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α-CLOSURE IN FUZZY TOPOLOGY

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ABSTRACT. Let X be an L-fuzzy topological space, let $\alpha \in L$, and let A be a crisp subset of X. The α-closure of A is the set of points x for which $G(x) > \alpha$ implies $G(a) \neq 0$ for some $a \in A$ whenever G is fuzzy open. With appropriate restrictions on α (which always are satisfied if L is a chain), α-closure is a semi-closure operator but may not be a closure operator. Relations between α-closure and recently introduced α-level properties are studied and a characterization of α-closure in the fuzzy unit interval is obtained. The non-suitability of the fuzzy unit interval and fuzzy open unit interval follows as a simple corollary.

Introduction. Recently Gantner et al. [2] and Rodabaugh [4, 5] have studied L-fuzzy topological spaces by considering properties which a space may have to a certain degree or at a certain α-level, where α is a member of the underlying lattice. As part of this approach in [5], the concept of α-closure was introduced. It is the purpose of this paper to study α-closure in more detail as a closure operator, to examine its relations with other α-level properties, and to characterize it in Hutton's fuzzy unit interval [3].

Throughout this paper L will denote a completely distributive lattice with 0, 1 ($0 \neq 1$) and with an order-reversing involution $\alpha \to \alpha'$. As in [2], $L^c = \{\alpha \in L$: α is comparable to each $\beta \in L\}$ and $L^a = \{\alpha \in L^c$: if $\beta > \alpha$ and $\gamma > \alpha$, then $\beta \wedge \gamma > \alpha\}$.

1. α-Closure as a semi-closure operator. Let (X, T) be an L-fuzzy topological space (L-*fts*). The following definition can easily be shown equivalent to the definition in [5].

DEFINITION 1.1. Let $\alpha \in L$-$\{1\}$ and let A be a crisp subset of X. $c_\alpha(A) = \{x$: if $G \in T$ and $G(x) > \alpha$, then $G \wedge \chi_A \neq 0\}$.

Clearly $c_\alpha(\emptyset) = \emptyset$ and $A \subseteqq c_\alpha(A)$ for every A. With a restriction on α one obtains the following lemma.

LEMMA 1.2. *Let* $\alpha \in L^a - \{1\}$ *and let* $A, B \subseteqq X$. *Then* $c_\alpha(A \cup B) = c_\alpha(A) \cup c_\alpha(B)$.

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The proof is routine with the L^a hypothesis needed only for the inclusion $c_\alpha(A \cup B) \subseteq c_\alpha(A) \cup c_\alpha(B)$.

Thus, in the terminology of Čech [1], c_α is a semi-closure operator on X provided $\alpha \in L^a - \{1\}$. It is easy to construct simple examples to show that, for $\alpha \neq 0$ in $L^a - \{1\}$, c_α need not be a closure operator, i.e., that $c_\alpha(A)$ may be a proper subset of $c_\alpha(c_\alpha(A))$ for some subsets A. Such examples appear later in connection with the fuzzy unit interval.

One can obtain a partial result by using α-compactness (defined in [2]) and the α-Hausdorff property (defined in [5]).

LEMMA 1.3. *Let $\alpha \in L - \{1\}$ and let $A \subseteq X$. If (X, T) is α-compact, then $c_\alpha(A)$ is α-compact.*

PROOF. Let $\mathscr{G}$ be an α-shading of $c_\alpha(A)$. For each $y \in X - c_\alpha(A)$ there $G_y \in T$ with $G_y(y) > \alpha$ and $G_y \wedge \chi_A = 0$. Then $\mathscr{G} \cup \{G_y: y \in X - c_\alpha(A)\}$ is an α-shading of X and so has a finite subshading $\mathscr{F}$. For $x \in c_\alpha(A)$ and $F \in \mathscr{F}$ with $F(x) > \alpha$, since $F \wedge \chi_A \neq 0$, $F \notin \{G_y: y \in X - c_\alpha(A)\}$. Thus $\mathscr{F} \cap \mathscr{G}$ is a finite subshading of $\mathscr{G}$.

