ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 3, Summer 1981

THE MODULAR FUNCTION AND THE MODULUS OF A DOUBLY-CONNECTED REGION

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ABSTRACT. Let D be a doubly connected region and let $K = K(z, \bar{z})$, $z \in D$, be its Szegö kernel. One then forms the conformal invariant $J(z) = K^{-2}\,\partial_z\bar{\partial}_z \log K$ and also finds $\alpha_D = \max_{z\in D} J(z)$. For $\beta = \beta_D = 4\pi^2/\alpha_D$, one has $\beta \in (0, 1)$. Let Mod $D \in (1, \infty)$ be the modulus of D. Then

$$(\text{Mod } D)^{-1} = \sum_{n=0}^{\infty} \frac{\delta_n}{2^{4n+1}}\left[\frac{1-(1-\beta)^{1/4}}{1+(1-\beta)^{1/4}}\right]^{4n+1},$$

where $\{\delta_n\}_{n=0}^{\infty}$ is a sequence of positive integers arising from the inversion of the modular function; thus $\delta_0 = 1$, $\delta_1 = 2$, $\delta_2 = 15$, $\delta_3 = 150, \ldots$. The series converges rapidly and usually the first two terms suffice. A truncation error analysis is provided.

1. **Introduction.** Our recent work [3] conceals in it a rather interesting relationship between the modular function, the analytic capacity and the modulus of a doubly-connected region. This relationship may be exploited to yield an efficient method for determining the modulus of a doubly-connected region. Basically, this relationship can be described in the following way. Let D be a doubly-connected region with no degenerate boundary component and let $C(z)$ be its analytic capacity at $z \in D$ (see definition below). One then forms the well-defined conformal invariant

$$J(z) = \pi^2 C^{-2} \Delta \log C, \quad C = C(z)$$

where Δ denotes the usual Laplace operator $\Delta = 4\partial_z\bar{\partial}_z$. We shall use the fact (see [3]) that $J(z) \geqq 4\pi^2$ for all $z \in D$ and that, within a proper approach, $J(z) = 4\pi^2$ for $z \in \partial D$. We define $\alpha_D = \max_{z\in D} J(z)$, $\beta_D = 4\pi^2/\alpha_D$ and thus $\beta \equiv \beta_D \in (0, 1)$. Let r^{-1} $(0 < r < 1)$ be the modulus of D. Then

$$r = \sum_{n=0}^{\infty} \frac{\delta_n}{2^{4n+1}}\left[\frac{1-(1-\beta)^{1/4}}{1+(1-\beta)^{1/4}}\right]^{4n+1}, \tag{1.1}$$

where $\{\delta_n\}_{n=0}^{\infty}$ is a sequence of positive integers arising from the well-known inversion of the modular function (see Weierstrass [10, p. 276]). Thus $\delta_0 = 1$, $\delta_1 = 2$, $\delta_2 = 15$, $\delta_3 = 150$, $\delta_4 = 1{,}707$, $\delta_5 = 20{,}910$,

Received by the editors on August 6, 1979.

$\partial_6 = 261{,}416, \ldots$. The series converges rapidly and usually the first two terms of this series suffice.

The modulus of D, therefore, is determined at once via (1.1) provided the value of β or of α_D is known. It is required, therefore, to effectively compute the invariant $J(z)$. For this purpose we shall assume that the boundary ∂D of D is rectifiable. Then, by a familiar identity due to Garabedian (cf. [**1**, p. 118]), $K(z, \bar{z}) = 2\pi C(z)$, where $K(z, \bar{\xi})$ is the Szegö kernel for D, $z, \xi \in D$. Since $K(z, \bar{\xi})$ may be expressed as a bilinear expansion of an orthonormal basis of analytic functions belonging to the Hardy-Szegö space $H_2(\partial D)$, we evidently have an effective method for computing $C(z)$ and therefore also $J(z)$.

In numerical computations we must replace the infinite sum defining $K(z, \bar{z})$ by partial finite sums. The truncation error committed by this replacement, however, has already been estimated by Nehari (see, for example, Nehari [**7**, p. 392]). Such an estimate is not available for the analogous reduced Bergman kernel function for D. Since we actually are using $J(z)$ for computing the modulus of D, it is desirable to provide a bound for the truncation error in computing $J(z)$. This will be done here for the more general case where D may even be allowed to be a p-connected region, $1 \leqq p < \infty$. As a result of the above one obtains an effective and rapid procedure for computing the modulus of a doubly-connected region D. Numerical examples based on this algorithim, smilar to those employed in [**2**, **4**, **5**, **9**], will be elaborated elsewhere.

