ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 1, Winter 1981

CONSTRUCTIONS OF WEAK INVERSE PROPERTY LOOPS

K. W. JOHNSON AND B. L. SHARMA

1. **Introduction.** Examples of loops which satisfy the weak inverse property and are not Moufang are scarce in the literature ([2], [4]). In a previous paper ([3]) the authors gave several constructions for families of commutative loops which satisfy the weak inverse property. In the following, certain families of extensions of groups are given which are non-commutative non-Moufang weak inverse property loops. A non-Moufang weak inverse property loop will be called a WIP loop. The method of construction arises from an examination of an elementary abelian 2-group of "G-actions" for an arbitrary group G. This approach is a special case of that of Albert ([1]), which was found to be too general to be used in specific constructions. The method of construction, in contrast to that for the extensions constructed in [3], bears very little relation to that for any group extension constructions.

In §2 we discuss the extension theory, and in §3 the explicit constructions of various families are given, together with examples. In §4 are given some conclusions.

2. **Extensions.** Let G, H be loops, with identities I, e respectively. A loop extension (H, E, G) of G by H is defined to be an exact sequence

$$e \to H \to E \to G \to I \tag{1}$$

in the category of loops, with morphisms of extensions and equivalence of extensions being defined as in the theory of group extensions. To summarise the results of Albert, such an extension can be constructed by defining for each element (g_1, g_2) of $G \times G$ a map $\phi(g_1, g_2) : H \times H \to H$ which is, in effect, a binary operation under which H is a groupoid, $H(g_1, g_2)$. The further conditions necessary on the map $\phi(g_1, g_2)$ are that for each $g \in G$, $H(g, I)$ be a loop with identity e, that e is a left identity for every groupoid $H(I, g)$ and that $H(I, I) = H$. Conversely, given an extension (H, E, G), maps $\phi(g_1, g_2)$ satisfying the above properties can be defined in an obvious manner. Thus an extension can be represented by an element of $\mathrm{Hom}_S(G \times G, \mathrm{Hom}_S(H \times H, H))$ with certain properties, where Hom_S denotes the homomorphism functor in the category of sets.

Such a classification appears to be too general to actually construct

Received by the editors on June 2, 1977.

extensions, and a more restricted approach is used here. Let G and H both be groups, and define on the set $H \times G$ a multiplicaton given by

$$(2) \qquad (h_1, g_1)(h_2, g_2) = ([h_1L(g_1, g_2), h_2R(g_1, g_2)]T(g_1, g_2)\Delta, g_1g_2)$$

where $L(g_1, g_2)$ and $R(g_1, g_2)$ are 1:1 maps from H onto H, $T(g_1, g_2)$ is a 1:1 map of $H \times H$ onto $H \times H$, and Δ is the map from $H \times H$ to H defined by $(h_1, h_2) \to h_1 \circ h_2$, where $\circ$ denotes the original group operation in H. Thus, if there are no restrictions on L, R and T the above is a more complicated version of Albert's multiplication, up to a few minor modifications. However, if we restrict L and R to be either the identity map on H or the map ρ defined by $\rho(h) = h^{-1}$ for $h \in H$, and restrict T to be either the identity map or the map $\tau : (h, k) \to (k, h)$, then considerable simplification is possible.

We define a triple (h, R, T) to be a *G-action* if the following three conditions are satisfied.

(i) For each element (g_1, g_2) of $G \times G$, $L(g_1, g_2)$, $R(g_1, g_2)$ and $T(g_1, g_2)$ are defined, where L and R take on values in the cyclic group of order 2, $\{i, \rho\}$, and T takes on values in the cyclic group of order 2, $\{i, \tau\}$.

(ii) $L(I, I) = R(I, I) = i$, $T(I, I) = i$.

(iii) $L(g, I) = i$ and $R(I, g) = i$ for all $g \in G$.

Define the product of G-actions (L_1, R_1, T_1) and (L_2, R_2, T_2) to be the triple (L_1L_2, R_1R_2, T_1T_2), where $L_1L_2(g_1, g_2) = L_1(g_1, g_2)L_2(g_1, g_2)$, etc. Under this product the G-actions form an elementary abelian 2-group.

Given a group H, every G-action defines a loop extension of the form (1). The loop E is the set $H \times G$ with multiplication defined as in (2), where ρ and τ are the maps described in the discussion subsequent to (2). If the operation $*$ on H is defined by

$$(3) \qquad h_1 * h_2 = (h_1L(g_1, g_2), h_2R(g_1, g_2))T(g_1, g_2)\Delta,$$

then $h_1 * h_2$ is one of $h_1 \circ h_2$, $h_2 \circ h_1$, $h_1^{-1} \circ h_2$, $h_2 \circ h_1^{-1}$, $h_1 \circ h_2^{-1}$, $h_2^{-1} \circ h_1$, $h_1^{-1} \circ h_2^{-1}$, $h_2^{-1} \circ h_1^{-1}$, where $\circ$ is the operation of H, and under all these operations H is a groupoid, and this implies that $H \times G$ is a groupoid. Condition (iii) ensures that (e, I) is the identity element, and condition (ii) ensures that H is a normal subloop of E. The map $H \to E$ is the usual embedding $h \to (h, I)$ and the map $E \to G$ is the usual projection $(h, g) \to g$. In contrast to the discussion in **[1]** it is not necessary in our treatment to have the groupoid $H(g, I)$ defined by the operation $*$ for the pair (g, I) as in (3) to be a loop.

