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THE REDUCED THEORY OF QUADRATIC FORMS

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Two central problems in the theory of quadratic forms over fields are the computation of the set of equivalence classes of quadratic forms over a field (which reduces to the computation of the Witt ring) and the computation of the set of values taken on by a given quadratic form (or equivalently, the determination of when a given quadratic form represents zero nontrivially). Recently, a "reduced theory" of quadratic forms has provided a partial solution to these problems by the computation of the Witt ring modulo its nil radical and the computation of the additive semigroup generated by the value set of a quadratic form. Our intention here is to provide an efficient and fairly self-contained exposition of these results to the reader knowing the rudiments of the algebraic theory of quadratic forms (mainly, the definition of the Witt ring) and Pfister's local-global principle. These prerequisites can all be found in a few chapters of either of the books of Lam, Scharlau or Milnor-Husemoller [7, **14**, **10**]. The necessary valuation theory can be found, for example, in Ribenboim's book [**13**].

Some of the results and many of the arguments here are new. Little use is made of the formalism of residue class forms and none of semiorderings. We do emphasize real-valued places (which allow applications of the Stone-Weierstrass theorem as well as valuation theory) and closely examine the maximal preorders over which a form is anisotropic. In spite of innovations, however, our main goals are expository and we have borrowed on accasion from the arguments as well as the results of others (especially including Becker and Bröcker [**1**]).

In §1 we introduce the notions of equivalence and isotropy of forms with respect to a preorder. The questions of isotropy and representability with respect to an arbitrary preorder are reduced in §2 to preorders consistent with only finitely many real-valued places. The main theorems on the structure of the reduced Witt ring and on representability are in §§3 and

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4, respectively. Isotropy is discussed in §5, which may be read immediately after §2.

Throughout the paper, K will denote a formally real field. **Z** and **R** denote the sets of integers and real numbers, respectively. "$A^{\cdot}$" denotes the group of units of a unitary ring A (so, $\mathbf{Z}^{\cdot} = \{1, -1\}$). "$f|B$" denotes the restriction of a map f to a subset B of its domain. Our notation involving quadratic forms is standard. For $a_1, \ldots, a_n \in K^{\cdot}$, $\langle a_1, \ldots, a\rangle_n$ will denote the equivalence (i.e., isometry) class of the form $\sum_i a_i x_i^2$ and $\langle\langle a_1, \ldots, a_n\rangle\rangle = \otimes_{i=1}^n \langle 1, a_i\rangle$. $\rho \perp \rho'$ denotes the orthogonal sum of the forms ρ and ρ'. $X = X_K$ will denote the set of orderings of K, given the coarsest topology with $X(A) = \{P \in X\colon P \supseteq A\}$ open for all finite subsets A of K. X is compact and Hausdorff. (It can be regarded as a closed subset of the product space $\{1, -1\}^{K^{\cdot}}$.) For $P \in X$ and a form ρ, $\operatorname{sign}_P(\rho)$ denotes the signature of ρ at P.

Throughout the paper, T will denote a fixed "preorder" of K, i.e., an additively and multiplicatively closed subset of K containing $K^2 = \{a^2\colon a \in K\}$. Let $T^{\cdot} = T\backslash\{0\}$. $T^{\cdot}$ is a subgroup of $K^{\cdot}$ ($x^{-1} = x(x^{-1})^2$). We will assume throughout that $T \neq K$. Since the form $\langle 1, -1\rangle$ is universal, this is equivalent to assuming that $-1 \notin T$, and hence that $T^{\cdot}$ is additively closed. We are mainly interested in the case that T is $D(\infty)$, the set of sums of squares in K. The use of arbitrary preorderings allows inductive (Zorn's lemma) arguments. Also, much of the "local" theory can be conveniently phrased in terms of preorders (orderings are preorders and real-valued places can be thinly disguised as preorders).

1. **Form theory modulo preorders.** Throughout this section, $\rho = \langle a_1, \ldots, a_n\rangle$ and $\rho' = \langle b_1, \ldots, b_m\rangle$ denote forms over K. We let $D_T(\rho) = Ta_1 + \cdots + Ta_n$ denote the "T-value set of ρ". The forms ρ and ρ' are called *T-similar* (written $\rho \sim \rho'$ (mod T)) if $\operatorname{sign}_P(\rho) = \operatorname{sign}_P(\rho')$ for all $P \in X(T)$. They are called *T-equivalent* if they are T-similar and have the same dimension. The T-equivalence class of ρ is denoted ρ_T.

We now show that many basic properties of equivalence classes of quadratic forms hold for T-equivalence classes. For more details see **[2]**.

The reader might note our use of Pfister's local-global principle in the proof of our first lemma. This will be our only application of it, except for its use in interpreting our results in the case $T = D(\infty)$.

LEMMA 1.1. *Let $a \in K^{\cdot}$. Then $a \in D_T(\rho)$ if and only if $\rho_T = (\langle a\rangle \perp \rho'')_T$ for some form ρ''.*

PROOF. ($\Rightarrow$) There exist $t_i \in T$ with $a = \sum a_i t_i$. Let $a_1' = a_i$ if $t_i = 0$ and $a_1' = a_i t_i$ otherwise. Then a is represented by $\langle a_1', \ldots, a_n'\rangle$, so $\rho_T = \langle a_1', \ldots, a_n'\rangle_T = (\langle a\rangle \perp \rho'')_T$ for some ρ'' [7, p. 9].

($\Leftarrow$) Since $P \mapsto \operatorname{sign}_P(\rho)$, $P \in X$, is continuous, there is an open set

$U \supseteqq X(T)$ such that ρ and $\rho'' \perp \langle a\rangle$ have the same signature on U. Since $X\backslash U$ is compact, it has a finite cover of the form $X(-t_1), \ldots, X(-t_s)$ where $t_i \in T$. Then $\rho \otimes \langle\langle t_1, \ldots, t_s\rangle\rangle$ and $(\langle a\rangle \perp \rho'') \otimes \langle\langle t_1, \ldots, t_s\rangle\rangle$ have the same signature throughout X (the signatures are zero off of U). Pfister's local-global principle **[11**, Satz 22] then says $2^r(\rho \otimes \langle\langle t_1, \ldots, t_s\rangle\rangle)$ is equivalent to $2^r(\langle a\rangle \perp \rho'') \otimes \langle\langle t_1, \ldots, t_s\rangle\rangle$ for some r. Since a is represented by the latter form, it is represented by the former (which has the same value set as ρ with respect to T) and is therefore in $D_T(\rho)$.

As a corollary we obtain the following lemmas.

LEMMA 1.2. *If* $\rho_T = \rho'_T$, *then* $D_T(\rho) = D_T(\rho')$.

LEMMA 1.3. $\bigcap_{P\in X(T)} P = T$.

PROOF. "$\supseteqq$" is obvious. If $a \notin T$, then by 1.2, $\langle a\rangle$ and $\langle 1\rangle$ are not T-equivalent and so have different signatures at some element of $X(T)$. That is, $a \notin \bigcap_{P\in X(T)} P$.

LEMMA 1.4. *If* $\rho_T = \rho'_T$, *then* $\det\rho \cdot T = \det\rho' \cdot T$.