A somewhat similar analysis for determining the modulus of D was also conducted by Zarankiewicz [**12**] by considering the reduced Bergman kernel for D. However, in contrast to the results in [**12**] the results here have the following additional features: (i) an estimate for the truncation error in computing $K(z, \bar{z})$ or $J(z)$ is available and, moreover, (ii) the modulus of D is explicitly given by a rapidly convergent series (1.1) arising from the inversion of the modular function.

We should remark here that, since doubly connected regions arise as tori cut along the image of a line, the connection with modular functions is rather natural, of one takes into account Kelvin's method of images.

To make this paper self contained as much as possible, some of the notions and the results of [**3**] are repeated here. Consequently, this paper can be read, to some extent, independently of [**3**].

2. **General theory.** Let D be a plane region $\notin O_{AB}$ (i.e., D carries a non-constant bounded analytic function.). The class $H_\infty(D)$ designates the Banach space of bounded analytic functions f in D nomred by $\|f\|_\infty = \sup_{z \in D}|f(z)|$. For $\xi \in D$, the analytic capacity $C(\xi) = C_D(\xi)$ is defined by

$$C(\xi) = \max\{|f'(\xi)| : f \in H_\infty(D), \|f\|_\infty \leqq 1, f(\xi) = 0\}.$$

The maximum is uniquely (up to a rotation) attained by the Ahlfors function $F(z) = F(z\colon \xi)$. Thus, $\|F\|_\infty = 1$, $F(\xi) = 0$ and $F'(\xi) = C(\xi)$. Evidently, the metric $C(z)|dz|$ is conformally invariant and one can show that $C(z)$ is real analytic in D (see [**8**]). Consequently, one can introduce the curvature of this metric

$$\kappa(z) = -C^{-2}\,\Delta \log C,\ C = C(z).$$

This is, of course, a conformal invariant. It is perhaps more convenient, however, to redefine this invariant by introducing $J(z) = -\pi^2\kappa(z)$.

In [**3**, **8**] it was shown that $J(\xi) \geqq 4\pi^2$ for each $\xi \in D$. Moreover, if the Ahlfors function $F(z\colon\xi)$ has a zero ξ_0 in D other than ξ, then $J(\xi) > 4\pi^2$. Especially, if $\mathscr{D}_p$, $1 \leqq p < \infty$, denotes the class of all p-connected regions with no degenerate boundary component, then for $D \in \mathscr{D}_p$ and any $\xi \in D$, $J(\xi) \geqq 4\pi^2$ with equality if and only if $p = 1$ (see [**3**]). Other properties of the curvature of the analytic capacity were studied in greater detail in [**3**].

For future reference we denote by $\mathscr{D}_p^{(r)}$ the subclass of $\mathscr{D}_p$, $1 \leqq p < \infty$, consisting of those regions whose boundary components are rectifiable. Also, we let $\mathscr{D}_p^{(a)}$ designate the subclass of $\mathscr{D}_p^{(r)}$ consisting of those regions whose boundary components are closed analytic Jordan curves.

Let $D \in \mathscr{D}_p^{(r)}$ and denote by $H_2(\partial D)$ the Hardy-Szegö space of D. This a Hilbert space of analaytic functions in D with the scalar product

$$(f, g) = \int_{\partial D} f(z)\overline{g(z)}\,|dz|,\ (\|f\|^2 = (f, f) < \infty).$$

The integration is carried over the boundary values of the analytic functions f and g (this refers to an arbitrary non-tangential approach). The space $H_2(\partial D)$ admits a reproducing kernel $K(z, \bar{\xi})$ which is the classical Szegö kernel for D.

