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A CONSTRUCTIVE PROXIMINALITY PROPERTY OF FINITE-DIMENSIONAL LINEAR SUBSPACES

DOUGLAS S. BRIDGES

We recall that a subspace X of a metric space E is *proximinal* (in E) if, to each element in E, there corresponds a best approximant in X. Classically, every finite-dimensional linear subspace X of a normed linear space E is proximinal [**5**, Ch. 1, p. 20]. Our interest in a constructive development of best approximation theory was first aroused by the observation that all known proofs of this last proposition are nonconstructive. In fact, as we pointed out in §1 and Proposition 2.1 of [**3**], the constructive content of these proofs is the computability of $\text{dist}(a, X)$ for each a in E. The further assertion that there exists a best approximant of a in X seems to depend on the attainment of the infimum of a continuous, real-valued function on a compact space, an essentially nonconstructive property of such functions (cf. [**6**, pp. 115-116]).

As this state of affairs appears to compromise the position of classical approximation theory as the foundation of a vast and powerful branch of numerical analysis, we believe that the systematic redevelopment of the classical theory along constructive lines provides a worthwhile and interesting mathmatical activity. We began this activity in [**3**], where we derived a result [**3**, Theorem 2.2] which was strong enough to yield a constructive proof of existence of minimax polynomial approximants of elements of $C[0, 1]$. Unfortunately, that result is not strong enough to cover other situations, such as that of best uniform approximation by linear combinations of functions in a general Tchebychev set, in which the classical theory proves existence and uniqueness of best approximants. In this paper, we present a constructive theorem which certainly disposes of the general Tchebychev approximation problem, and may well enable us to handle other unique existence problems in the bargain.

For general background material on constructive analysis, we refer to [**2**]. (A fuller, but less up-to-date, exposition of constructive mathematics is to be found in Bishop's fundamental treatise [**1**].) For our present purposes, an appreciation of the following facts and definitions is certainly necessary.

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If f is a uniformly continuous mapping of a compact (that is, totally bounded and complete) metric space K into $\mathbf{R}$, then $\sup f$ and $\inf f$ are computable (although not necessarily attained!); for all but countably many real numbers $\alpha > \inf f$, the set $\{x \in K: f(x) \leqq \alpha\}$ is then compact.

A subset X of a metric space E is *located* (in E) if $\operatorname{dist}(a, X) \equiv \inf\{d(a, x): x \in X\}$ is computable for each a in E. Compact subspaces, and finite-dimensional linear subspaces of a normed space, are located. If X is located in E, we say that $a \in E$ has *at most one best approximant* in X if

$$\max(d(a, x), d(a, x')) > \operatorname{dist}(a, X)$$

whenever $x \in X$, $x' \in X$ and $d(x, x') > 0$.

The one original definition of this paper is the following. We say that a located subset X of E is *quasi-proximinal* (in E) if any element of E with at most one best approximant in X has a (clearly unique) best approximant in X. It is trivial to prove classically that quasi-proximinality and proximinality are equivalent properties. That quasi-proximinal subspaces are of constructive significance is shown by the following theorem, the main result of this paper.

THEOREM. *A finite-dimensional linear subspace of a normed linear space over* $\mathbf{R}$ *is quasi-proximinal.*

The proof proceeds by induction on the dimension of the subspace, and hinges on several applications of the following lemma.

LEMMA. *Let x, e be elements of the normed linear space E over* $\mathbf{R}$, *with* $\|e\| > 0$. *Let* $d \geqq 0$, *and suppose that*

$$\max(\|x - \lambda e\|, \|x - \lambda' e\|) > d$$

whenever $\lambda \in \mathbf{R}$, $\lambda' \in \mathbf{R}$ *and* $|\lambda - \lambda'| > 0$. *Then there exists* $\zeta \in \mathbf{R}$ *such that* $\|x - \zeta e\| > d$ *implies* $\operatorname{dist}(x, \mathbf{R}e) > d$.