LEMMA 1.4. *Let $\alpha, 0 \in L^a - \{1\}$ and let $A \subseteq X$. If (X, T) is α-Hausdorff and A is α-compact, then $c_\alpha(A) = A$.*

PROOF. Let $x \in c_\alpha(A)$. If $x \notin A$, then for each $a \in A$ there exist $U_a, V_a \in T$ such that $U_a(x) > \alpha$, $V_a(a) > \alpha$, and $U_a \wedge V_a = 0$ by α-Hausdorff. By α-compactness there is a finite subshading $\{V_a: a \in \Delta\}$. For $U = \wedge \{U_a: a \in \Delta\}$, since $\alpha \in L^a$, $U(x) > \alpha$. Thus there is $t \in A$ with $U(t) > 0$. But for some $a \in \Delta$, $V_a(t) > \alpha$ and so, since $0 \in L^a$, $U_a \wedge V_a(t) > 0$, a contradiction.

THEOREM 1.5. *Let $\alpha, 0 \in L^a - \{1\}$. If (X, T) is α-Hausdorff and α-compact, then c_α is a closure operator.*

2. Relations to the α-property and suitability. Čech [1] has shown that for any semi-closure operator k the set of k-fixed subsets is the set of closed subsets of a topology; moreover $k(A) \subseteq \mathrm{Cl}(A)$ (the closure of A in this topology) with equality for every A if and only if k is a closure operator. Thus, in considering an L-fts at the α-level where $\alpha \in L^a - \{1\}$, one can consider the topology generated by c_α. Throughout this section let (X, T) denote an L-fts and let W_α denote the topology generated by c_α. There is also another natural α-level topology.

DEFINITION 2.1. Let $\alpha \in L - \{1\}$ and let $G \in T$. $\alpha(G) = \{x: G(x) > \alpha\}$.

LEMMA 2.2. *Let $\alpha \in L^a - \{1\}$. Then $\{\alpha(G): G \in T\}$ is a topology for X.*

For $\alpha \in L^a - \{1\}$ let T_α denote $\{\alpha(G): G \in T\}$. It is natural to ask whether W_α and T_α are related. First recall from [5] the definition of the α-property.

DEFINITION 2.3. Let $\alpha \in L - \{1\}$. (X, T) has the α-property provided, for $A \subseteqq X$, $c_\alpha(A) = A$ if and only if there is $U \in T$ with $A = \{x: U(x) \leqq \alpha\}$.

THEOREM 2.4. *Let* $\alpha \in L^a - \{1\}$. *Then*

(i) $W_\alpha \subseteqq T_\alpha$

and

(ii) $W_\alpha = T_\alpha$ *if and only if* (X, T) *has the* α-*property.*

PROOF. For i), given $U \in W_\alpha$, $c_\alpha(X - U) = X - U$. Let $x \in U$. Then there is $G \in T$ with $G(x) > \alpha$ and $G \wedge \chi_{X-U} = 0$. Clearly, $\alpha(G) \subseteqq U$. For ii), note that the definition of the α-property simply identifies the W_α-closed sets and the T_α-closed sets.

THEOREM 2.5. *Let* $\alpha \in L^a - \{1\}$. *If* (X, T) *has the* α-*property, then* c_α *is a closure operator.*

PROOF. Let $A \subseteqq X$ and let $x \notin c_\alpha(A)$. Then there is $G \in T$ such that $G(x) > \alpha$ and $G \wedge \chi_A = 0$. Since (X, T) has the α-property, $c_\alpha(\{y: G(y) \leqq \alpha\}) = \{y: G(y) \leqq \alpha\}$. Then $c_\alpha(c_\alpha(A)) \subseteqq \{y: G(y) \leqq \alpha\}$ and so $x \notin c_\alpha(c_\alpha(A))$.

Examples in the fuzzy unit interval will show that the converse of 2.5 is false. However, with additional hypotheses, one can obtain partial results.