According to the previously quoted result of Garabedian [**1**, p. 118], $C(\xi) = 2\pi K(\xi, \bar{\xi})$ and $F(z\colon \xi) = K(z, \bar{\xi})/L(z, \xi)$. Here, $L(z, \xi)$ is the adjoint kernel of $K(z, \bar{\xi})$ satisfying the boundary relation

$$i\overline{K(z, \bar{\xi})}|dz| = L(z, \xi)dz,\ \xi \in D, \tag{2.1}$$

for almost all $z \in \partial D$. Therefore, $|F(z)| \equiv 1$ for almost all $z \in \partial D$ and, obviously, $|F(z)| < 1$ throughout D. Moreover, the function

$$\ell(z, \xi) = L(z, \xi) - \frac{1}{2\pi}\left(\frac{1}{z - \xi}\right) \tag{2.2}$$

is antisymmetric and analytic for $(z, \xi) \in D \times D$. Also, for a fixed $\xi \in D$, $(z - \xi)L(z, \xi)$ does not vanish in D and $F(z) = F(z\colon\xi)$ vanishes at $z = \xi$

and $z = \overline{b_j(\xi)}$, $1 \leqq j \leqq p - 1$. Here, $\overline{b_j(\xi)} \in D$, $1 \leqq j \leqq p - 1$, are the $p - 1$ (possibly repeated) zeros of $K(z, \bar{\xi})$ and $b_j(\xi)$ are analytic in $\xi \in D$.

We have, of course,

$$f(\xi) = (f, K(\cdot, \bar{\xi})) = \int_{\partial D} f(z)\overline{K(z, \bar{\xi})}|dz|, \; \xi \in D, \tag{2.3}$$

for each $f \in H_2(\partial D)$. Let $\{\phi_n\}$ be any orthonormal basis of $H_2(\partial D)$. Then

$$K(z, \bar{\xi}) = \sum_{n=1}^{\infty} \phi_n(z)\overline{\phi_n(\xi)}; \; z, \xi \in D, \tag{2.4}$$

where the bilinear sum on the right converges absolutely and uniformly on compacta of D.

For practical computation of the kernel function by means of the bilinear expansion (2.4) we may assume, to avoid unessential difficulties, that $D \in \mathscr{D}_p^{(a)}$, $1 \leqq p < \infty$. These smoothness requirements on ∂D can be relaxed considerably. For example, it is sufficient to require that each boundary component of D is of class piecewise-C^1. Let $\Gamma_1, \ldots, \Gamma_p$ be the boundary components of D, and let α_j be a point inside the "hole" of D which is surrounded by Γ_j. It is then well-known that the functions $\{(z - \alpha_j)^{-n}\}$, $1 \leqq j \leqq p$; $n = 0, 1, \ldots$, span $H_2(\partial D)$. Moreover, when $\infty \notin D$ we may take, say, $\alpha_p = \infty$ and thus $(z - \alpha_p)^{-n}$ is replaced by z_n (see, for example, [7, p. 372]). A standard application of the Gram-Schmidt procedure on the above set of functions yields an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ of $H_2(\partial D)$, where we may also assume that $\phi_1(z) \equiv d^{-1/2}$, d being the length of ∂D.

We conclude this section by observing the following simple proposition.

PROPOSITION 1. *Let $z, \xi \in D$. Then*

$$\int_{\partial D} \ell(t, \xi)\overline{f(t)}|dt| = -\frac{1}{2\pi}\int_{\partial D} \frac{1}{t - \xi}\overline{f(t)}|dt|, \; f \in H_2(\partial D), \tag{2.5}$$

and

$$\ell(z, \xi) = -\frac{1}{2\pi}\int_{\partial D} \frac{1}{t - \xi} K(z, \bar{t})|dt|. \tag{2.6}$$

Moreover,

$$K(z, \bar{\xi}) = \Gamma(z, \bar{\xi}) - (\ell(z, \cdot), \ell(\xi, \cdot)) \tag{2.7}$$

where $\Gamma(z, \bar{\xi})$ is the "geometric" kernel

$$\Gamma(z, \bar{\xi}) = \frac{1}{4\pi^2}\int_{\partial D} \frac{|dt|}{(t - z)\overline{(t - \xi)}}. \tag{2.8}$$