Let (L, R, T) be a *restricted G-action* if $L(g, g^{-1}) = R(g, g)^{-1}$ for all $g \in G$. This ensures that the two-sided inverse of the element (h, g) in the corresponding extension E is (h^{-1}, g^{-1}), and of course implies that right and left inverses coincide. We calculate the conditions on a restricted G-action (L, R, T) such that for an arbitrary group H the extension E

defined by (2) is weak inverse. Thus we take arbitrary elements $x = (h_1, g_1)$ and $y = (h_2, g_2)$ of E, and consider the identity $x(yx)^{-1} = y^{-1}$.

$$
\begin{aligned}
yx &= ([h_2L(g_2, g_1), h_1R(g_2, g_1)]T(g_2, g_1)\Delta, g_2g_1),\\
(yx)^{-1} &= ([h_1^{-1}R(g_2, g_1), h_2^{-1}L(g_2, g_1)]T(g_2, g_2)\Delta, (g_2g_1)^{-1}),\\
x(yx)^{-1} &= ([h_1L(g_1, (g_2g_1)^{-1}), [h_1^{-1}R(g_2, g_1), h_2^{-1}L(g_2, g_1)]T((g_2, g_1)\Delta\\
&\quad R(g_1, (g_2g_1)^{-1})]T(g_2g_1)^{-1})\Delta, g_2^{-1}).
\end{aligned} \tag{4}
$$

Thus the condition that the right-hand side of (4) is y^{-1} is that

$$
\begin{aligned}
&[h_1L(g_1, (g_2g_1)^{-1}), [h_1^{-1}R(g_2, g_1), h_2^{-1}L(g_2, g_1)]\\
&\quad T(g_2, g_1)\Delta R(g_1, (g_2g_1)^{-1})]T(g_1, (g_1g_2)^{-1})\Delta = h_2^{-1}.
\end{aligned} \tag{5}
$$

We tabulate in table I the various possibilities for (5) to be satisfied.

TABLE I

$L(g_2, g_1)$	$R(g_2, g_1)$	$T(g_2, g_1)$	$L(g_1, (g_2g_1)^{-1})$	$T(g_1,(g_2g_1)^{-1})$	$R(g_1, (g_2g_1)^{-1})$
i	i	i	i	i	i
i	i	τ	i	i	τ
i	ρ	i	ρ	i	i
ρ	i	i	ρ	ρ	τ
i	ρ	τ	ρ	i	τ
ρ	i	τ	ρ	ρ	i
ρ	ρ	i	i	ρ	τ
ρ	ρ	τ	i	ρ	i

Thus it may be seen from the above that if the restricted G-actions (L_1, R_1, T_1) and (L_2, R_2, T_2) satisfy (5), then so does the product (L_1L_2, R_1R_2, T_1T_2). Thus such actions form a subgroup of the group of all G-actions.

3. **Constructions**. From the discussion in §2, in order to construct a WIP loop extension of a group G by a group H, it is sufficient to give a restricted G-action (L, R, T) such that for arbitrary (g_1, g_2) in $G \times G$ the values of L, R and T on (g_2, g_1) and $(g_1, (g_2g_1)^{-1})$ coincide with one line of table I.

(a) Let H be an arbitrary non-commutative group, and let G be C_3, the cyclic group of order 3 generated by α. Let

$$
\begin{aligned}
L(I, \alpha^2) &= L(\alpha^2, \alpha) = \rho\\
R(\alpha, I) &= R(\alpha^2, \alpha) = \rho\\
T(\alpha^2, \alpha) &= T(\alpha^2, \alpha^2) = \tau\ .
\end{aligned}
$$

Let, L, R and T take on the value i on all other pairs.

Then in the corresponding extension E right and left inverse coincide, and the proof that E is WIP is contained in tabel II.

TABLE II

$(g_2, g_1) = \beta$	$(g_1, (g_2g_1)^{-1}) = \gamma$	$L(\beta)$	$R(\beta)$	$T(\beta)$	$L(\gamma)$	$R(\gamma)$	$T(\gamma)$
(I, α^2)	(α^2, α)	ρ	i	i	ρ	ρ	τ
(α, I)	(I, α^2)	i	ρ	i	ρ	i	i
(α^2, α)	(α, I)	ρ	ρ	τ	i	ρ	i
(α^2, α^2)	(α^2, α^2)	i	i	τ	i	i	τ

For all other pairs the appropriate line in the table consists of i only.

The non-commutativity of H is necessary for E to be non-associative. If h and k are elements of H which do not commute, then

$$((h, \alpha^2)(k, \alpha^2))^{-1} = (kh, \alpha)^{-1} = (h^{-1}k^{-1}, \alpha^2)$$
$$(k, \alpha^2)^{-1}\,(h, \alpha^2)^{-1} = (k^{-1}, \alpha)(h^{-1}, \alpha) = (k^{-1}h^{-1}, \alpha^2).$$

Thus the loop E does not satisfy the inverse property, and hence is not Moufang.