THEOREM 2.6. *Let* α, $0 \in L^a - \{1\}$. *If* (X, T) *is* α-*Hausdorff,* T_α *is minimal Hausdorff and* c_α *is a closure operator, then* (X, T) *has the* α-*property.*

PROOF. Let $x \neq y$ and let U, $V \in T$ with $U(x) > \alpha$, $V(y) > \alpha$ and $U \wedge V = 0$. Let $A = \{t: U(t) = 0\}$ and $B = \{t: V(t) = 0\}$. Since $0 \in L^a$, $A \cup B = X$. Thus $X - c_\alpha(A)$, $X - c_\alpha(B)$ are disjoint, W_α-open subsets with $x \in X - c_\alpha(A)$ and $y \in X - c_\alpha(B)$. Then (X, W_α) is Hausdorff. Since (X, T_α) is minimal Hausdorff, $W_\alpha = T_\alpha$ and so (X, T) has the α-property.

In [5] Rodabaugh gives a direct proof of the following corollary, which is immediate from 1.5 and 2.6.

COROLLARY 2.7. *Let* 0, $\alpha \in L^a - \{1\}$. *If* (X, T) *is* α-*compact and* α-*Hausdorff, then* (X, T) *has the* α-*property.*

Suitable closed subsets of an L-fts were introduced in [4] as proper, crisp, fuzzy-closed subsets and were studied there in connection with fuzzy extension theorems. Suitable spaces are those which contain a suitable closed subset. The following result has an interesting application in the fuzzy unit interval.

THEOREM 2.8. *Let* $\alpha \in L - \{1\}$ *and let* $A \subseteqq X$. *If* A *is suitable closed, then* $c_\alpha(A) = A$.

PROOF. For $x \notin A$ and $G = (\chi_A)'$, $G(x) > \alpha$ and $G \wedge \chi_A = 0$. Since A is suitable closed, $G \in T$ and $x \notin c_\alpha(A)$.

3. α-Closure in the fuzzy unit interval. The first two lemmas for a general space will be used implicitly in much of what follows. The concept of an L-fuzzy subspace is defined in **[6]**. Both proofs are routine.

LEMMA 3.1. *Let* (X, T) *be an L-fts and* $\alpha \in L - \{1\}$. *Let* $A \subseteqq X$ *and let* c_α^A *denote the* α*-closure in the L-fuzzy subspace* A. *Then, for* $B \subseteqq A$, $c_\alpha^A(B) = A \cap c_\alpha(B)$.

LEMMA 3.2. *Let* (X, T) *be an L-fts and let* $\alpha \in L^c - \{1\}$. *Let* $\mathscr{B}$ *be a base for* T *and let* $A \subseteqq X$. *Then* $c_\alpha(A) = \{x$: *if* $B \in \mathscr{B}$ *and* $B(x) > \alpha$, *then* $B \wedge \chi_A \neq 0\}$.

Throughout this section the notation of **[2]** will be used for $I(L)$ and (0, 1) (L). It is easy to verify that the closed intervals in the next lemma do not depend on the choice of representative from the equivalence class.

LEMMA 3.3. *Let* $\alpha \in L^c - \{1\}$, *let* $\lambda \in I(L)$, *and let* $s \in R$. *Then*
i) $R_s(\lambda) > \alpha$ *if and only if there is* $\delta > 0$ *with* $s + \delta \in \mathrm{Cl}\{x: \lambda(x) > \alpha\}$.
ii) $L_s(\lambda) > \alpha$ *if and only if there is* $\delta > 0$ *with* $s - \delta \in \mathrm{Cl}\{x: \lambda(x) < \alpha'\}$

PROOF. $\bigvee_{x>s} \lambda(x) > \alpha$ if and only if $\lambda(t) > \alpha$ for some $t > s$ and so i) holds. $(\bigwedge_{x<s}\lambda(x))' > \alpha$ if and only if $\bigwedge_{x<s} \lambda(x) < \alpha'$, which holds if and only if $\lambda(t) < \alpha'$ for some $t < s$ and so ii) holds.