PROOF. For $f \in H_2(\partial D)$ we have, in view of (2.1),

$$\int_{\partial D} L(t, \xi)\overline{f(t)}|dt| = i\int_{\partial D}\overline{f(t)K(t, \bar{\xi})dt} = 0,$$

and using (2.2), (2.5) follows. Putting $f(t) = K(t, \bar{z})$ in (2.5) and using (2.3) we deduce that

$$\ell(z, \xi) = \int_{\partial D} \ell(t, \xi)\overline{K(t, \bar{z})}|dt| = -\frac{1}{2\pi}\int_{\partial D}\frac{1}{t-\xi}\overline{K(t, \bar{z})}|dt|,$$

and, since $K(t, \bar{z})$ is Hermitian, (2.6) follows. Next, putting $f(t) = \ell(t, z)$ in (2.5) and noting that $\ell(t, z)$ is antisymmetric we obtain

$$\begin{aligned}
(\ell(z, \cdot), \ell(\xi, \cdot)) &= \int_{\partial D} \ell(z, t)\overline{\ell(\xi, t)}|dt| \\
&= -\int_{\partial D} \ell(t, z)\overline{\ell(\xi, t)}|dt| \\
&= \frac{1}{2\pi}\int_{\partial D}\frac{1}{t-z}\overline{\ell(\xi, t)}|dt| \\
&= \frac{1}{2\pi}\int_{\partial D}\frac{1}{t-z}\left[\overline{L(\xi, t)} - \frac{1}{2\pi}\overline{\frac{1}{\xi - t}}\right]|dt| \\
&= \Gamma(z, \bar{\xi}) - \frac{1}{2\pi}\int_{\partial D}\frac{\overline{L(t, \xi)}}{t-z}|dt| \\
&= \Gamma(z, \bar{\xi}) - \frac{1}{2\pi i}\int_{\partial D}\frac{K(t, \bar{\xi})}{t-z}\,dt \\
&= \Gamma(z, \bar{\xi}) - K(z, \bar{\xi}),
\end{aligned}$$

where (2.1) and (2.2) have been used. This concludes the proof.

Corollary 1. *Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $H_2(\partial D)$. Then, for $z, \xi \in D$,*

$$\ell(z, \xi) = \sum_{n=1}^{\infty} a_n(\xi)\phi_n(z)$$

with

$$a_n(\xi) = -\frac{1}{2\pi}\int_{\partial D}\frac{\overline{\phi_n(t)}}{t-\xi}|dt|,$$

and

$$\Gamma(z, \bar{\xi}) - K(z, \bar{\xi}) = \sum_{n=1}^{\infty} a_n(z)\overline{a_n(\xi)}.$$

Proof. This follows from (2.6) and (2.7).

3. **Error estimates.** Let

$$K_m(z, \bar{\xi}) = \sum_{n=1}^{m} \phi_n(z)\overline{\phi_n(\xi)},$$

and, write

$$R_m(z, \bar{\xi}) = K(z, \bar{\xi}) - K_m(z, \bar{\xi}) = \sum_{n=m+1}^{\infty} \phi_n(z)\overline{\phi_n(\xi)}$$

for the truncation error committed by the replacement of $K(z, \bar{\xi})$ by its partial sum $K_m(z, \bar{\xi})$; $z, \xi \in D$. Further, since

$$J(z) = \frac{1}{4}K^{-2}\Delta\log K = K^{-2}\partial_z\bar{\partial}_z\log K;\ K = K(z, \bar{z}),$$

we let

$$J_m(z) = K_m^{-2}\partial_z\bar{\partial}_z\log K_m;\ K_m = K_m(z, \bar{z}).$$

The main purpose of this section is to provide a bound for the error $S_m(z) = J(z) - J_m(z)$ in computing $J(z)$, $z \in D$. A bound for the error $R_m(z, \bar{\xi})$ is by now classical and is due to Nehari (see, for example, [7, p. 392]). However, since, eventually, the computation of $S_m(z)$ involves that of $R_m(z, \bar{z})$, we find it convenient to also provide here a somewhat different proof for Nehari's result.

Evidently, the kernel

$$Q(z, \bar{\xi}) = \Gamma(z, \bar{\xi}) - K(z, \bar{\xi}) \tag{3.1}$$

is positive definite in D and so is the error kernel $R_m(z, \bar{\xi})$. This leads to Nehari's estimate, namely, the following proposition.