This follows from 1.3 and the identity (for all $P \in X(T)$):

$$\operatorname{sign}_P(\det\rho) = (-1)^{(1/2)(\operatorname{sign}_P(\rho)-\dim(\rho))}. \tag{1}$$

LEMMA 1.5. *If* ρ'' *is a form with* $(\rho \perp \rho'')_T = (\rho' \perp \rho'')_T$, *then* $\rho_T = \rho'_T$.

This follows immediately from the definition of T-equivalence. This cancellation lemma is also valid for T-similarity classes.

We call ρ *T-isotropic* if $\sum a_i t_i = 0$ for some $t_i \in T$, not all zero. We call ρ *T-anisotropic* if it is not T-isotropic. It is important to note that ρ T-anisotropic implies $n\rho$ is T-anisotropic for any positive integer n.

LEMMA 1.6. ρ *is T-isotropic if and only if* $\rho_T = (\langle -1, 1\rangle \perp \rho'')_T$ *for some form* ρ''.

PROOF. ($\Leftarrow$) Since $\rho_T = (\langle a_1, -a_1\rangle \perp \rho'')_T$, by 1.5, $\langle a_2, \ldots, a_n\rangle_T = (\langle -a_1\rangle \perp \rho'')_T$. Hence $-a_1 \in D_T(\langle a_2, \ldots, a_n\rangle)$ by 1.2. Hence $a_1 1 + \sum_{i>1} a_i t_i = 0$ for some $t_i \in T$.

($\Rightarrow$) $\sum a_i t_i = 0$ for some $t_i \in T$, not all zero. We may assume $t_1 \neq 0 \neq t_2$. Then $-a_1 \in D_T(\langle a_2, \ldots, a_n\rangle)$, so $\langle a_1, \ldots, a_n\rangle_T = (\langle a_1\rangle \perp \langle -a_1\rangle \perp \rho'')_T$ for some ρ'' (Lemma 1.1).

LEMMA 1.7. *There exists a T-anisotropic form* ρ'', *unique up to T-equivalence, and a unique integer* $s \geqq 0$ *with* $\rho_T = (\rho'' \perp s\langle 1, -1\rangle)_T$.

The existence of ρ'' and s follows from Lemma 1.6, while their uniqueness follows from Lemmas 1.5 and 1.6.

LEMMA 1.8. *If* ρ *and* ρ' *are T-anisotropic and T-similar, then they are T-equivalent.*

PROOF. We may suppose $\dim(\rho) \geqq \operatorname{dio}(\rho')$. Also $\dim(\rho) \equiv \operatorname{sign}_P(\rho') \equiv \operatorname{sign}_P(\rho') \equiv \dim(\rho') \pmod 2$ for any $P \in X(T)$. (Since $T \neq K$, Lemma 1.3 implies $X(T)$ is nonempty.) Hence $\rho_T = (\rho' \perp s\langle 1, -1\rangle)_T$, where $s = (1/2)(\dim \rho - \dim \rho')$. Since ρ is T-anisotropic, we must have $s = 0$ (Lemma 1.6).

The reader should note that, as expected, the calculation of the value set of a T-equivalence class of quadratic forms is closely connected with the problem of T-isotropy: *$a \in D_T(\rho)$ if and only if $\langle -a\rangle \perp \rho$ is T-isotropic* (Lemmas 1.1 and 1.6). Further, the calculation of the set of T-equivalence classes of forms reduces to the calculation of the set of T-similarity classes of (T-anisotropic) forms (Lemmas 1.7 and 1.8).

2. **Reduction to semilocal theory.** Let M denote the set of all places from K to the field of real numbers $\mathbf{R}$ (i.e., "real-valued places"); and let $M(T)$ denote the set of $\sigma \in M$ satisfying $\sigma(T) \geqq 0$. (We shall agree that $\infty \geqq 0$.) The main result of this section can now be stated.

THEOREM 2.1. *Let ρ be a T-anisotropic form of dimension n. Then ρ is S-anisotropic for some preorder $S \supseteqq T$ with $|M(S)| \leqq n/2$.*

Note that the theorem says that a form ρ is T-isotropic if and only if it is S-isotropic for all preorders $S \supseteqq T$ with $M(S)$ finite. We will also show in this section that a continuous map $f\colon X(T) \to Z$ is "representable" (by a form—see below) if and only if its restriction $f|X(S)$ is representable for all preorders $S \supseteqq T$ with $M(S)$ finite. Further results about T-isotropy and representability will follow from the relatively easy analysis in later sections of the preorders S with $M(S)$ finite.

We begin the proof of 2.1 with some basic lemmas.

LEMMA 2.2. *Let σ be a place on K with valuation ring A. Let S be a subgroup of $K^\cdot$ with $K^{\cdot 2}\sigma^{-1}(1) \subseteqq S$ and $\sigma(S \cap A^\cdot)$ additively closed. Then $S \cup \{0\}$ is a preorder, and $-1 \notin S$. Now suppose $\sigma \in M(T)$. Then $-1 \notin T\sigma^{-1}(R^{\cdot 2})$, and if U is any subgroup of $K^\cdot$ with $-1 \notin U \supseteqq T^\cdot\sigma^{-1}(R^{\cdot 2})$, then $U \cup \{0\}$ is a preorder.*

PROOF. Let $a, b \in S$. We may assume $ab^{-1} \in A$, so $\sigma(ab^{-1} + 1) = \sigma(s)$ for some $s \in S \cap A^\cdot$. Then

$$a + b = sb(s^{-1}(ab^{-1} + 1)) \in S\sigma^{-1}(1) \subseteqq S.$$

Thus S is additively closed, so $S \cup \{0\}$ is a preorder, and $-1 \notin S$. Now suppose $\sigma \in M(T)$. Then $\sigma(T) \geqq 0$, so $-1 \notin T\sigma^{-1}(R^{\cdot 2})$. Also $\sigma(U \cap A^\cdot) = \sigma(A) \cap R^{\cdot 2}$ which is additively closed. (If $\sigma(u) < 0$, $u \in U$, then $-1 = u(-u^{-1}) \in U\sigma^{-1}(R^{\cdot 2}) \subseteqq U$, a contradiction.) Thus $U \cup \{0\}$ is a preorder.

M is given the coarsest topology with the evaluation maps $\sigma \mapsto \sigma(a)$

$\cdot$ ($\sigma \in M$) continuous for all $a \in K^{\cdot}$. Here $R \cup \{\infty\}$ is regarded as the onepoint compactification of R.

