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EQUICONVERGENCE OF INTERPOLATING PROCESSES

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1. **Introduction.** The following theorem is due to J. L. Walsh [3, p. 153].

THEOREM. *Let f be analytic for* $|z| < R$, $R > 1$. *Let* $p_n(z)$ *be the polynomial of degree n which coincides with f at the roots of the unity* $e^{2k\pi i/(n+1)}$, $k = 0, 1, \ldots, n$, *and let* $q_n(z)$ *be the n-th Taylor polynomial of f around the origin. Then* $p_n(z) - q_n(z) \to 0$, $n \to \infty$, *for* $|z| < R^2$.

It was a lecture of A. Sharma, in which he presented several extensions of this theorem, extensions obtained jointly by A. S. Cavaretta Jr., A. Sharma and R. S. Varga [1], that has directed our attention to this topic. In this paper we present a generalization of Walsh's theorem that goes in a different direction than the results in [1].

2. **Notation.** The infinite triangular matrix $[z_{kn}]$, $k = 1, 2, \ldots, n$; $n = 1, 2, \ldots$, where the entries z_{kn} are complex numbers, defines a Hermite interpolation process. We assume that we are given two such matrices, $[z_{kn}]$ and $[\tilde{z}_{kn}]$, and that $|z_{kn}| \leqq d$, $|\tilde{z}_{kn}| \leqq d$ for all k, n. We write

$$w_n(z) = \prod_{k=1}^{n}(z - z_{kn}) = \sum_{r=0}^{n} A_{n,r} z^{n-r}$$

and similarly we define the polynomials $\tilde{w}_n(z)$ and the coefficients $\tilde{A}_{n,r}$. If f is a function analytic in $|z| < R$, and $R > d$, then the interpolating polynomials to f based on the systems $[z_{kn}]$ and $[\tilde{z}_{kn}]$ are denoted by $p_n(z, f)$ and $\tilde{p}_n(z, f)$, respectively.

3. Let us formulate a heuristic principle. If two systems of interpolating nodes, $[z_{kn}]$ and $[\tilde{z}_{kn}]$, are "close", then the set of z's for which

$$p_n(z, f) - \tilde{p}_n(z, f) \to 0, \; n \to \infty, \tag{1}$$

is "large". In particular, it can be larger than the set on which f is analytic. This principle looks quite natural, because if the two systems are identical, then certainly (1) holds for all z.

We can look at Walsh's theorem in the light of this heuristic principle. Taylor's polynomials of f at the origin are the interpolating polynomials corresponding to the system of the nodes $[\tilde{z}_{kn}]$, where $\tilde{z}_{kn} = 0$ for every k

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and n. That system of nodes and the system $[z_{kn}]$, $z_{kn} = \exp(2k\pi i/n)$ $k = 1, 2, \ldots, n$; $n = 1, 2, \ldots$ are "close" because their n-th rows have the first $n - 1$ moments identical, i.e.,

$$\sum_{k=1}^{n} z_{kn}^{r} = \sum_{k=1}^{n} \tilde{z}_{kn}^{r}$$

for $r = 1, 2, \ldots, n - 1$; $n = 1, 2, \ldots$, and only the n-th moments are different. So it should not be considered surprising that the set of z's for which (1) holds is "large", i.e., contains all z such that $|z| < R^2$. On the contrary, one has to wonder why this set is not even larger, in view of the fact that the systems of nodes are so close.

The key step in transforming the heuristic principle into a precise mathematical statement is to decide what meaning to give to "closeness" of two systems of nodes. If one does that in the most straightforward manner, one obtains the not very interesting Theorem 3; if one approaches that task more delicately, one obtains Theorem 1, which is a generalization of Walsh's theorem. (It should be noted, however, that Theorem 1 contains as a special case only a weak form of Walsh's theorem: setting in Theorem 1 $d = \gamma = 1$, we obtain overconvergence not for $|z| < R^2$, but only for $|z| < (R - 2)^2/4$, and that provided $R > 4$).

Let us make one more remark. It is of no importance whether we formulate our results in terms of the moments $\sum_{k=1}^{n} z_{kn}^{r}$ or in terms of the elementary symmetric functions of $\{z_{kn}\}$, i.e., in terms of the coefficients $A_{n,r}$.