PROPOSITION 2. *Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $H_2(\partial D)$ and let $z, \xi \in D$. Then $|R_m(z, \bar{\xi})|^2 \leqq R_m(z, \bar{z})R_m(\xi, \bar{\xi})$ and*

$$R_m(z, \bar{z}) \leqq \Gamma(z, \bar{z}) - \sum_{n=1}^{m} [|a_n(z)|^2 + |\phi_n(z)|^2]. \tag{3.2}$$

Equality in (3.2) *holds if and only if* $m = \infty$.

PROOF. We have

$$\begin{aligned} R_m(z, \bar{z}) &= K(z, \bar{z}) - K_m(z, \bar{z}) \\ &= \Gamma(z, \bar{z}) - Q(z, \bar{z}) - K_m(z, \bar{z}) \\ &= \Gamma(z, \bar{z}) - \sum_{n=1}^{\infty} |a_n(z)|^2 - K_m(z, \bar{z}) \\ &\leqq \Gamma(z, \bar{z}) - \sum_{n=1}^{m} |a_n(z)|^2 - K_m(z, \bar{z}) \end{aligned}$$

and the result follows.

We proceed now to obtain an estimate for $J_m(z)$. For a positive C^2 function $f(z)$, defined in an open subset of the plane, we define

$$J(f) = f^{-2}\partial_z\bar{\partial}_z \log f, f = f(z).$$

clearly, for $K = K(z, \bar{z})$ and $K_m = K_m(z, \bar{z})$, $z \in D$, we have $J(z) = J(K)$ and $J_m(z) = J(K_m)$. Let $g(z)$ be another positive C^2 function defined in the same open set as that of $f(z)$. We first observe the following identity (see, for example, [**6**, p. 7]):

$$(3.3)\quad fg(f+g)[(f+g)^3J(f+g) - f^3J(f) - g^3J(g)] = |f\partial_z g - g\partial_z f|^2.$$

Putting $f = K_m$ and $g = R_m$, $R_m = R_m(z, \bar{z})$ in (3.3) we deduce that

$$\begin{aligned} &J(K) - J(K_m) \\ &\quad = -[1 - (K_m/K)^3]J(K_m) \\ &\qquad + (R_m/K)^3J(R_m) + K_m^{-1}R_m^{-1}K^{-4}|K_m\partial_z R_m - R_m\partial_z K_m|^2. \end{aligned}$$

Clearly,

$$J(K_m) = K_m^{-4}[K_m\partial_z\bar{\partial}_z K_m - |\partial_z K_m|^2] \geqq 0$$

and, similarly,

$$J(R_m) = R_m^{-4}[R_m\partial_z\bar{\partial}_z R_m - |\partial_z R_m|^2] \geqq 0.$$

Therefore,

$$\begin{aligned} &|J(K) - J(K_m)| \\ &\leqq [1 - (K_m/K)^3]J(K_m) + (R_m/K)^3J(R_m) \\ &\qquad + 2K_m^{-1}R_m^{-1}K^{-4}[K_m^2|\partial_z R_m|^2 + R_m^2|\partial_z K_m|^2] \end{aligned}$$

or

$$\begin{aligned} |J(z) - J_m(z)| &\leqq [1 - (K_m/K)^3]K_m^{-3}\partial_z\bar{\partial}_z K_m + K^{-3}\partial_z\bar{\partial}_z R_m \\ &\quad + R_m^{-1}K^{-3}(2K_m/K - 1)|\partial_z R_m|^2 \\ &\quad + K_m^{-1}[2R_m/K^4 - K_m^{-3}(1 - (K_m/K)^3)]|\partial_z K_m|^2 \\ &\leqq [1 - (K_m/K)^3]K_m^{-3}\partial_z\bar{\partial}_z K_m \\ &\quad + K^{-3}\partial_z\bar{\partial}_z R_m + K_m R_m^{-1}K^{-4}|\partial_z R_m|^2 \\ &\leqq K_m^{-3}K^{-3}[K^3 - K_m^3]\partial_z\bar{\partial}_z K_m + 2K^{-3}\partial_z\bar{\partial}_z R_m \\ &\leqq 3K_m^{-3}K^{-1}R_m\partial_z\bar{\partial}_z K_m + 2K^{-3}\partial_z\bar{\partial}_z R_m. \end{aligned}$$

We are now in a position to state the following theorem.