LEMMA 2.3. *There eixsts a continuous surjection $\lambda: X \to M$ with $\lambda(P) = \sigma$ if and only if $\sigma(P) \geqq 0$ (for all $P \in X$, $\sigma \in M$). λ maps $X(T)$ onto $M(T)$.*

PROOF. (See **1**, §6; **6**, p. 259.) Let $p \in X$. P induces an Archimedean ordering on the residue class field, call it F, of the valuation ring $\{a \in K$: $n + a \in P$ and $n - a \in P$ for some $n \in Z\}$ [**8**, p. 272]. There is a unique embedding of F into R preserving this ordering. Let σ be the composition of this embedding with the canonical place $K \to F \cup \{\infty\}$. Then $\sigma \in M$ and $\sigma(P) \geqq 0$. If $\sigma \neq \tau \in M$, then $\sigma(x) \neq \tau(x)$ for some $x \in K^{\cdot}$. We may assume $\sigma(x) \neq \infty \neq \tau(x)$ (otherwise replace x by $(x + r)^{-1}$ for some appropriate integer r) and even that $\sigma(x) < 0 < \tau(x) < \infty$ (if necessary replace x by $\pm x + s$, for a suitable reational s). Then $-x \in P$ (otherwise $x \in P$, and $\sigma(x) \geqq 0$), so $\tau(P) \ngeqq 0$. This proves the existence of λ. Clearly, if $P \in X(T)$, then $\lambda(P) \in M(T)$. To prove continuity note that the inverse image under λ of the subbasic open set $\{\sigma \in M$: $\sigma(a) < 0\}$ of M (where $a \in K^{\cdot}$) is $\bigcup_{n \geqq 1} X(n + a, n + a^{-1}, -a)$. The surjectivity assertions follow from Lemma 2.2. (Given $\sigma \in M(T)$, let $P = U \cup \{0\}$ where U is as in 2.2 with $[K^{\cdot}: U^{\cdot}] = 2$. Then $P \in X(T)$, $\lambda(P) = \sigma$. Take $T = D(\infty)$ to get the surjectivity of λ.)

LEMMA 2.4. [**6**, Lemma 5.13]. *M is compact and Housdorff. If L is any compact subset of M, then the image of the evaluation map $\bigcap_{\sigma \in M} \sigma^{-1}(R) \to C(L, R)$ is dense in the sup-norm.*

PROOF. The compactness of M follows from the previous lemma. The closure of the image of the evaluation map contains all rationals and hence all reals. Thus, by the Stone-Weierstrass theorem it suffices to show that points are separated. Let $\sigma \neq \tau$, σ, $\tau \in L$. As in the proof of Lemma 2.3, we have $a \in K$ with $\sigma(a) < 0 < \tau(a) < \infty$. Replacing a by $a(1 + a^2)^{-1}$ if necessary, we can assume $a \in \bigcap_{\rho \in M} \rho^{-1}(R)$ (note that $\rho(1 + a^2) > 0$ for all $\rho \in M$ since ρ is real-valued). Clearly then, a separates σ and τ.

We now prove Theorem 2.1. There is a maximal (with respect to inclusion) preorder $S \supseteqq T$ with ρ S-anisotropic (Zorn's Lemma). Suppose $\sigma_1, \sigma_2, \ldots, \sigma_r$ are distinct elements of $M(S)$. Let $b_0 = 1$. We claim there exist $b_1, \ldots, b_r$ in $\bigcap_{\sigma \in M} \sigma^{-1}(R)$ such that for all $1 \leqq i \leqq r$,

i) $S + b_{i-1}S \subsetneqq S + b_iS$, and

ii) $\sigma_j(b_i) < 0$ if $j \leqq i$ and $\sigma_j(b_i) > 0$ if $j > i$ (for all $j \leqq r$).

Suppose inductively that such $b_1, \ldots, b_{t-1}$ have been found, where $1 \leqq t \leqq r$. Applying 2.4 (with $L = \{\sigma \in M(S)$: $\sigma(b_{t-1}) \leqq 0\} \cup \{\sigma_1, \ldots, \sigma_r\}$) we find $b_t \in \bigcap_{\sigma \in M} \sigma^{-1}(R)$ with $\sigma(b_t) < 0$ if $\sigma(b_{t-1}) \leqq 0$ and with condition ii) above satisfied with $i = t$. If $P \in X(S)$ has $b_{t-1} \notin P$, then $\lambda(P)(b_{t-1}) \leqq 0$, so by construction $\lambda(P)(b_t) < 0$. Hence $b_t \notin P$. Thus $S + b_{t-1} S \subseteqq Sb_t + S$ (by Lemma 1.3 applied to the preorder $S + Sb_t$) The inclusion is proper since $b_t \notin S + Sb_{t-1}$. (If $b_t = s + s'b_{t-1}$ for $s, s' \in S$, then $\sigma_t(s) = \infty$ and $\sigma_t(s'b_{t-1}/s) \geqq 0$, since $\sigma_t(b_t) < 0, \sigma_t(b_{t-1}) \geqq 0$ and $\sigma_t(S) \geqq 0$. But then $0 = 1 + \sigma_t(s'b_{t-1}/s)$, a contradiction.) This proves the claim. We may suppose $b_r = -1$.

Set $c_i = b_i b_{i-1}$ for $1 \leqq i \leqq r$. Note that $c_1 = b_1$. For $1 \leqq i \leqq r, \langle b_{i-1}, c_i\rangle_S = \langle 1, b_i\rangle_S$ (Lemmas 1.1 and 1.4). Repeated application of this identity shows that $\langle c_1, \ldots, c_r\rangle_S = \langle 1, \ldots, 1, -1\rangle_S$. For each $1 \leqq i \leqq r$ there exists a form g_i over K of dimension $n - 2$ with $\rho \sim g_i (\text{mod } S + c_i S)$ (since $S + c_i S$ is a proper extension of S; use Lemma 1.6). Then

$$\rho \otimes \langle 1, c_i\rangle \sim g_i \otimes \langle 1, c_i\rangle \ (\text{mod } S)$$

(both forms have signature zero off of $X(S + c_i S)$). Hence

$$(2r - 2)\, \rho \sim \textstyle\sum_{i \leqq r} \langle 1, c_i\rangle \otimes \rho \sim \sum_{i \leqq r} \langle 1, c_i\rangle \otimes g_i (\text{mod } S).$$

But ρ and hence $(2r - 2)\rho$ is S-anisotropic, so the dimension of the latter is no more than that of $\sum_{i \leq r} \langle 1, c_i\rangle \otimes g_i$ (Lemmas 1.7 and 1.8). That is $(2r - 2)n \leqq 2r(n - 2)$. Hence $r \leqq n/2$. Thus $|M(S)| \leqq n/2$ as claimed.

We say $f \in C(X(T), \mathbf{Z})$ is *representable* when there exists a form ρ with $f(P) = \text{sign}_P(\rho)$ for all $P \in X(T)$. The computation of the set of representable functions is equivalent, then, to the computation of the set of T-similarity classes of forms.

THEOREM 2.5. *Suppose $f \in C(X(T), \mathbf{Z})$. Then f is representable if and only iff $|X(S)$ is representable for all preorders $S \supseteqq T$ with $M(S)$ finite.*

We begin the proof of 2.5 with a lemma.

LEMMA 2.6. *Let $f \in C(X(T), \mathbf{Z})$. There exists a finite set $A = \{a_1, \ldots, a_n\}$ in K with $f(P) = f(P')$ for all $P, P' \in X(T)$ with $A \cap P = A \cap P'$. Further, $2^n f$ is represented by $\sum_\varepsilon f_\varepsilon \langle\langle a_1\varepsilon_1, \ldots, a_n\varepsilon_n\rangle\rangle$ where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ runs over $\{1, -1\}^n$ and f_ε denotes the value f takes on $X(T) \cap X(a_1\varepsilon_1, \ldots, a_n\varepsilon_n)$ (and $f_\varepsilon = 0$ if this set is empty).*

PROOF. The existence of A follows from the fact that f is uniformly continuous. The uniformity on $X(T)$ has a basis consisting of all sets $\{(P, P'): P \cap A = P' \cap A\}$ where A runs over the finite subsets of K. The rest of 2.6 follows by inspection.