DEFINITION 3.4. Let $\alpha \in L^c - \{1\}$ and let $\lambda \in I(L)$.

$$H_\alpha(\lambda) = \begin{cases} \mathrm{Cl}\{x:\ \lambda(x) < \alpha'\} \cap \mathrm{Cl}\{x:\ \lambda(x) > \alpha\} & \text{if } \alpha < \alpha' \\ \mathrm{Cl}(R - \{x: \lambda(x) < \alpha'\}) \cap \mathrm{Cl}(R - \{x: \lambda(x) > \alpha\}) & \text{if } \alpha \geqq \alpha' \end{cases}$$

LEMMA 3.5. *Let* $\alpha \in L^c - \{1\}$ *and let* $\lambda \in I(\lambda)$. *Then* $H_\alpha(\lambda)$ *is a non-smpty closed subinterval of* [0, 1]

PROOF. Suppose $\alpha \geqq \alpha'$. *Let* $\mathrm{Cl}\{x:\ \lambda(x) < \alpha'\} = [b, \infty]$ and $\mathrm{Cl}\{x:\ \lambda(x) > \alpha\} = (-\infty, a]$. If $b < a$, then for $b < y < a$, $\lambda(y) < \alpha'$ and $\lambda(y) > \alpha$ which contradicts $\alpha \geqq \alpha'$. Thus $a \leqq b$ and $H_\alpha(\lambda) = [a, b]$. If $b > 1$, then there is $x > 1$ with $\lambda(x) \geqq \alpha' > 0$. Thus $b \leqq 1$. Similarly $a \geqq 0$. The case $\alpha < \alpha'$ is similar.

It is worth noting that the endpoints of $H_\alpha(\lambda)$ are the numbers $a(\lambda, \alpha)$ and $b(\lambda, \alpha)$ which were used extensively in [4] and [5].

LEMMA 3.6. *Let $\alpha \in L^a - \{1\}$ with $\alpha \geqq \alpha'$, let $s, t \in R$, and let $\lambda \in I(L)$. Then $R_s \wedge L_t(\lambda) > \alpha$ if and only if $s < t$ and $H_\alpha(\lambda) \subseteqq (s, t)$.*

PROOF. Suppose $R_s \wedge L_t(\lambda) > \alpha$. By 3.3 there is $\delta > 0$ with $(-\infty, s + \delta) \subseteqq \mathrm{Cl}\{x: \lambda(x) > \alpha\}$ and $(t - \delta, \infty) \subseteqq \mathrm{Cl}\{x: \lambda(x) < \alpha'\}$. Then $H_\alpha(\lambda) \subseteqq [s + \delta, \infty) \cap (-\infty, t - \delta]$ and so, for $x \in H_\alpha(\lambda)$, $x > s$ and $x < t$. If $s \geqq t$, then $\lambda(s) \leqq \lambda(t)$. From above $\lambda(s) > \alpha$ and $\lambda(t) < \alpha'$. With $\alpha \geqq \alpha'$, $\lambda(s) > \lambda(t)$, a contradiction. To see the sufficiency of the condition, let $H_\alpha(\lambda) = [a, b]$ where $\mathrm{Cl}\{x: \lambda(x) > \alpha\} = (-\infty, a]$ and $\mathrm{Cl}\{x: \lambda(x) < \alpha'\} = [b, \infty)$. Since $s < a$ and $b < t$, by 3.3, $R_s(\lambda) > \alpha$ and $L_t(\lambda) > \alpha$. Since $\alpha \in L^a$, $R_s \wedge L_t(\lambda) > \alpha$.

LEMMA 3.7. *Let $\alpha \in L^a - \{1\}$ with $\alpha < \alpha'$. Let $s, t \in R$ and let $\lambda \in I(L)$. Then*

i) *if $s < t$, $R_s \wedge L_t(\lambda) > \alpha$ if and only if $H_\alpha(\lambda) \cap (s, t) \neq \emptyset$, and*

(ii) *if $s \geqq t$, $R_s \wedge L_t(\lambda) > \alpha$ if and only if $[t, s] \subseteqq \mathrm{Int}\, H_\alpha(\lambda)$.*

PROOF. Let $H_\alpha(\lambda) = [a, b]$ where $\mathrm{Cl}\{x: \lambda(x) > \alpha\} = (-\infty, b]$ and $\mathrm{Cl}\{x: \lambda(x) < \alpha'\} = [a, \infty)$. Since $\alpha \in L^a$ and 3.3 applies, $R_s \wedge L_t(\lambda) > \alpha$ if and only if $s < b$ and $a < t$. Then i) and ii) are immediate.