THEOREM 1. *Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $H_2(\partial D)$ with $\phi_1(z) \equiv d^{-1/2}$, where d is the length of ∂D. Let $z \in D$. Then*

$$(3.4)\qquad |S_m(z)| = |J(z) - J_m(z)| \leqq d^3(3dR_m\partial_z\bar{\partial}_zK_m + 2\partial_z\bar{\partial}_zR_m),$$

where,

$$R_m = R_m(z, \bar{z}) \leqq \Gamma(z, \bar{z}) - \sum_{n=1}^{m} [|a_n(z)|^2 + |\phi_n(z)|^2],$$

and

$$\partial_z\bar{\partial}_zK_m = \sum_{n=1} |\phi_n'(z)|^2$$

$$\partial_z\bar{\partial}_zR_m \leqq \frac{1}{4\pi^2}\int_{\partial D} \frac{|dt|}{|t - z|^4} - \sum_{n=1}^{m} [|a_n'(z)|^2 + |\phi_n'(z)|^2].$$

PROOF. We clearly have $K \geqq K_m = \sum_{n=1}^{m}|\phi_n(z)|^2 \geqq d^{-1}$. Therefore, using the above previous inequality, (3.4) follows. Next, since

$$R_m = \Gamma(z, \bar{z}) - \sum_{n=1}^{\infty} |a_n(z)|^2 - K_m(z, \bar{z}),$$

we obtain

$$\partial_z\bar{\partial}_zR_m = \partial_z\bar{\partial}_z\Gamma(z, \bar{z}) - \sum_{n=1}^{\infty} |a_n'(z)|^2 - \sum_{n=1}^{m} |\phi_n'(z)|^2$$

and the theorem follows.

Of course, once the orthonormal basis $\{\phi_n\}$ of $H_2(\partial D)$ is constructed, the invariant $J(z)$ may be formed by elementary operations, where the truncation error is estimated via Theorem 1. For the actual computation of $J(z)$ we may also use (3.1) and (3.3). In fact, let $Q = Q(z, \bar{z})$ and $\Gamma = \Gamma(z, \bar{z})$. By (3.1) $\Gamma = Q + K$ and therefore, by (3.3), we have

$$J(z) = (\Gamma/K)^3J(\Gamma) - (Q/K)^3J(Q) - \frac{1}{Q\Gamma K^4}|Q\partial_zK - K\partial_zQ|^2.$$

Here $Q = Q(z, \bar{z}) = \sum_{n=1}^{\infty}|a_n(z)|^2$ and $\Gamma = \Gamma(z, \bar{z})$ is a geometric kernel as given in (2.8).

4. **Analytic capacity on annulus.** Of particular interest, as far as problems of applications are concerned, is when D is a doubly connected region, i.e., when $D \in \mathscr{D}_2$. We shall assume that the modulus of D is r^{-1} so that D can be conformally represented as the annulus $A = \{z: r < |z| < 1\}$, $0 < r < 1$. In this case we have at our disposal the theory of elliptic functions. Using this theory, we take $\omega_1 = \pi i$ and $\omega_2 = \log r$ as half-periods of the Weierstrass $\mathscr{P}$-function, $\mathscr{P}(u) = \mathscr{P}(u; \omega_1, \omega_2)$.

As usual, we set $\omega_3 = \omega_1 + \omega_2$ and $e_j = \mathscr{P}(\omega_j)$, $j = 1, 2, 3$. Now, on the boundary of the rectangle $0, \pi i, \log r + \pi i, \log r$, $\mathscr{P}(u)$ attains values increasing monotonically from $-\infty$ to $+\infty$ and thus $e_1 < e_3 < e_2$. Further, for $\mathscr{P} = \mathscr{P}(u)$, we have