PROOF. Let $\phi(\lambda) \equiv \|x - \lambda e\|$ $(\lambda \in \mathbf{R})$, and assume without loss of generality that $\|e\| = 1$. We first observe that, if $t_1 > t_2 > 0$ and

$$S_k \equiv \{\lambda \in \mathbf{R}: \phi(\lambda) \leqq \operatorname{dist}(x, \mathbf{R}e) + t_k\}$$

is compact for $k = 1, 2$, then (as ϕ is uniformly continuous)

$$\phi(\inf S_k) = \operatorname{dist}(x, \mathbf{R}e) + t_k = \phi(\sup S_k) \qquad (k = 1, 2).$$

As $S_2 \subset S_1$, it follows that

$$\inf S_1 < \inf S_2 \leqq \sup S_2 < \sup S_1.$$

Hence as S_k is convex, $S_k = [\inf S_k, \sup S_k]$ (k = 1, 2).

We now construct a sequence $(\alpha_n)_{n\geqq 1}$ of positive numbers such that, for each n,

$$\begin{aligned} A(n) &\equiv \{\lambda \in \mathbf{R}: \phi(\lambda) \leqq \operatorname{dist}(x, \mathbf{R}e) + \alpha_n\} \\ &= \{\lambda \in \mathbf{R}: |\lambda| \leqq \|x\| + \operatorname{dist}(x, \mathbf{R}e) + \alpha_n, \phi(\lambda) \leqq \operatorname{dist}(x, \mathbf{R}e) + \alpha_n\} \end{aligned}$$

is compact and

$$\alpha_{n+1} < \alpha_n \leqq \left(\frac{2}{3}\right)^{n-1} (\sup A(1) - \inf A(1)).$$

Then

$$A(n+1) \subset A(n) = [\inf A(n), \sup A(n)]$$

and

$$\inf A(n) < \inf A(n+1) < \sup A(n+1) < \sup A(n).$$

We next construct a sequence $(\nu_n)_{n\geqq 1}$ of positive integers with the properties:

(i) $\nu_1 \leqq \nu_2 \leqq \nu_3 \leqq \dots$

(ii) $\nu_{n+1} > \nu_n \Rightarrow \sup A(\nu_{n+1}) - \inf A(\nu_{n+1}) < (\frac{2}{3})(\sup A(\nu_n) - \inf A(\nu_n))$

(iii) $\nu_{n+1} = \nu_n \Rightarrow \forall k \geqq n(\nu_k = \nu_n)$

(iv) $\nu_{n+1} = \nu_n \Rightarrow \operatorname{dist}(x, \mathbf{R}e) > d$.

Having computed $\nu_1 \equiv 1, \dots, \nu_k$, we set $m_j \equiv \inf A(\nu_j)$, $M_j \equiv \sup A(\nu_j)$ $(1 \leqq j \leqq k)$. If $k > 1$ and $\nu_k = \nu_{k-1}$, we set $\nu_{k+1} \equiv \nu_k$. If $k = 1$, or $k > 1$ and $\nu_k > \nu_{k-1}$, we compute ξ with

$$\frac{1}{2}(m_k + M_k) < \xi < m_k + \frac{2}{3}(M_k - m_k)$$

and $\phi(\xi) > d$. Then either $d < \operatorname{dist}(x, \mathbf{R}e)$, in which case we set $\nu_{k+1} \equiv \nu_k$; or $\operatorname{dist}(x, \mathbf{R}e) < \phi(\xi)$. In the latter case, choosing $\nu_{k+1} > \nu_k$ so that $\operatorname{dist}(x, \mathbf{R}e) + \alpha_{\nu_{k+1}} < \phi(\xi)$, we see from the definition of $A(\nu_{k+1})$ and the uniform continuity of ϕ that either $\xi < \inf A(\nu_{k+1})$ or $\sup A(\nu_{k+1}) < \xi$; whence, certainly condition (ii) is satisfied. This completes the inductive construction of (ν_n).