Note that the necessity of the conditions in 3.6 and 3.7 requires only $\alpha \in L^c - \{1\}$.

THEOREM 3.8. *Let α, $0 \in L^a - \{1\}$ with $\alpha \geqq \alpha'$. Let $A \subseteqq I(L)$. Then $\lambda \in c_\alpha(A)$ if and only if $H_\alpha(\lambda) \cap \mathrm{Cl}(\bigcup\{H_0(\sigma): \sigma \in A\}) \neq \emptyset$.*

PROOF. $\lambda \notin c_\alpha(A)$ if and only if there exist $s, t \in R$ such that $R_s \wedge L_t(\lambda) > \alpha$ and $R_s \wedge L_t(\sigma) = 0$ for every $\sigma \in A$. By 3.6 and 3.7 i), $\lambda \notin c_\alpha(A)$ if and only if there exist $s < t$ in R with $H_\alpha(\lambda) \subseteqq (s, t)$ and $H_0(\sigma) \cap (s, t) = \emptyset$ for every $\sigma \in A$, i.e., $\lambda \notin c_\alpha(A)$ if and only if $H_\alpha(\lambda) \cap \mathrm{Cl}(\bigcup\{H_0(\sigma): \sigma \in A\}) = \emptyset$.

COROLLARY 3.9. *Let α, $0 \in L^a - \{1\}$ with $\alpha \geqq \alpha'$.*

i) *If $\sigma \in I(L)$ is such that $0 < \sigma(x) < 1$ for all $x \in (0, 1)$, then $c_\alpha(\{\sigma\}) = I(L)$.*

ii) *If $\sigma \in I(L)$ is such that $\alpha \geqq \sigma(x) \geqq \alpha'$ for all $x \in (0, 1)$, then for every non-empty $A \subseteqq I(L)$, $\sigma \in c_\alpha(A)$.*

PROOF. In i) $H_0(\sigma) = [0, 1]$ and in ii) $H_\alpha(\sigma) = [0, 1]$.

The second and third parts of the following corollary are obtained in [5] and [4] by different methods.

COROLLARY 3.10. *Let* $\alpha, 0 \in L^a - \{1\}$ *with* $\alpha \geqq \alpha'$. *Then*
i) c_α *is not a closure operator*;
ii) $I(L)$ *does not have the* α-*property*; *and*
iii) $I(L)$ *is not suitable.*

PROOF. Let θ, λ be the canonical images of 1/4, 1/2 respectively in $I(L)$. $H_\alpha(\theta) = \{1/4\}$ and $H_0(\lambda) = \{1/2\}$. By 3.8, $\theta \notin c_\alpha(\{\lambda\})$ and by 3.9, with $\sigma(x) = \alpha$ for $x \in (0, 1)$, $c_\alpha(c_\alpha(\{\lambda\})) = I(L)$. Part ii) now follows from 2.5. Lastly by 3.9 no proper subset of $I(L)$ is α-closed and so iii) follows from 2.8.

With a slight modification of these methods one obtains analogous results for $(0, 1)(L)$. In 3.12 and 3.13, c_α refers to the α-closure in the subspace $(0, 1)(L)$.

LEMMA 3.11. *Let* $\alpha \in L - \{0, 1\}$ *with* $\alpha \geqq \alpha'$ *and* $0 < a \leqq b < 1$. *Then there is* $\lambda \in (0, 1)(L)$ *with* $H_\alpha(\lambda) = [a, b]$.