(4.1) $$(\mathscr{P}')^2 = 4(\mathscr{P} - e_1)(\mathscr{P} - e_2)(\mathscr{P} - e_3) = 4\mathscr{P}^3 - g_2\mathscr{P} - g_3$$

and thus $e_1 + e_2 + e_3 = 0$,

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = 2(e_1^2 + e_2^2 + e_3^2),$$

and $g_3 = 4e_1e_2e_3$. Consequently, $e_1 < 0$, $e_2 > 0$, $g_2 > 0$ and

(4.2) $$g_2 - 12e_3^2 = -4(e_3 - e_1)(e_3 - e_2) > 0$$

Since the sequence $\{z^n/[2\pi(1 + r^{2n+1})]^{1/2}\}_{n=-\infty}^{\infty}$ forms an orthonormal basis for $H_2(\partial A)$, the Szegö kernel for A is given by

$$K_A(z, \bar{\xi}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(z\bar{\xi})^n}{1 + r^{2n+1}}.$$

With the aid of this expression one can, of course, compute the adjoint kernel $L(z, \xi)$ and, therefore, the Ahlfors function $F(z\colon \xi)$ of A, by just appealing to (2.6). Now, using some known identities from the theory of elliptic functions, one can show (see [3]) that

(4.3) $$K_A^2(z, \bar{\xi}) = \frac{1}{4\pi^2 z\bar{\xi}}\{\mathscr{P}(\log z\bar{\xi}; \omega_1, \omega_2) - e_3\}.$$

From (4.3) we obtain

$$2\partial_z \log K_A(z, \bar{\xi}) = \frac{1}{z}\left(-1 + \frac{\mathscr{P}'}{\mathscr{P} - e_3}\right),$$

where $\mathscr{P} = \mathscr{P}(u; \omega_1, \omega_2)$ and $u = \log z\bar{\xi}$. Hence

(4.4) $$2\partial_z\bar{\partial}_\xi \log K_A(z, \bar{\xi}) = \frac{1}{z\bar{\xi}} \frac{\mathscr{P}''(\mathscr{P} - e_3) - (\mathscr{P}')^2}{(\mathscr{P} - e_3)^2}$$

Using (4.1), from which also follows that $2\mathscr{P}'' = 12\mathscr{P}^2 - g_2$, and (4.4) we deduce that

$$J_A(z, \bar{\xi}) \equiv \frac{1}{K_A^2(z, \bar{\xi})} \partial_z\bar{\partial}_\xi \log K_A(z, \bar{\xi}) = 4\pi^2\left[1 + \frac{1}{4}\frac{g_2 - 12e_3^2}{(\mathscr{P} - e_3)^2}\right]$$

Finally, in view of (4.2),

$$J_A(z, \bar{\xi}) = 4\pi^2\left[1 + \frac{(e_3 - e_1)(e_2 - e_3)}{[\mathscr{P}(\log z\bar{\xi}; \omega_1, \omega_2) - e_3]^2}\right]$$

for $z, \xi \in A$ and we note that $J_A(z) = J_A(z, \bar{z})$ is the previously considered invariant of A.

Let $\rho = |z|$, and hence $\rho \in (r, 1)$ for $z \in A$. Evidently, $J_A(z) = J_A(\rho) = J_A(r/\rho)$, $z \in A$, where

$$J_A(\rho) = 4\pi^2\left[1 + \frac{(e_3 - e_1)(e_2 - e_3)}{(\mathscr{P}(2\log\rho) - e_3)^2}\right],$$

$$\mathscr{P}(2\log\rho) = \mathscr{P}(2\log\rho; \omega_1, \omega_2).$$

From all these we easily obtain the following crucial theorem.

THEOREM 2. *The function $J_A(\rho)$ satisfies $J_A(\rho) > 4\pi^2$ for $\rho \in (r, 1)$ and $J_A(r) = J_A(1) = 4\pi^2$. It has only one extremal point in $(r, 1)$. This extremal point is at $\sqrt{r}$ and it is a maximal point. The value of this maximum is given by*

$$J_A(\sqrt{r}) = 4\pi^2 \frac{e_2 - e_1}{e_2 - e_3}.$$

Now, we observe that, as a function of $r \in (0, 1)$,

$$\lambda\left(\frac{1}{\pi i} \log r\right) = \frac{e_3 - e_2}{e_1 - e_2},$$

where $\lambda(\tau)$; $\operatorname{Im} \tau > 0$, is the familiar modular function. Further, using Jacobi theta functions notation (as found in [**11**, pp. 462–490]) we can write

$$Q(r) \equiv k^2(r) = \left[\frac{\theta_2(0, r)}{\theta_3(0, r)}\right]^4 = \lambda\left(\frac{1}{\pi i} \log r\right)$$

and, therefore, $J_A(\sqrt{r}) = 4\pi^2/Q(r)$, $r \in (0, 1)$.