4. With the previously introduced notation, we can state our main result.

THEOREM 1. *Let the two systems of nodes, $[z_{kn}]$ and $[\tilde{z}_{kn}]$, and the number γ, $0 < \gamma \leqq 1$, satisfy the conditions*:

$$|z_{kn}| \leqq d,\ |\tilde{z}_{kn}| \leqq d \text{ for all } k, n \text{ and} \tag{2}$$

$$\Big(\sum_{s \leqq \lambda n} |A_{n,s} - \tilde{A}_{n,s}|\Big)^{1/n} \to 0,\ n \to \infty,\ \text{for every } \lambda < \gamma. \tag{3}$$

Let $R > 2d + (1 + d)^{2/\gamma}$ and let f be a function analytic for $|z| < R$. Then $p_n(z, f) - \tilde{p}_n(z, f) \to 0$, $n \to \infty$, for $|z| < C(R - 2d)^{1+\gamma}$, where $C = (1 + d)^{-2}$.

PROOF. We first choose ρ so that $2d + (1 + d)^{2/\gamma} < \rho < R$, then we choose λ between 0 and γ, but sufficiently close to γ so that

$$2d + (1 + d)^{2/\lambda} < \rho. \tag{4}$$

Since for fixed ρ and d

$$\frac{(\rho - 2d)^{1+\lambda}}{(1 + d)^2} \neq \rho, \tag{5}$$

except for at most one value of λ, we can make sure that (5) holds by chososing λ sufficiently close to γ.

To prove the theorem it is sufficient, because of the maximum modulus principle, to show that $p_n(z,f) - \tilde{p}_n(z,f) \to 0, n \to \infty$ when z is restricted to the circumference $|z| = (1 + d)^{-2}(\rho - 2d)^{1+\lambda}$, for every fixed ρ and λ which satisfy the conditions stated above. Let us also observe that these conditions imply

(6) $$|z| > 1, \rho > 1$$

and

(7) $$\rho > d, |z| \neq \rho,$$

the last fact following from (5).

Because of (2) and (7) we have, by Hermite's formula,

(8) $$p_n(z,f) - \tilde{p}_n(z:f) = \frac{1}{2\pi} \int_{|t|=\rho} \frac{f(t)}{t - z} \left(\frac{\tilde{w}_n(z)}{\tilde{w}_n(t)} - \frac{w_n(z)}{w_n(t)} \right) dt.$$

We shall show that for $|t| = \rho$, with our choice of ρ and $|z|$ we obtain

(9) $$\left| \frac{\tilde{w}_n(z)}{\tilde{w}_n(t)} - \frac{w_n(z)}{w_n(t)} \right| \leqq Aq^n$$

where $0 < q < 1$, and A is a constant.

From (8) and (9) it follows that

$$|p_n(z,f) - \tilde{p}_n(z,f)| \leqq \frac{A\rho M_\rho(f)}{||z| - \rho|} q^n,$$

where $M_\rho(f) = \text{Max}_{|z|=\rho} |f(z)|$, and so $p_n(z,f) - \tilde{p}_n(z,f) \to 0, n \to \infty$, which proves the theorem. So we have only to establish (9).

For that purpose we observe first that

$$|w_n(t)| = |\prod_{k=1}^{n}(t - z_{kn})| \geqq \prod_{k=1}^{n}(|t| - |z_{kn}|) \geqq (\rho - d)^n,$$

and similarly for $|\tilde{w}_n(t)|$, so that

(10) $$|w_n(t)\, \tilde{w}_n(t)| \geqq (\rho - d)^{2n}.$$

We write

(11) $$\begin{aligned} &\tilde{w}_n(z)w_n(t) - w_n(z)\tilde{w}_n(t) \\ &= \sum_{r,s=0}^{n} (\tilde{A}_{n,r}A_{n,s} - A_{n,r}\tilde{A}_{n,s})z^{n-r}t^{n-s} \\ &= \Sigma_1 + \Sigma_2, \end{aligned}$$

where Σ_1 is the sum of the terms for which both indices r and s are $\leqq \lambda n$, and Σ_2 is the sum of the remaining terms.