PROOF. Use

$$\lambda(t) = \begin{cases} 1 \text{ if } t < a \\ \alpha \text{ if } a \leqq t \leqq b \\ 0 \text{ if } t > b. \end{cases}$$

COROLLARY 3.12. *Let* $\alpha, 0 \in L^a - \{1\}$ *with* $\alpha \geqq \alpha'$. *For* $n \geqq 3$ *choose* $\lambda_n \in (0, 1)(L)$ *with* $H_\alpha(\lambda_n) = [1/n, (n-1)/n]$. *Then*
i) *for any* $k \geqq 3$, $c_\alpha(\{\lambda_n : n \geqq k\}) = (0, 1)(L)$, *and*
ii) *for any* $\sigma \in (0, 1)(L)$, *there is some* $k \geqq 3$ *with* $\{\lambda_n : n \geqq k\} \subseteqq c_\alpha(\{\sigma\})$.

COROLLARY 3.13. *Let* $\alpha, 0 \in L^a - \{1\}$ *with* $\alpha \geqq \alpha'$. *Then*
i) c_α *is not a closure operator*,
ii) $(0, 1)(L)$ *does not have the* α-*property*, *and*
iii) $(0, 1)(L)$ *is not suitable.*

The next example shows that the converse of 2.5 fails and that the α-Hausdorff hypothesis in 2.7 is necessary.

EXAMPLE 3.14. Let $\alpha, 0 \in L^a - \{1\}$ with $\alpha \geqq \alpha'$. Choose $\lambda, \mu \in I(L)$ with $H_\alpha(\lambda) = [1/2, 3/4]$ and $H_\alpha(\mu) = [1/4, 3/4]$ and let $A = \{\lambda, \mu\}$. By 3.8, $\lambda \in c_\alpha(\{\mu\})$ and $\mu \in c_\alpha(\{\lambda\})$ and so, in the notation of §2 for the subspace A, W_α is indiscrete. By 3.6, if $1/4 < s < 1/2 < 3/4 < t$, $R_s \wedge L_t(\lambda) > \alpha$ while $R_s \wedge L_t(\mu) \leqq \alpha$ and so $\{\lambda\} \in T_\alpha$ where T is the fuzzy topology for the subspace A. By 2.4, A does not have the α-property. However, c_α is a closure operator in any two-point space.

For the case $\alpha < \alpha'$ the characterization of the α-closure is quite different. The second parts of 3.18 and 3.20 were obtained by different methods in [5].

THEOREM 3.15. *Let* $\alpha, 0 \in L^a - \{1\}$ *with* $\alpha < \alpha'$. *Let* $A \subseteqq I(L)$. *Then* $\lambda \in c_\alpha(A)$ *if and only if* $H_\alpha(\lambda) \subseteqq \mathrm{Cl}(\bigcup\{H_0(\sigma): \sigma \in A\})$ *and, for every compact* $K \subseteqq \mathrm{Int}\, H_\alpha(\lambda)$, *there is* $\sigma \in A$ *with* $K \subseteqq \mathrm{Int}\, H_0(\sigma)$.

PROOF. Let $\lambda \in c_\alpha(A)$ and let $x \in H_\alpha(\lambda)$. For any $s < t$ with $x \in (s, t)$, by 3.7, $R_s \wedge L_t(\lambda) > \alpha$ and so there is $\sigma \in A$ with $R_s \wedge L_t(\sigma) > 0$. Again by 3.7, $(s, t) \cap H_0(\sigma) \neq \emptyset$ and so $x \in \mathrm{Cl}(\bigcup\{H_0(\sigma): \sigma \in A\})$. Now let $K \neq \emptyset$ be compact with $K \subseteqq \mathrm{Int}\, H_\alpha(\lambda)$. Then, for $s \geqq t$ with $K \subseteqq [t, s] \subseteqq \mathrm{Int}\, H_\alpha(\lambda)$, $R_s \wedge L_t(\lambda) > \alpha$ and so there is $\sigma \in A$ with $R_s \wedge L_t(\sigma) > 0$. Then by 3.7, $K \subseteqq [t, s] \subseteqq \mathrm{Int}\, H_0(\sigma)$. For the converse let the two conditions hold for λ relative to A and let $R_s \wedge L_t(\lambda) > \alpha$. If $s < t$, $(s, t) \cap (\bigcup\{H_0(\sigma): \sigma \in A\}) \neq \emptyset$ and so by 3.7, $R_s \wedge L_t(\sigma) > 0$ for some $\sigma \in A$. If $s \geqq t$, $[t, s] \subseteqq \mathrm{Int}\, H_0(\sigma)$ for some $\sigma \in A$ and so $R_s \wedge L_t(\sigma) > 0$. Thus $\lambda \in c_\alpha(A)$.