PROOF OF 2.5. (compare with the proof of [**1**, Prop. 5.1]). Suppose $f|X(S)$ is representable for all $S \supseteqq T$ with $M(S)$ finite. First note that $f(P) \equiv f(P') (\text{mod } 2)$ for all $P, P' \in X(T)$. (After all, $\{P, P'\} = X(P \cap P')$, and $\lambda(X(P \cap P')) = M(P \cap P')$, so by hypothesis $f|\{P, P'\}$ is representable.) We may suppose that $2f$ is representable (by the above lemma and induction on the least s with $2^s f$ representable), say by a T-anisotropic form $\rho = \langle a_1, \ldots, a_n \rangle$. By hypothesis, for each $S \supseteqq T$ with $M(S)$ finite we can find an S-anisotropic form $\rho(S)$ and a (necessarily nonnegative) integer r_S with ρ and $2\rho(S) + r_S \langle 1, -1 \rangle$ S-equivalent (Lemmas 1.6 and 17.). For S as above and $P \in X(S)$,

$$(-1)^{n/2 - f(P)} P^{\cdot} = (\det \rho) P^{\cdot} = (-1)^{r_S} P^{\cdot}$$

(cf., formula (1) and Lemma 1.4). Since f has constant parity on $X(T)$, $r_S \equiv r_{S'} (\mathrm{mod}\ 2)$ holds for all preorders $S, S' \supseteqq T$ with $M(S)$ and $M(S')$ finite. If r_S were odd for any $S \supseteqq T$ with $M(S)$ finite, r_S would be odd, and hence nonzero, for all such S. Theorem 2.1 would then say that ρ is T-isotropic, a contradiction. Hence r_S is even for all such S. Since $a_1 \in D_S(\rho) = D_S(\rho(S) \perp r_S/2\langle 1, -1 \rangle)$, for some form $g(S)$ we have $\rho(S) \perp r_S/2\langle 1, -1 \rangle)$, and $\langle a_1 \rangle \perp g(S)$ S-equivalent. Thus $\langle a_2, \ldots, a_n \rangle$ is S-equivalent to $\langle a_1 \rangle \perp 2g(S)$, so $\langle -a_1, a_2, \ldots, a_n \rangle$ is S-isotropic for all S with $M(S)$ finite. Hence (Theorem 2.1), it is T-isotropic. Thus $\rho_T = (2\langle a_1 \rangle \perp \rho')_T$ for some form ρ' of dimension $n - 2$ (Lemma 1.1). The theorem now follows by induction on n. (The map $P \mapsto 2(f(P)) - \mathrm{sign}_P(a_1))$, $P \in X(T)$, is represented by ρ'.)

The reader may wish to skip over the next three lemmas until they are needed and proceed directly to either §3 or §5. (They are put here so that §5 may be read directly after §2.)

For $\sigma \in M(T)$ and $B \subseteqq M(T)$, set $T_\sigma = T\sigma^{-1}(R^{\cdot 2})$ and $T_B = \bigcap_{\sigma \in B} T_\sigma$. By 2.2, T_σ and T_B are preorders.

LEMMA 2.7. *If B is closed, then $M(T_B) = B$.*

PROOF. If $\tau \notin B$, then by 2.4 there exists $a \in \tau^{-1}(-R^{\cdot 2}) \cap (\bigcap_{\sigma \in B} \sigma^{-1}(R^{\cdot 2}))$. Then $a \in T_B$ but $\tau(a) < 0$, so $\tau \notin M(T_B)$.

LEMMA 2.8. *Let $\sigma_1, \ldots, \sigma_n \in M$, $a_1, \ldots, a_n \in K^{\cdot}$. Suppose for all i, j, $a_i a_j^{-1}$ is a unit in the valuation ring $\sigma_i^{-1}(R)\sigma_j^{-1}(R)$. Then for any $\varepsilon > 0$ there exists $a \in K^{\cdot}$ with $|\sigma_i(1 - a_i a^{-1})| < \varepsilon$ for all $i \leqq m$.*

PROOF. This special case of [**5**, Th. 2.1A] can also be deduced as follows. By [**13**, Th. 1, p. 135] there exists $b \in K^{\cdot}$ such that $u_i = a_i b^{-1} \in \sigma_i^{-1}(R^{\cdot})$ for all $i \leqq n$. By Lemma 2.4 applied to $L = \{\sigma_1, \ldots, \sigma_n\}$ there exists $c \in \bigcap_{\sigma \in L} \sigma^{-1}(R^{\cdot})$ such that $1 - \varepsilon < \sigma_i(u_i)\sigma_i(c)^{-1} < 1 + \varepsilon$ for all $i \leqq n$. Take $a = bc$.

Let $A(T)$ denote the (valuation) subring of K generated by $\bigcup_{\sigma \in M(T)} \sigma^{-1}(R)$.

LEMMA 2.9. *If τ is the place associated with $A(T)$, then $\tau^{-1}(1) \subseteqq T$.*

PROOF. By Lemmas 1.3 and 2.3,

$$\tau^{-1}(1) \subseteqq \bigcap_{\sigma \in M(T)} \sigma^{-1}(1) \subseteqq \bigcap_{R \in X(T)} P = T.$$

3. **The reduced Witt ring.** The set of T-similarity classes of quadratic forms over K can be regarded in a natural way as a ring, which we will denote by $W(K/T)$. The operations on $W(K/T)$ are induced by orthogonal sum and Kronecker product, so that $W(K/T)$ is a factor ring of the Witt ring $W(K)$. Indeed, if T is the set of sums of squares, $D(\infty)$, then $W(K/T)$ is the Witt ring modulo its nil radical [**11**, Satz 22]. We compute $W(K/T)$ here for arbitrary T in terms of basic arithmetic invariants of K, such as its orderings and their Archimedean class groups.

For each $\sigma \in M(T)$, $T_\sigma \supseteqq T$ (cf., the paragraph preceeding 2.7), so T-similarity implies T_σ-similarity. Hence we have a natural homomorphism

$$\theta\colon W(K/T) \longrightarrow \prod_{\sigma \in M(T)} W(K/T_\sigma).$$

θ is clearly injective (if $P \in X(T)$, then $\lambda(P) \in M(T)$ and $P \in X(T_{\lambda(P)})$). Thus, in order to compute $W(K/T)$ it in some sense suffices to compute the image of θ and each of the "localizations" $W(K/T_\sigma)$. We begin with the second of these tasks.

