEOREM 2. We will derive a contradiction from the assumption that an arbitrary ideal I has property (iii) but is not normal. If I is not normal, then by Lemma 1, there is an $A \in I^+$ and an f: $A \to \kappa$ which is regressive and I-small. We may conclude from (iii) that $B = \{\inf(f^{-1}(\{\alpha\}))| \alpha < \kappa\} \in (I|A)^*$. Thus $B \in I^+$ and we may apply Lemma 3 to the function $g = f|B$ to conclude the existence of a subset $C \subseteqq B$ with $C \in I^+$ such that $f|C$: $C \to \kappa - C$. Now let us consider the function $h =$

$f|C \cup \text{id}|$ (ran $f|C$). [Here, id is the identity function.] For example, if $b_\alpha = \inf(f^{-1}(\{\alpha\})) \in C$, then $\langle b_\alpha, \alpha\rangle \in f|C$ and so both $\langle b_\alpha, \alpha\rangle$ and $\langle \alpha, \alpha\rangle$ belong to h. Letting $D = \text{ran} f|C$ and noting that $D \subseteq \kappa - C$, we observe that dom h, which is the disjoint union of C and D, has positive I-measure and that h is still I-small; so we may invoke property (iii) once again to conclude that $Z = \{\inf(h^{-1}(\{\alpha\}))| \alpha < \kappa\} \in (I|C \cup D)^*$. Consequently, by Lemma 2, $Z \cup (\kappa - (C \cup D)) = (Z \cup (\kappa - C)) \cap (Z \cup (\kappa - D)) \in I^*$; in particular, $Z \cup (\kappa - C) \in I^*$. Note however that because f is regressive on C, $\inf(h^{-1}(\{\alpha\}))$ is equal either to 0 by convention if $\alpha \notin D$ or to α itself if $\alpha \in D$. This implies that $Z \subseteq \kappa - C \cup \{0\}$, and hence that $\kappa - C \in I^*$, contradicting the fact that $C \in I^+$.

PROOF OF THEOREM 3. Suppose $A \in I^+$ and f: $A \to \kappa$ is I-small. Clearly we may assume $|\kappa\text{-Range}(f)| = \kappa$. Let h map $\kappa - A$ one-to-one into κ-Range(f) and let $g = f \cup h$. Then by property (ii), $\{\inf(g^{-1}(\{\alpha\})\,|\,\alpha < \kappa\} \in I^*$. Therefore $\{\inf(f^{-1}(\{\alpha\}))|\alpha < \kappa\} = A \cap \inf(g^{-1}(\{\alpha\}))|\alpha < \kappa\} \in (I|A)^*$.

PROOF OF THEOREM 4. Theorem 4 is well-known for κ-complete ideals and it is easy to verify that the standard proof (for example, see Baumgartner, Taylor, Wagon [1], Chapter I, Theorem 3.9 (a)) actually applies to arbitrary ideals. In fact, given an I-small f: $\kappa \to \kappa$, to show that $X = \{\inf f^{-1}(\{\alpha\})\,|\,\alpha < \kappa\} \in I^*$, we observe that the map defined on $\kappa - X$ which sends each element of $f^{-1}(\{\alpha\})$ to $\inf f^{-1}(\{\alpha\})$ is I-small and regressive; thus $\kappa - X \in \nabla I$, and the normality of I then yields that $\kappa - X \in I$.

4. Observation. Let $\psi(I)$ and $\chi(I)$ be properties of ideals. It frequently happens (e.g., with certain other characterizations of normality such as that in Lemma 1) that the equivalence, $\psi(I)$ if and only if $\chi(I)$, has a straightforward proof in the sense that any witness to $\psi(I)$ is already *itself* (or yields in an obvious way) a witness to $\chi(I)$, and vice versa. The following example rules out the possibility of a straightforward proof of Weglorz' Theorem in this sense.

Let κ be a measurable cardinal and let I arise from a non-normal measure on κ such that $\{\alpha < \kappa | \alpha \text{ is a successor}\} \in I^*$; then the function

$$f(\alpha) = \begin{cases} \alpha - 1 & \text{if } \alpha \text{ is a successor} \\ 0 & \text{if } \alpha = 0 \text{ or } \alpha \text{ is a limit ordinal} \end{cases}$$

is an I-small, regressive function, i.e., a witness to the non-normality of I by Lemma 1. But f itself is not a witness to the failure of Weglorz' condition since

$$\{\inf(f^{-1}(\{\alpha\}))\,|\,\alpha < \kappa\} = \{\alpha < \kappa|\alpha \text{ is a successor}\} \cup \{0\} \in I^*.$$

REFERENCES

1. J. Baumgartner, A. Taylor, and S. Wagon, *Structural properties of ideals*, to appear in Dissertationes Mathematicae.

2. B. Weglorz, *Some properties of filters*, Lecture Notes in Mathematics **619**, Springer-Verlag, Berlin, 1977, 311–328.

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