LEMMA 3.16. *Let* $\alpha \in L^c - \{0, 1\}$ *with* $\alpha < \alpha'$.

i) *If* $0 \leqq a \leqq 1$, *then there is* $\lambda \in I(L)$ *with* $H_\alpha(\lambda) = \{a\}$ *and* $H_0(\lambda) = [0, 1]$.

ii) *If* $0 < a \leqq b \leqq c < 1$, *then there is* $\lambda \in (0, 1)(L)$ *with* $H_\alpha(\lambda) = \{b\}$ and $H_0(\lambda) = [a, c]$.

PROOF. For i) let

$$\lambda(t) = \begin{cases} 1 & \text{if } t < 0 \\ \alpha' & \text{if } 0 \leqq t \leqq a \\ \alpha & \text{if } a < t \leqq 1 \\ 0 & \text{if } t > 1. \end{cases}$$

For ii) let

$$\lambda(t) = \begin{cases} 1 & \text{if } t < a \\ \alpha' & \text{if } a \leqq t \leqq b \\ \alpha & \text{if } b < t \leqq c \\ 0 & \text{if } t > c. \end{cases}$$

COROLLARY 3.17. *Let* $\alpha, 0 \in L^a - \{1\}$ *with* $\alpha \neq 0$ and $\alpha < \alpha'$. *Let* $\emptyset \neq A \subseteqq I(L)$. *Then* $c_\alpha(A)$ *is* α-*closed if and only if* $c_\alpha(A) = I(L)$.

PROOF. Let $\gamma \in A$ with $H_0(\gamma) = [a, b]$. Let $\lambda \in I(L)$ with $H_0(\lambda) = [0, 1]$ and $H_\alpha(\lambda) = \{a\}$. By 3.15, $\lambda \in c_\alpha(A)$ and $I(L) = c_\alpha(\{\lambda\}) \subseteqq c_\alpha(c_\alpha(A))$.

COROLLARY 3.18. *Let* $\alpha, 0 \in L^a - \{1\}$ *with* $\alpha \neq 0$ *and* $\alpha < \alpha'$. *Then*

i) c_α *is not a closure operator on* $I(L)$, *and*

ii) $I(L)$ *does not have the* α-*property*.

PROOF. Let θ, σ be the canonical images of 1/4, 1/2 respectively in $I(L)$. By 3.15, $\sigma \notin c_\alpha(\{\theta\})$ and i) follows from 3. 17.

COROLLARY 3.19. *Let* α, $0 \in L^a - \{1\}$ *with* $\alpha \neq 0$ *and* $\alpha < \alpha'$. Let $\emptyset \neq A \subseteqq (0, 1)(L)$. *Then* $c_\alpha(A)$ *is* α*-closed if and only if* $c_\alpha(A) = (0, 1)(L)$.

PROOF. Let $\sigma \in A$ and let $b \in H_0(\sigma)$. Using 3.16 pick a sequence $\lambda_n \in (0, 1)(L)$ with $H_0(\lambda_n) = [1/n, (n-1)/n]$ and $H_\alpha(\lambda_n) = \{b\}$ (for n sufficiently large). Then by 3.15, $\lambda_n \in c_\alpha(\{\sigma\}) \subseteqq c_\alpha(A)$ and $c_\alpha(\{\lambda_n : n \geqq k\}) = (0, 1)(L)$.

COROLLARY 3.20. *Let* α, $0 \in L^a - \{1\}$ *with* $\alpha \neq 0$ *and* $\alpha < \alpha'$. *Then*

i) c_α *is not a closure operator on* $(0, 1)(L)$, *and*

ii) $(0, 1)(L)$ *does not have the* α*-property.*

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