PROPOSITION 3.1. *Let $\sigma \in M(T)$ and $P \in X(T_\sigma)$. Then the map s induced by $a(\pm T_\sigma^\cdot) \mapsto \operatorname{sign}_P(a)\langle a\rangle$, $a \in K^\cdot$, is a ring isomorphism from the integral group ring $\mathbf{Z}(K^\cdot/\pm T_\sigma^\cdot)$ onto $W(K/T_\sigma)$.*

Here, $\pm T_\sigma^\cdot$ denotes $T^\cdot\sigma^{-1}(R^\cdot) = T_\sigma^\cdot \cup -T_\sigma^\cdot$. Thus $K^\cdot/\pm T_\sigma^\cdot$ is the value group of σ modulo the values of elements of $T^\cdot$. When $T = D(\infty)$, $W(K/T_\sigma)$ is naturally isomorphic to $W(K_\sigma)$ for an appropriate "completion" K_σ of K at σ, and the isomorphism s of 3.1 is given in [**6** Theorem 2.5]. Also $K^\cdot/\pm D(\infty)_\sigma^\cdot$ is simply the value group of σ modulo squares.

PROOF OF 3.1. Everything is clear except the injectivity of s. For $a \in K^\cdot$ denote by $\bar{a}$ its image modulo $\pm T_\sigma^\cdot$. Suppose $\alpha = \sum_i n_i \bar{a}_i \in \ker s$, where $n_i \in \mathbf{Z}$ and $a_i \in K^\cdot$ for all i. Let $\gamma \in G = \operatorname{Hom}(K^\cdot/\pm T_\sigma^\cdot, \mathbf{Z}^\cdot)$. Let $S = \{a \in K\colon a = 0 \text{ or } \operatorname{sign}_P(a) = \gamma(\bar{a})\}$. Then $S \in X(T)$ (apply Lemma 2.2). Thinking of α as a linear combination of characters on G, we have

$$\alpha(\gamma) = \textstyle\sum_i n_i \gamma(\bar{a}_i) = \sum_i n_i \operatorname{sign}_P(a_i)\operatorname{sign}_S(a_i) = \operatorname{sign}_S(s(\alpha)) = 0.$$

Thus $n_i = 0$ for all i [**8**, Theorem 7, p. 209].

We now turn to the computation of the image of θ. For $\sigma, \tau \in M(T)$, let $G_{\sigma\tau} = K^\cdot/T^\cdot(\sigma^{-1}(R)\tau^{-1}(R))^\cdot$ and let $v_{\sigma\tau}\colon K^\cdot \to G_{\sigma\tau}$ be the natural map. (Thus $G_{\sigma\tau}$ is the value group of the finest common coarsening of σ and τ, modulo the values of elements of $T^\cdot$, and $v_{\sigma\tau}$ is induced by the valuation map.) We have a homomorphism $\phi_{\sigma\tau} = \phi_{\sigma\tau}^T\colon W(K/T_\sigma) \to Z/2Z(G_{\sigma\tau})$ with $\phi_{\sigma\tau}(\langle a\rangle) = v_{\sigma\tau}(a)$ for all $a \in K^\cdot$ (cf., Proposition 3.1). The following theorem was proved in [6] in some special cases, and then, in general, in [1].

THEOREM 3.2. $(g_\sigma) \in \Pi_{\sigma \in M(T)} W(K/T_\sigma)$ *is in the image of* θ *if and only if* $\phi_{\sigma\tau}(g_\sigma) = \phi_{\tau\sigma}(g_\tau)$ *for all* $\sigma, \tau \in M(T)$ *and the map* $P \mapsto \mathrm{sign}_P(g_{\lambda(P)})$, $P \in X(T)$, *is continuous.*

PROOF. Necessity follows from the continuity of the total signature map of a form, and the commutativity for all $\sigma, \tau \in M(T)$ of the diagram

$$\begin{array}{ccc} W(K/T) & \longrightarrow & W(K/T_\sigma) \\ \downarrow & & \downarrow \\ W(K/T_\tau) & \longrightarrow & Z/2Z(G_{\sigma\tau}). \end{array}$$

(The continuity requirement is equivalent to saying that (g_σ) is continuous, if $\dot{\bigcup} W(K/T_\sigma)$ is given the "open path topology" [6, Lemma 5.10].)

Now suppose (g_σ) satisfies the above conditions. It suffices to show the map $f(P) = \mathrm{sign}_P(g_{\lambda(P)})$, $P \in X(T)$, is representable. Let S be a preorder containing T with $M(S) = \{\sigma_0, \sigma_1, \ldots, \sigma_m\}$ finite. It suffices to show $f \mid X(S)$ is representable (Theorem 2.5). We induct on m, and suppose $m \geqq 1$. For each $i \leqq m$ set $A_i = \sigma_i^{-1}(R)$. We may (since valuation rings lying over a given valuation ring are linearly ordered) assume that $\sigma_1, \ldots, \sigma_m$ are indexed so that $A_0A_1 \subseteqq A_0A_i$ for $1 \leqq i \leqq m$. Let $B = \{\sigma_1, \ldots, \sigma_m\}$, so that $M(S_B) = B$ (Lemma 2.7) has only m elements. By induction we can find a form representing f over S_B. This form may be decomposed as

$$\sum\nolimits_{\gamma \in G_{\sigma_0\sigma_1}} \sum\nolimits_{i=1}^{n(\gamma)} \langle b_{i\gamma} \rangle$$

with $n(\gamma) = 0$ (i.e., we have the empty sum) for all but a finite number of γ and with $b_{i\gamma} \in v_{\sigma_0\sigma_1}^{-1}(\gamma)$ for all i, γ. Similarly write

$$g_0 = \sum\nolimits_\gamma \sum\nolimits_{i=1}^{n(\gamma)'} \langle a_{i\gamma} \rangle$$

where $a_{i\gamma} \in v_{\sigma_0\sigma_1}^{-1}(\gamma)$ for all i, γ. By hypothesis $n(\gamma)' \equiv n(\gamma) \pmod 2$ so we may assume $n(\gamma)' = n(\gamma)$ (paste on a few hyperbolic planes if necessary). Scaling by elements of T we may also assume that for all i, γ, $a_{i\gamma}$ and $b_{i\gamma}$ have the same value in the value group of A_0A_1, and hence in the value group of A_0A_s for all $s \geqq 1$. Hence, we can find

$$c_{i\gamma} \in a_{i\gamma}\, \sigma_0^{-1}(R^{\cdot 2}) \cap \left(\bigcap\nolimits_{j=1}^{m} b_{i\gamma} \sigma_j^{-1}(R^{\cdot 2})\right)$$

for all i, γ (Lemma 2.8). Then $\sum_\gamma \sum_i \langle c_{i\gamma} \rangle$ represents f over $X(S_B) \cup X(S_{\sigma_0}) = X(S)$ (any ordering containing S induces σ_i for some i and hence contains S_{σ_i} for some i). This completes the induction and the proof of the theorem.

4. **Representable functions.** The set of representable functions in $C(X(T), \mathbf{Z})$ (cf., §2) form a subring naturally isomorphic to $W(K/T)$. We now apply the structure theory in §3 to give Becker and Bröcker's criterion [**1**, Th. 5.3] for representability. (A short more elementary proof, valid for ab stract spaces of orderings, can be found in [**9**].)