Because of (6) we get

(12) $$|\Sigma_1| \leqq |z|^n \rho^n \sum_{r,s \leqq \lambda n} |\tilde{A}_{n,r} A_{n,s} - A_{n,r} \tilde{A}_{n,s}|.$$

Since

$$\tilde{A}_{n,r} A_{n,s} - A_{n,r} \tilde{A}_{n,s} = \tilde{A}_{n,r}(A_{n,s} - \tilde{A}_{n,s}) + \tilde{A}_{n,s}(\tilde{A}_{n,r} - A_{n,r}),$$

we obtain

(13) $$\sum_{r,s \leqq \lambda n} |\tilde{A}_{n,r} A_{n,s} - A_{n,r} \tilde{A}_{n,s}| \leqq 2 \sum_{r=0}^{n} |\tilde{A}_{n,r}| \sum_{s \leqq \lambda n} |A_{n,s} - \tilde{A}_{n,s}|.$$

From (3) it follows that

(14) $$\sum_{s \leqq \lambda n} |A_{n,s} - \tilde{A}_{n,s}| \leqq \varepsilon_n^n,$$

where $\varepsilon_n \to 0$, $n \to \infty$. On the other hand, since for any polynomial

$$\prod_{j=1}^{n} (z - z_j) = \sum_{k=0}^{n} a_k z^k$$

we have

$$\sum_{k=0}^{n} |a_k|\,|z|^k \leqq \prod_{j=1}^{n} (|z| + |z_j|),$$

we get in particular

(15) $$\sum_{s=0}^{n} |A_{n,s}| \leqq \prod_{k=1}^{n} (1 + |z_{kn}|) \leqq (1 + d)^n,$$

and similarly

(15′) $$\sum_{s=0}^{n} |\tilde{A}_{n,s}| \leqq (1 + d)^n,$$

so that from (12), (13), (14) and (15′) we obtain that

(16) $$|\Sigma_1| \leqq 2|z|^n \rho^n (1 + d)^n \varepsilon_n^n = 2\delta_n^n,$$

where $\delta_n = (1 + d)\,\rho|z|\,\varepsilon_n \to 0$, $n \to \infty$.

To estimate Σ_2 the crucial observation is that, with our choice of ρ and $|z|$, the following estimate holds

(17) $$|z|^{n-r}|t|^{n-s} \leqq \left(\frac{\rho(\rho - 2d)}{(1 + d)^2}\right)^n$$

if $r \geqq \lambda n$ or $s \geqq \lambda n$. To prove (17) we observe first that because of (6), for $r \geqq \lambda$ or $s \geqq \lambda n$,

$$|z|^{n-r}|t|^{n-s} \leqq \operatorname{Max}\{|z|^{n-\lambda n}|t|^n, |z|^n|t|^{n-\lambda n}\}$$

so that we need only to verify that both $|z|^{n-\lambda n}\rho^n$ and $|z|^n\rho^{n-\lambda n}$ are $\leqq$ the right hand side in (17), i.e., we need to verify that

$$(17') \qquad |z|^{1-\lambda} \leqq \frac{\rho - 2d}{(1 + d)^2}$$

and

$$(17'') \qquad |z|\rho^{-\lambda} \leqq \frac{\rho - 2d}{(1 + d)^2}$$

Since $|z| = (\rho - 2d)^{1+\lambda}/(1 + d)^2$, we have

$$|z|^{1-\lambda} = \frac{\rho - 2d}{(1 + d)^2}\left(\frac{(1 + d)^2}{(\rho - 2d)^\lambda}\right)^\lambda$$

from which (17′) follows, because $(1 + d)^2 < (\rho - 2d)^\lambda$ due to (4). On the other hand, (17″) is obvious because $\rho > 2d$ and so

$$|z|\rho^{-\lambda} = \frac{\rho - 2d}{(1 + d)^2}\left(\frac{\rho - 2d}{\rho}\right)^\lambda \leqq \frac{\rho - 2d}{(1 + d)^2}.$$