5. **Determination of the modulus.** We proceed now to determine the modulus r^{-1} of $D \in \mathscr{D}_2$. The doubly connected region D is conformally equivalent to the annulus $A = \{\omega : r < |\omega| < 1\}$, $0 < r < 1$. We consider the invariant $J(z)$ of D. Thus, in view of the conformal invariance and Theorem 2, we have

$$\alpha_D \equiv \max_{z \in D} J(z) = J_A(\sqrt{r}). \tag{5.1}$$

Because of (4.5) and the monotonicity of $Q(r)$ in $r \in (0, 1)$, the modulus of D is determined at once from equation (5.1). We can, however, say much more; namely, we will determine the modulus explicitly.

We write $\beta = \beta_D = 4\pi^2/\alpha_D$ and note that $\beta \in (0, 1)$. Therefore, using (4.5) and (5.1), the determination of the modulus is reduced to solving for $r \in (0, 1)$ the classical equation

$$Q(r) = \beta; \ \beta = \beta_D \in (0, 1). \tag{5.2}$$

Following a familiar device due to Weierstrass [**10**, p. 276] (see also [**11**, p. 486]), this equation admits the unique solution

$$r = r(\varepsilon) = \sum_{n=0}^{\infty} \delta_n \varepsilon^{4n+1}, \tag{5.3}$$

where

$$\varepsilon = \frac{1}{2} \frac{1 - (1 - \beta)^{1/4}}{1 + (1 - \beta)^{1/4}}; \ 0 < \varepsilon < 1/2.$$

The series in (5.3) converges for $|\varepsilon| \leqq 1/2$ and clearly,

$$1 = \sum_{n=0}^{\infty} \delta_n 2^{-4n-1}. \tag{5.4}$$

The $\{\delta_n\}_{n=0}^{\infty}$ are well determined positive integers (see [**10**, p. 276]) with $\delta_0 = 1$, $\delta_1 = 2$, $\delta_2 = 15$, $\delta_3 = 150$, $\delta_4 = 1707$, $\delta_5 = 20{,}910$, $\delta_6 = 261{,}416, \ldots$. One also notes that the series (5.3) converges rapidly and that usually the first two terms of this expansion suffice (see [**11**, p. 486]).

An estimate for the truncation error of the series (5.3) can be given as follows:

$$P_m(\varepsilon) \equiv \sum_{n=0}^{\infty} \delta_n \varepsilon^{4n+1} - \sum_{n=0}^{m-1} \delta_n \varepsilon^{4n+1} = \sum_{n=m}^{\infty} \delta_n \varepsilon^{4n+1}$$

and using (5.4) we deduce that

$$P_m(\varepsilon) \leqq (2\varepsilon)^{4m+1} \sum_{n=m}^{\infty} \delta_n 2^{-4n-1} = (2\varepsilon)^{4m+1}\Big[1 - \sum_{n=0}^{m-1} \delta_n 2^{-4n-1}\Big].$$

All these lead to the following corollary to Theorem 2.

COROLLARY 2. *Let* $r^{-1} = \operatorname{Mod} D$, $r \in (0, 1)$, *be the modulus of* $D \in \mathscr{D}_2$ *and let* $\alpha_D = \max_{z \in D} J(z)$, $\beta \equiv \beta_D = 4\pi^2/\alpha_D$. *Then* $\beta \in (0, 1)$ *and*

$$(\operatorname{Mod} D)^{-1} = \sum_{n=0}^{\infty} \frac{\delta_n}{2^{4n+1}} \left[\frac{1 - (1-\beta)^{1/4}}{1 + (1-\beta)^{1/4}}\right]^{4n+1},$$

where the δ_n*'s are as before. The truncation error* $P_m(\beta)$ *of the above series satisfies*

$$0 < P_m(\beta) \leqq \left[1 - \sum_{n=0}^{m-1} \frac{\delta_n}{2^{4n+1}}\right] \cdot \left[\frac{1 - (1-\beta)^{1/4}}{1 + (1-\beta)^{1/4}}\right]^{4m+1}; \; m = 1, 2, \ldots .$$

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