A preorder $S \neq K$ is called a *fan* if $U \cup \{0\}$ is a preorder (i.e., is additively closed) for all subgroups U of $K^\cdot$ satisfying $-1 \notin U \supseteqq S^\cdot$. For example, Lemma 2.2 says T_σ is a fan for all $\sigma \in M(T)$.

REMARK 4.1. Suppose S is a fan. Then $U \cup \{0\}$ is an ordering for every subgroup U of index two in $K^\cdot$ satisfying $-1 \notin U \supseteqq S^\cdot$. Thus $P \mapsto \mathrm{sign}_P$ gives a bijection from $X(S)$ to $\{\gamma \in \mathrm{Hom}(K^\cdot/S^\cdot, \mathbf{Z}^\cdot)\colon \gamma(-S^\cdot) = -1\}$. Also $W(K/S) \cong \mathbf{Z}(K^\cdot/\pm S^\cdot)$ by the same proof used in Proposition 3.1. Finally, if $a \in K^\cdot$, $-a \notin S$, then $S \cup aS$ is a preorder, so $S + aS = S \cup aS$. Each of these assertions can, in fact, be shown to be equivalent to the assertion that S is a fan.

THEOREM 4.2. *Let $f \in C(X(T), \mathbf{Z})$. The following are equivalent:*
(i) *f is representable,*
(ii) *$f\,|X(S)$ is representable for all fans $S \supseteqq T$ with $[K^\cdot : S^\cdot) < \infty$,*
(iii) $\sum_{P\in X(S)} f(P) \equiv 0 \pmod{|X(S)|}$ *for all fans $S \supseteqq T$ with $[K^\cdot : S^\cdot] < \infty$.*

PROOF. Clearly (i) $\Rightarrow$ (ii). Now assume (ii) and let $S \supseteqq T$ be a fan of finite index. We may assume f is represented by $\langle a\rangle$ for some $a \in K^\cdot$. Then $\sum_{P\in X(S)} f(P)$ equals either $\pm\,|X(S)|$ or 0 according as $a \in \pm S$ or $a \notin \pm S$ (in the last case, note that exactly half the orderings of $X(S)$ contain a, cf. 4.1). Now assume (iii). Let $S \supseteqq T$ be a fan. Let A be as in Lemma 2.6 and assume the a_i are indexed so that $-S^\cdot, a_1S^\cdot, \ldots, a_mS^\cdot$ (where $m \leqq n$) is a basis for the subspace of $K^\cdot/S^\cdot$ spanned by $-S^\cdot, a_1S^\cdot, \ldots, a_nS^\cdot$. Since S is a fan we can (by linear algebra) find a preorder (and hence a fan) $U \supseteqq S$ with $-U^\cdot, a_1U^\cdot, \ldots, a_mU^\cdot$ a basis for $K^\cdot/U^\cdot$. Now for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$ in $\{\pm 1\}^m$ let P_ε be the unique ordering in $X(U \cup \{a_1\varepsilon_1, \ldots, a_m\varepsilon_m\})$. Note as ε varies, P_ε runs through all of $X(U)$. For $\delta = (\delta_1, \ldots, \delta_m) \in \{0, 1\}^m$, let $a^\delta = a_1^{\delta_1} \cdots a_m^{\delta_m}$. Then by Lemma 2.6, $2^m f|X(S)$ is represented by

$$\begin{aligned}\textstyle\sum_\varepsilon f(P_\varepsilon)\, \langle\!\langle a_1\varepsilon_1, \ldots, a_m\varepsilon_m\rangle\!\rangle &= \textstyle\sum_\varepsilon\sum_\delta f(P_\varepsilon)\langle a^\delta\varepsilon^\delta\rangle \\ &= \textstyle\sum_\delta\sum_\varepsilon f(P_\varepsilon)\,\mathrm{sign}_{P_\varepsilon}(a^\delta)\langle a^\delta\rangle \\ &= \textstyle\sum_\delta\sum_{P\in X(U)} f(P)\,\mathrm{sign}_P(a^\delta)\langle a^\delta\rangle \\ &= \textstyle\sum_\delta\big(\sum_{P\in X(U)} f(P) - 2\sum_{P\in X(U\cup -a^\delta U)} f(P)\big)\,\langle a^\delta\rangle\end{aligned}$$

(where ε and δ range over $\{\pm 1\}^m$ and $\{0, 1\}^m$, respectively) which is in $2^m W(K/S)$ by hypothesis. This shows $f|X(S)$ is representable for all fans $S \supseteqq T$. In particular, for all $\sigma \in M(T)$ we can find a form g_σ representing $f|X(T_\sigma)$. f is the map $P \mapsto \mathrm{sign}_P(g_{\lambda(P)})$. Let $\sigma, \tau \in M(T), \sigma \neq \tau$. By Theorem

3.2, it remains only to show $\phi^T_{\sigma\tau}(g_\sigma) = \phi^T_{\tau\sigma}(g_\tau)$. Let η, with residue class field F, be the place associated with the valuation ring $A = \sigma^{-1}(R)\,\tau^{-1}(R)$. Let $\bar{\sigma}$, $\bar{\rho}$, and $\bar{T}$ denote the real-valued places and the preordering which σ, τ and T induce on F. Let P_σ and P_τ be orderings of F containing $\bar{T}_{\bar{\sigma}}$ and $\bar{T}_{\bar{\tau}}$, respectively. Define $S = T\eta^{-1}(P^\cdot_\sigma \cap P^\cdot_\tau)$. Note $-1 \notin S$. Suppose U is any subgroup of $K^\cdot$ such that $-1 \notin U \supseteqq S^\cdot$ Then $-1 \notin \eta(A \cap U)$. Since $P_\sigma \cap P_\tau$ is a fan in F ($P^\cdot_\sigma \cap P^\cdot_\tau$ has index 4 in $F^\cdot$) it follows that $\eta(A \cap U)$ is a preorder of F. Thus, by Lemma 2.2, U is additively closed. This proves S is a fan. Note that $T\eta^{-1}(F^\cdot) = S\eta^{-1}(F^\cdot)$. Since $f|X(S)$ is representable, we have

$$\phi^T_{\sigma\tau}(g_\sigma) = \phi^S_{\sigma\tau}(g_\sigma) = \phi^S_{\tau\sigma}(g_\tau) = \phi^T_{\tau\sigma}(g_\tau),$$

as required.

REMARK 4.3. A crucial step in 4.2 is the proof that "enough" fans exist. On the other hand, there aren't "too many" fans [**4**]. In fact, if T is a fan, then $|M(T)| \leqq 2$ (by the claim early in the proof of 2.1) and T induces a fan of index $\leqq 4$ on the residue class field of $A(T)$. Additional knowledge of the map f (such as a bound on the "n" of Lemma 2.6) can allow one to considerably restrict the number of fans one must consider in 4.2 (iii). For the more explicit computation of the image of the total signature map of some special fields, see [**6**, 7.5].

5. **Isotropy and the value set of a form.** Throughout this section $\rho = \langle a_1, \ldots, a_n\rangle$ will denote a form over K. We let A_σ denote the valuation ring of any place σ on K, and K_σ denote the Henselization of K at σ. Our next theorem combines (and slightly strengthens) results of Becker, Bröcker, and Prestel on T-isotropy [**5**, **12**, **1**].