We obtain now form (17), (15) and (15′) that

$$\begin{aligned} |\Sigma_2| &\leqq \sum_{r\geqq\lambda n \text{ or } s\geqq\lambda n} |\tilde{A}_{n,r}A_{n,s} - A_{n,r}\tilde{A}_{n,s}||z|^{n-r}|t|^{n-s} \\ &\leqq \left(\frac{\rho(\rho - 2d)}{(1 + d)^2}\right)^n \sum_{r\geqq\lambda n \text{ or } s\geqq\lambda n} (|\tilde{A}_{n,r}||A_{n,s}| + |A_{n,r}||\tilde{A}_{n,s}|) \\ &\leqq 2\left(\frac{\rho(\rho - 2d)}{(1 + d)^2}\right)^n \sum_{r=0}^{n} |\tilde{A}_{n,r}| \sum_{s=0}^{n} |A_{n,s}| \\ &\leqq 2\rho^n(\rho - 2d)^n. \end{aligned}$$

From this estimate and from (16) and (11) we deduce that for n large we have

$$|\tilde{w}_n(z)w_n(t) - w_n(z)\tilde{w}_n(t)| \leqq 3\rho^n(\rho - 2d)^n.$$

This together with (10) shows that

$$\left|\frac{\tilde{w}_n(z)}{\tilde{w}_n(t)} - \frac{w_n(z)}{w_n(t)}\right| \leqq 3\left(\frac{\rho(\rho - 2d)}{(\rho - d)^2}\right)^n = 3q^n,$$

where $q = \rho(\rho - 2d)/(\rho - d)^2 < 1$, which proves (9).

5. What happens in case $\gamma = 0$, in other words, if the condition (3) of the Theorem 1 is simply dropped? This means that we no longer assume that the two systems of interpolating nodes are "close". So we have no reason to expect overconvergence, i.e., that $p_n(z, f) - \tilde{p}_n(z, f) \to 0$, $n \to \infty$, in a strictly bigger disc than the disc in which both p_n and $\tilde{p}_n$ converge to f. (To confirm that, take f analytic in $|z| < R$, with singulari-

ties at R and $-R$, and take for p_n and $\tilde{p}_n$ the Taylor polynomials of f around d and $-d$, respectively.)

However, there is a very simple convergence theorem, that is worth mentioning.

THEOREM 2. *If* $|z_{kn}| \leqq d$, $k = 1, 2, \ldots$; $n = 1, 2, \ldots$, *if* $R > 2d$, *and if* f *is analytic in* $|z| < R$, *then* $p_n(z, f) \to f(z)$, $n \to \infty$ *for* $|z| < R - 2d$.

This result is a special case of a very general theorem [**2**, page 55].

It can also be quite easily proved directly. Fix $|z|$, and let $|z| + 2d < \rho < R$. By Hermite's formula

$$p_n(z, f) - f(z) = \frac{1}{2\pi i} \int_{|t|=\rho} \frac{f(t)}{z - t} \frac{w_n(z)}{w_n(t)} \, dt,$$

and since

$$|w_n(z)| = \prod |z - z_{kn}| \leqq (|z| + d)^n,$$

and

$$|w_n(t)| = \prod |t - z_{kn}| \geqq (\rho - d)^n,$$

we have

$$\left| \frac{w_n(z)}{w_n(t)} \right| \leqq \left(\frac{|z| + d}{\rho - d} \right)^n \leqq q^n, q < 1,$$

from which the result follows immediately.

6. What happens if the condition (3) in Theorem 1 is replaced by the very simple condition

$$(\sum_{s=0}^{n} |A_{n,s} - \tilde{A}_{n,s}|)^{1/n} \to 0, \; n \to \infty, \tag{18}$$

which is obviously stronger than $\gamma = 1$?

It is not difficult to see that (18) is equivalent to the following condition. There exists a sequence ε_n, $\varepsilon_n \to 0$, and there exists for every n a $1 - 1$ correspondence between $\{z_{kn}\}$ and $\{\tilde{z}_{kn}\}$ such that the distance of corresponding elements is $\leqq \varepsilon_n^n$. Informally said, this means that one system of interpolating nodes has been obtained from the other by a perturbation, a perturbation that tends to zero faster than exponentially as $n \to \infty$.