Define a sequence $(\zeta_n)_{n\geqq 1}$ in $\mathbf{R}$ as follows. If $\nu_2 = \nu_1$, define $\zeta_n \equiv M_1$ for each n. If $\nu_{n+1} > \nu_n$, define $\zeta_n \equiv M_n$; if $\nu_2 > \nu_1$ and $\nu_{n+1} = \nu_n$, define $\zeta_n \equiv \zeta_r$, where r is that unique integer such that $\nu_{r+1} = \nu_r > \nu_{r-1}$. We show that (ζ_n) is a Cauchy sequence. Let k be a positive integer. If $\nu_{k+2} > \nu_{k+1}$, then $\nu_{j+1} > \nu_j$ for each j in $\{1, \dots, k+1\}$; whence (by (ii) and the definition of (ζ_n))

$$\begin{aligned} 0 \leqq \zeta_k - \zeta_{k+1} &= M_k - M_{k+1} \\ &< M_k - m_k \\ &\leqq \left(\frac{2}{3}\right)^{k-1} (M_1 - m_1). \end{aligned}$$

Thus

$$|\zeta_k - \zeta_{k+1}| < \left(\frac{2}{3}\right)^{k-1}(M_1 - m_1),$$

an inequality which holds also if $\nu_{k+2} = \nu_{k+1}$, when $\zeta_{k+1} = \zeta_k$. If p, q are positive integers with $q > p$, we now have

$$\begin{aligned}|\zeta_p - \zeta_q| &\leqq \sum_{k=p}^{q-1} |\zeta_k - \zeta_{k+1}| \\ &\leqq \sum_{k=p}^{q-1} \left(\frac{2}{3}\right)^{k-1}(M_1 - m_1) \\ &\leqq 3\left(\frac{2}{3}\right)^{p-1}(M_1 - m_1).\end{aligned}$$

Hence (ζ_n) is a Cauchy, and therefore convergent, sequence in $\mathbf{R}$. Let ζ be its limit, and note that

$$|\zeta - \zeta_n| \leqq 3\left(\frac{2}{3}\right)^{n-1}(M_1 - m_1) \qquad (n \geqq 1).$$

We shall show that $\phi(\zeta) > d$ entails dist $(x, \mathbf{R}e) > d$. To this end, suppose that $\phi(\zeta) > d$, and choose a positive integer n so that

$$\phi(\zeta) > d + 4\left(\frac{2}{3}\right)^{n-1}(M_1 - m_1).$$

If $\nu_{n+1} = \nu_n$, then we certainly have dist $(x, \mathbf{R}e) > d$. If $\nu_{n+1} > \nu_n$, then $\nu_j > \nu_{j-1}$ for each j in $\{2, \ldots, n+1\}$; so that $\nu_n \geqq n$. As $\zeta_n = \sup A(\nu_n)$ in this case, we have

$$\begin{aligned}\phi(\zeta_n) &= \operatorname{dist}(x, \mathbf{R}e) + \alpha_{\nu_n} \\ &\leqq \operatorname{dist}(x, \mathbf{R}e) + \alpha_n \\ &\leqq \operatorname{dist}(x, \mathbf{R}e) + \left(\frac{2}{3}\right)^{n-1}(M_1 - m_1).\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{dist}(x, \mathbf{R}e) &\geqq \phi(\zeta_n) - \left(\frac{2}{3}\right)^{n-1}(M_1 - m_1) \\ &\geqq \phi(\zeta) - |\zeta - \zeta_n| - \left(\frac{2}{3}\right)^{n-1}(M_1 - m_1) \\ &> d + 4\left(\frac{2}{3}\right)^{n-1}(M_1 - m_1) - 3\left(\frac{2}{3}\right)^{n-1}(M_1 - m_1) \\ &\quad - \left(\frac{2}{3}\right)^{n-1}(M_1 - m_1) = d.\end{aligned}$$

Hence, in both possible cases, dist $(x, \mathbf{R}e) > d$.

We now prove our Theorem. Let $a \in E$ have at most one best approximant in the finite-dimensional linear subspace X of the normed space E over $\mathbf{R}$. If $X = \{0\}$, the theorem is trivial. If $X = \mathbf{R}e$ has dimension 1, we need only take $x = a$, $d = \operatorname{dist}(a, X)$ in our lemma, to compute $\zeta \in \mathbf{R}$ such that $\|a - \zeta e\| = \operatorname{dist}(a, X)$.