THEOREM 5.1. *The following statements are equivalent*:

(i) *ρ is T-isotropic*;

(ii) *ρ is P-isotropic for all $P \in X(T)$ and ρ is $T\sigma^{-1}(1)$-isotropic for all places σ on K satisfying $-1 \notin \sigma(T)$ and $a_ia_j \notin A^\cdot_\sigma T^\cdot$ for some i, j*;

(iii) *ρ is isotropic in the real closure of K at P for all $P \in X(T)$, and ρ is K^2_σ T-isotropic in K_σ for all places σ on K satisfying $-1 \notin \sigma(T)$ and $a_ia_j \notin A^\cdot_\sigma T^\cdot$ for some i, j; and*

(iv) *ρ is S-isotropic for all preorders $S \supseteqq T$ such that $[K^\cdot : S^\cdot] \leqq 2^{n-1}$ and either $S = K$, $S \in X$, or $a_ia_j \notin A(S)^\cdot S^\cdot$ for some i, j.*

REMARK 5.2. A) That $T\sigma^{-1}(1)$ and TK^2_σ are preorders (for σ as in 5.1 (ii) and (iii)) follows from Lemma 2.2. Lemma 5.4 below shows that the question of isotropy for such preorders reduces to a corresponding (but lower dimensional) question on the residue class field of σ.

B) In 5.1 (ii) and (iii) one may restrict attention to those σ with valuation rings of the form $A_\alpha A_\beta$ for $\alpha, \beta \in M(T)$ (see the proof of 5.1) and to those P with $a_i a_j \in A(P)^{\cdot} T^{\cdot}$ for all i, j (otherwise consider $\sigma = \lambda(P)$).

C) If $[K^{\cdot} : S^{\cdot}] < \infty$, then one can determine in a finite number of steps whether or not ρ is S-isotropic. (For each of the sequences $b_3 S^{\cdot}, \ldots, b_n S^{\cdot}$ of elements of $K^{\cdot}/S^{\cdot}$, one must test the S-equivalence of ρ and $\langle 1, -1, b_3, \ldots, b_n \rangle$ by computing signatures at elements of $X(S)$.)

D) If K is an algebraic function field in one variable over $\mathbf{Q}$ or $\mathbf{R}$, then there are only finitely many preorders S with $a_i a_j \notin S^{\cdot} A(S)^{\cdot}$ for some i, j and only finitely many equivalence classes of places σ with $-1 \notin \sigma(T)$ and $a_i a_j \notin A_\sigma^{\cdot} T^{\cdot}$ for some i, j.

E) Each isotropy criterion in 5.1 gives at least a formal computation of $D_T(\rho)$ (cf., the last paragraph of §1). A corollary is: The additive semigroup generated by the value set of ρ (i.e., by $D(\rho)$) is $\bigcap_S D_S(\rho)$ where S ranges over all preorders of K with $[K^{\cdot} : S^{\cdot}] < \infty$.

We begin the proof of 5.1 with two lemmas implicitly involving the "residue class forms" of ρ **[1, 3, 12]**. In both lemmas, σ will denote a place on K "compatible" with T, i.e., with $\sigma^{-1}(1) \subseteq T$. We write $\bar{a}$ for $\sigma(a)$, $a \in A_\sigma$, $\bar{T}$ for $\sigma(A_\sigma \cap T)$, and $\bar{\rho}$ for $\langle \bar{a}_1, \ldots, \bar{a}_n \rangle$ if $a_i \in A_\sigma^{\cdot}$ for all i.

LEMMA 5.3. *Let $c_i, d_i \in A_\sigma^{\cdot}$ for all $i \leqq m$. Then $\langle c_1, \ldots, c_m \rangle$ and $\langle d_1, \ldots, d_m \rangle$ are T-equivalent if $\langle \bar{c}_1, \ldots, \bar{c}_m \rangle$ and $\langle \bar{d}_1, \ldots, \bar{d}_m \rangle$ are $\bar{T}$-equivalent.*

PROOF. Apply the definition of T-equivalence. (Note that if $P \in X(T)$ and $d \in A_\sigma^{\cdot}$, then $\bar{P} \in X(\bar{T})$ and $\mathrm{sign}_P(d) = \mathrm{sign}_{\bar{P}}(\bar{d})$.)

The converse of 5.3 is true, but is not needed here. For the next lemma, note that ρ is T-equivalent to a form $\sum_{i=1}^{m} \sum_{j=1}^{n(i)} \langle b_i a_{ij} \rangle$ with $a_{ij} \in A_\sigma^{\cdot}$ for all i, j and $b_1, \ldots, b_m$ representing distinct cosets in $K^{\cdot}/A_\sigma^{\cdot} T^{\cdot}$. (Group terms and scale by elements of T.) With this notation we have the following lemma.

LEMMA 5.4. *ρ is T-isotropic if and only if $\langle \bar{a}_{i1}, \ldots, \bar{a}_{in(i)} \rangle$ is $\bar{T}$-isotropic for some $i \leqq m$.*

PROOF. ($\Rightarrow$) Suppose $\sum_i \sum_j b_i a_{ij} t_{ij} = 0$ for some $t_{ij} \in T$, not all zero. We may assume $b_1 = t_{11} = 1$ and that $b_i a_{ij} t_{ij} \in A_\sigma$ for all i, j (scale by the multiplicative inverse of a term of least value, and reindex). Then $b_i a_{ij} t_{ij} \notin A_\sigma^{\cdot}$ for all $i > 1$, so $\sum_{j=1}^{n(1)} \bar{a}_{1j} \bar{t}_{1j} = 0$.

($\Rightarrow$) Immediate from Lemmas 1.6 and 5.3.

We now prove Theorem 5.1.

(i) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (ii). Clearly if ρ is isotropic in the real closure of K at $P \in X(T)$, it is P-isotropic. Next suppose σ is a place as in (ii). Then ρ is TK_σ^2-isotropic, and hence $\langle \bar{a}_{i1}, \ldots, \bar{a}_{in(i)} \rangle$ is $\bar{T}$-isotropic for some $i \leqq m$ (we use the notation of 5.4 with T replaced by $T\sigma^{-1}(1)$). Hence ρ is $T\sigma^{-1}(1)$-isotropic. (Again apply 5.4. Note that σ and its extension to K_σ have the same residue class field and value group, and that $T\sigma^{-1}(1)$ and TK_σ^2 induce the same preorder and subgroup, respectively, of them.)

(ii) $\Rightarrow$ (iv). Let S be a preorder as in (iv). We may suppose $[K^\cdot : S^\cdot] > 2$ (otherwise, ρ is S-isotropic by (ii), since either $S = K$ or $S \in X$). Then ρ is $T\sigma^{-1}(1)$-isotropic, where σ is the place associated with $A(S)$. (Note $-1 \notin \sigma(T)$, since $\sigma^{-1}(1) \subseteqq S$ by 2.9, and $-1 \notin S$.) Since $T\sigma^{-1}(1) \subseteqq S$, it follows that ρ is S-isotropic.