THEOREM 3. *If two systems of interpolating nodes* $[z_{kn}]$ *and* $[\tilde{z}_{kn}]$ *satisfy* $|z_{kn} - \tilde{z}_{kn}| \leqq \varepsilon_n^n$, *for* $k = 1, 2, \ldots, n$; $n = 1, 2, \ldots$ *where* $\varepsilon_n \to 0$, $n \to \infty$, *then* $p_n(z, f) - \tilde{p}_n(z, f) \to 0$, $n \to \infty$ *for every* z.

To prove this theorem, we have just to go through a part of the proof of Theorem 1, from formula (6) to formula (16), and to make only two modifications—to treat the whole sum in (11) as Σ_1, and to replace (14) by (18).

7. Let us make a final remark. The assumptions of Theorem 1 imply

$$\sum_{s=0}^{n} |A_{n,s} - \tilde{A}_{n,s}| \leqq 2(1+d)^n.$$

If we make one additional assumption in that theorem, namely

$$\sum_{s=0}^{n} |A_{n,s} - \tilde{A}_{n,s}| \leqq \alpha^n, \tag{19}$$

then if $\alpha < 1 + d$, this implies a stronger conclusion than in Theorem 1. More precisely, let (2), (3) and (19) hold, and let

$$R > 2d + [\alpha(1+d)]^{1/\gamma} > 1, \tag{20}$$

and also

$$R > 2d + [\alpha(1+d)]^{1/(1+\gamma)}. \tag{21}$$

Then, for every function f analytic in $|z| < R$, $p_n(z, f) - \tilde{p}_n(z, f) \to 0$, $n \to \infty$ for

$$|z| < \frac{1}{\alpha(1+d)} (R - 2d)^{1+\gamma}. \tag{22}$$

To obtain this refinement of Theorem 1, only minor modifications in the proof of that theorem are necessary. We replace (4) by the condition $\rho > 2d + [\alpha(1+d)]^{1/\lambda}$, and restrict z to the circumference $|z| = 1/\alpha(1+d)$ $(\rho - 2d)^{1+\lambda}$. The conditions (20) and (21) insure that (6) holds and that instead of (17) we have

$$|z|^{n-r}|t|^{n-s} \leqq \left(\frac{\rho(\rho - 2d)}{\alpha(1+d)}\right)^n \text{ if } r \geqq \lambda n \text{ or } s \geqq \lambda n. \tag{23}$$

We use (19) to obtain

$$\begin{aligned} &\sum_{r \geqq \lambda n \text{ or } s \geqq \lambda n} |\tilde{A}_{n,r}A_{n,s} - A_{n,r}\tilde{A}_{n,s}| \\ &\qquad \leqq 2\sum_r |\tilde{A}_{n,r}| \sum_s |A_{n,s} - \tilde{A}_{n,s}| \\ &\qquad \leqq 2(1+d)^n \alpha^n, \end{aligned}$$

and we use this last estimate and (23) to deduce that

$$\begin{aligned} |\Sigma_2| &\leqq \sum_{r \geqq \lambda n \text{ or } s \geqq \lambda n} |\tilde{A}_{n,r}A_{n,s} - A_{n,r}\tilde{A}_{n,s}|\, |z|^{n-r}|t|^{n-s} \\ &\leqq 2\rho^n(\rho - 2d)^n. \end{aligned}$$

This refinement of Theorem 1, which gives as the domain of overconvergence the disc (22) is of interest because it generalizes not only Theorem 1, but also Theorem 3. Namely, if the conditions (2) and (18) hold, if $R > 2d > 1$ and if f is analytic in $|z| < R$, then for every $\alpha > 0$ sufficiently small, (19) holds for all n sufficiently large, (20) and (21) are satisfied, and so $p_n(z, f) - \tilde{p}_n(z, f) \to 0$, $n \to \infty$ in every disc $|z| < 1/\alpha(1 + d)(R - 2d)^{1+r}$, with any $\alpha > 0$ sufficiently small. But that means that $p_n(z, f) - \tilde{p}_n(z, f) \to 0$, $n \to \infty$ for every z.

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