Now let n be a positive integer, suppose we have proved the Theorem for all n-dimensional linear subspaces of a normed space, and consider the case $\dim X = n + 1$. Let $\{e_1, \ldots, e_{n+1}\}$ be a basis of X, and $Y \equiv \operatorname{span}\{e_1, \ldots, e_n\}$. Define a new norm and equality on E by $\|x\|_1 \equiv \operatorname{dist}(x, \mathbf{R}e_{n+1})$, and $x =_1 x'$ if and only if $\|x - x'\|_1 = 0$. Note that

$$\begin{aligned}\inf_{y \in Y} \|x - y\|_1 &= \inf_{y \in Y} \inf_{\lambda \in \mathbf{R}} \|x - y - \lambda e_{n+1}\| \\ &= \operatorname{dist}(x, X).\end{aligned}$$

Also, if $y \in Y$, and λ_1, λ_2 are real numbers with $|\lambda_1 - \lambda_2| > 0$, then

$$\|(y + \lambda_1 e_{n+1}) - (y + \lambda_2 e_{n+1})\| = |\lambda_1 - \lambda_2| \, \|e_{n+1}\| > 0;$$

so that

$$\max_{k=1,2} \|a - y - \lambda_k e_{n+1}\| > \operatorname{dist}(a, X),$$

and we can apply our lemma with $x \equiv a - y$, $e \equiv e_{n+1}$ and $d \equiv \operatorname{dist}(a, X)$.

Now let y_1, y_2 belong to Y, with $\|y_1 - y_2\|_1 > 0$. Compute ζ_1, ζ_2 in $\mathbf{R}$ so that, for $k = 1, 2$,

$$\|a - y_k - \zeta_k e_{n+1}\| > \operatorname{dist}(a, X)$$

implies

$$\|a - y_k\|_1 = \operatorname{dist}(a - y_k, \mathbf{R}e_{n+1}) > \operatorname{dist}(a, X).$$

As

$$\begin{aligned}&\|(y_1 + \zeta_1 e_{n+1}) - (y_2 + \zeta_2 e_{n+1})\| \\ &\qquad = \|(y_1 - y_2) + (\zeta_1 - \zeta_2)e_{n+1}\| \\ &\qquad \geqq \|y_1 - y_2\|_1 > 0,\end{aligned}$$

we have

$$\max_{k=1,2} \|a - y_k - \zeta_k e_{n+1}\| > \operatorname{dist}(a, X).$$

Hence

$$\max_{k=1,2} \|a - y_k\|_1 > \operatorname{dist}(a, X) = \inf_{y \in Y} \|a - y\|_1.$$

Thus a has at most one best approximant in the n-dimensional linear subspace Y of $(E, \|.\|_1)$. By our induction hypothesis, it follows that there exists $\eta \in Y$ with

$$\|a - \eta\|_1 = \inf_{y \in Y} \|a - y\|_1 = \text{dist}\,(a, X).$$

Applying our lemma with $x \equiv a - \eta$, $e \equiv e_{n+1}$ and $d \equiv \text{dist}\,(a, X)$, we now compute $\zeta \in \mathbf{R}$ so that

$$\|a - \eta - \zeta e_{n+1}\| = \text{dist}\,(a - \eta, \mathbf{R}e_{n+1}) = \text{dist}\,(a, X).$$

Then $\eta + \zeta e_{n+1}$ is a best approximant of a in X.

Under the conditions of the induction step of this last proof, it is clear that if y_1, y_2 belong to Y, $\|y_1 - y_2\|_1 > 0$ and λ_1, λ_2 belong to $\mathbf{R}$, then

$$\|a - y_1 - \lambda_1 e_{n+1}\| + \|a - y_2 + \lambda_2 e_{n+1}\| > 2\,\text{dist}\,(a, X).$$

Classically, we could pass from this to the statement

$$\max_{k=1,2} \|a - y_k\|_1 > \text{dist}\,(a, X)$$

by two applications of the theorem which asserts that a uniformly continuous mapping of a compact metric space into the positive reals has positive infimum. As no constructive proof of this theorem is known, it is fortunate that the weaker result embodied in our lemma can be derived by constructive means.

Finally, for the application of our theorem in the general theory of best Tchebychev approximation, we refer the reader to [**4**].

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UNIVERSITY COLLEGE, BUCKINGHAM MK 18 1EG, ENGLAND