(iv) $\Rightarrow$ (i). Just suppose ρ is T-anisotropic. Let $S \supseteqq T$ be a maximal preorder with ρ S-anisotropic. It suffices to show S satisfies the conditions of (iv). We use the notation of Lemmas 5.3 and 5.4, with σ the place associated with $A(S)$ and with S in place of T (cf., Lemma 2.9). We may suppose $n \geqq 2$, and hence that $S \neq K$. Let us suppose $a_i a_j \in A(S)^\cdot S^\cdot$ for all i, j. We claim $S \in X(T)$. By Theorem 2.1, $M(S)$ is finite. We may assume $a_i a \in A(S)^\cdot$ for all i (scale by a_1^{-1} and elements of $S^\cdot$). Let $\tau \in M(S)$, and just suppose $M(S) \neq \{\tau\}$. Then $L = \{\eta \in M(S)\colon \eta \neq \tau \text{ and } A_\tau A_\eta = A(S)\}$ and $L' = M(S) \backslash L$ are both nonempty. (Recall that the valuation rings $A_\eta A_\tau$, $\eta \in M(T)$, all contain A_τ and hence are linearly ordered by inclusion.) Since $S \neq S_L$ (Lemma 2.7), ρ is S_L-isotropic. Hence $\bar{\rho}$ and $\langle -1, 1, \bar{h}_3, \ldots, \bar{h}_n \rangle$ are $\bar{S}_L$-equivalent for some $h_i \in A(S)$. (Lemma 1.6 and 5.4). Thus ρ and $\langle -1, 1, h_3, \ldots, h_n \rangle$ are S_L-equivalent (Lemma 5.3). Similarly ρ and $\langle -1, 1, c_3, \ldots, c_n \rangle$ are $S_{L'}$-equivalent for some $c_i \in A(S)^\cdot$. But $A_\eta A_{\eta'} = A(S)$ for all $\eta \in L$, $\eta' \in L'$. (For suppose $\eta \in L$, $\eta' \in L'$, $\eta' \neq \tau$. If $A_\eta A_{\eta'} \subseteqq A_{\eta'} A_\tau$, then $A_\eta A_\tau \subseteqq A_{\eta'} A_\tau \subsetneqq A(S)$, a contradiction. Thus $A_{\eta'} A_\tau \subseteqq A_\eta A_{\eta'}$, so $A(S) = A_\eta A_\tau \subseteqq A_\eta A_{\eta'} \subseteqq A(S)$.) Hence for each i, $3 \leqq i \leqq n$, we can find d_i in

$$\bigcap\nolimits_{\eta \in L} h_i \eta^{-1}(R^{\cdot 2}) \cap \left(\bigcap\nolimits_{\eta \in L'} c_i \eta^{-1}(R^{\cdot 2})\right)$$

(Lemma 2.8). Therefore $\rho_S = \langle -1, 1, d_3, \ldots, d_n \rangle_S$ (any $P \in X(S)$ has $\lambda(P)$ in L or L', hence $P \in X(S_L)$ or $P \in X(S_{L'})$). This contradicts that ρ is S-anisotropic and proves that $M(S) = \{\tau\}$. Hence $\bar{S}$ is an ordering. (Any two orderings containing $\bar{S}$ induce the same trivial real-valued place and hence are equal.) Let $P \in X(S)$. Then $\bar{P} = \bar{S}$. Since $\bar{\rho}$ is $\bar{P}$-anisotropic, it is P-anisotropic (Lemma 5.4). Hence $S = P \in X$ (by the maximality of S) and so $[K^\cdot : S^\cdot] \leqq 2^{n-1}$.

It remains to show $[K^\cdot : S^\cdot] \leqq 2^{n-1}$ in the case that $a_i a_j \notin A(S)^\cdot S^\cdot$ for some i, j. Then $n(i) < n$ for all $i \leqq m$. Hence we may suppose by induction on n that there exist preorders $S_i \supseteqq \bar{S}$ of $\overline{A(S)}$ with $\langle \bar{a}_{i1}, \ldots, \bar{a}_{in(i)} \rangle$ S_i-anisotropic and $[\overline{A(S)}^\cdot : S_i^\cdot] \leqq 2^{n(i)-1}$ for all $i \leqq m$. Let $S_0 = \overline{A(S)}$ if $n \neq m$ and pick $S_0 \in X(\overline{A(S)})$ if $n = m$. We may suppose $a_1 = b_1 = 1$. Let Γ denote the subgroup of $K^\cdot/A(S)^\cdot S^\cdot$ generated by the cosets of

$b_2, \ldots, b_m$. First suppose Γ has dimension $n-1$ as a vector space over $\mathbf{Z}/2\mathbf{Z}$. Then $n = m$ and $a_2, \ldots, a_m$ represent independent cosets in $K^{\cdot}/A(S)^{\cdot}S^{\cdot}$. Hence there exists an ordering S_* of K containing $S \cup \{a_1, \ldots, a_n\}$ (Lemma 2.2). Since ρ is clearly S_*-anisotropic, $S = S_*$ and so $[K^{\cdot}: S^{\cdot}] \leqq 2^{n-1}$. Now suppose $\dim \Gamma < n-1$. Then either $n(i) > 1$ for some i (if $m \neq n$) or $\dim \Gamma < m-1$ (if $m = n$). Hence $-1 \notin S_i$ for some $i \geqq 0$. Let U denote the subgroup of $K^{\cdot}$ generated by the union of S, $\{a \in A(S)^{\cdot}: \bar{a} \in \bigcap_{0 \leqq i \leqq m} S_i\}$, and a subset of $K^{\cdot}$ representing a basis for a subspace of $K^{\cdot}/A(S)^{\cdot}S^{\cdot}$ complementary to Γ. Then $U \cup \{0\}$ is a preorder excluding -1 (Lemma 2.2) and ρ is $U \cup \{0\}$-anisotropic (Lemma 5.4; the b_i represent distinct cosets of $K^{\cdot}/A(S)^{\cdot}U$ and for each i, $\langle \bar{a}_{i1}, \ldots, \bar{a}_{in(i)}\rangle$ is S_i-anisotropic, and hence $\bar{U} \cup \{0\}$-anisotropic). Hence $S^{\cdot} = U$, and so $\bar{S} = \bar{U} \cap \{0\} = \bigcap_{i=0}^{m} S_i$ and $K^{\cdot}/A(S)^{\cdot}S^{\cdot} = \Gamma$. Hence the dimension of $K^{\cdot}/S^{\cdot}$ as a $\mathbf{Z}/2\mathbf{Z}$-space is

$$\begin{aligned}
&\dim K^{\cdot}/S^{\cdot}A(S)^{\cdot} + \dim A(S)^{\cdot}/A(S)^{\cdot} \cap S^{\cdot} \\
&\quad = \dim \Gamma + \dim \overline{A(S)}^{\cdot}/\textstyle\bigcap_{i=1}^{m} S_i^{\cdot} \\
&\quad \leqq (\dim \Gamma + \dim \overline{A(S)}^{\cdot}/S_0^{\cdot}) + \textstyle\sum_{i=1}^{m} \dim \overline{A(S)}^{\cdot}/S_i^{\cdot} \\
&\quad \leqq (m-1) + (\textstyle\sum_{i=1}^{m} n(i) - 1) = n - 1
\end{aligned}$$

as required.

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