The loop of smallest order in this family is the loop of order 18, corresponding to $H = S_3$, the symmetric group on 3 letters. If we let $x_1 = e$, $x_2 = (123)$, $x_3 = (132)$, $x_4 = (12)$, $x_5 = (13)$, $x_6 = (23)$, and let $(x_i, \alpha^j) = 6j + i$, then the multiplication table of this loop is that given in table III.

TABLE III

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	3	1	6	4	5	8	9	7	12	10	11	15	13	14	17	18	16
3	1	2	5	6	4	9	7	8	11	12	10	14	15	13	18	16	17
4	5	6	1	2	3	10	11	12	7	8	9	16	17	18	13	14	15
5	6	4	3	1	2	11	12	10	9	7	8	17	18	16	15	13	14
6	4	5	2	3	1	12	10	11	8	9	7	18	16	17	14	15	13
7	9	8	10	11	12	13	14	15	16	17	18	1	2	3	4	5	6
8	7	9	12	10	11	15	13	14	17	18	16	2	3	1	6	4	5
9	8	7	11	12	10	14	15	13	18	16	17	3	1	2	5	6	4
10	12	11	7	8	9	16	17	18	13	14	15	4	5	6	1	2	3
11	10	12	9	7	8	17	18	16	15	13	14	5	6	4	3	1	2
12	11	10	8	9	7	18	16	17	14	15	13	6	4	5	2	3	1
13	14	15	16	17	18	1	3	2	4	5	6	7	8	9	10	11	12
14	15	13	18	16	17	3	2	1	6	4	5	8	9	7	11	12	10
15	13	14	17	18	16	2	1	3	5	6	4	9	7	8	12	10	11
16	17	18	13	14	15	4	5	6	1	3	2	10	12	11	7	9	8
17	18	16	15	13	14	5	6	4	2	1	3	11	10	12	8	7	9
18	16	17	14	15	13	6	4	5	3	2	1	12	11	10	9	8	7

(*b*) Let $G = C_{2n}$, generated by α. Let the pair (α^i, α^j) be abbreviated to (i, j). Let L take on the value i, except for the following pairs, on which it takes the value ρ; $(0, n+r)$, $0 \leqq r < n$, $(n, n+1)$, $(n, 2n-1)$, $(n, n)(n+1, 2n-1)$, $(2n-1, n+1)$, $(n+r, n-r)$, $0 \leqq r < n$. Let R take on the value ρ on the following pairs, and i otherwise; $(r, 0)$, $1 < r \leqq n, (n+1, n), (2n-1, n), (n, n)\,(n+1, 2n-1), (2n-1, n+1)$, $(n+r, n-r)$, $0 \leqq r < n$. Similarly, let T be i other than on the pairs below, on which T takes the value τ; $(n+r, n-r)$, $0 \leqq r < n$, $(n+1, 2n-1)$, $(2n-1, n+1)$, (n, n). Then $L(g^{-1}, g) = R(g^{-1}, g)$ is satisfied, and the condition that a corresponding extension be weak inverse is demonstrated by table IV.

TABLE IV

	$(g_2, g_1) = \beta$	$(g_1, (g_2g_1)^{-1}) = \gamma$	$L(\beta)$	$R(\beta)$	$T(\beta)$	$L(\gamma)$	$R(\gamma)$	$T(\gamma)$
$0 \leqq r < n$	$(0, n+r)$	$(n+r, n-r)$	ρ	i	i	ρ	ρ	τ
	$(n, n+1)$	$(n+1, 2n-1)$	ρ	i	i	ρ	ρ	τ
	$(n, 2n-1)$	$(2n-1, n+1)$	ρ	i	i	ρ	ρ	τ
$1 \leqq r < n$	$(n+r, n-r)$	$(n-r, 0)$	ρ	ρ	τ	i	ρ	i
	$(n+1, 2n-1)$	$(2n-1, n)$	ρ	ρ	τ	i	ρ	i
	$(2n-1, n+1)$	$(n+1, n)$	ρ	ρ	τ	i	ρ	i
	$(n+1, n)$	$(n, 2n-1)$	i	ρ	i	ρ	i	i
	$(2n-1, n)$	$(n, n+1)$	i	ρ	i	ρ	i	i
	(n, n)	$(n, 0)$	ρ	ρ	τ	i	ρ	i
$1 < r \leqq n$	$(r, 0)$	$(0, 2n-r)$	i	ρ	i	ρ	i	i

Thus, if we take H to be an arbitrary group which is not of exponent 2 (in which case the above construction is the direct product), we obtain a WIP loop. Then for arbitrary elements h, k of H,

$$[(h, \alpha^{n+1})(k, \alpha^{2n-1})]^{-1} = [(hk)^{-1}, \alpha^n]^{-1} = (hk, \alpha^n),$$
$$(k, \alpha^{2n-1})^{-1}(h, \alpha^{n+1})^{-1} = (h^{-1}, \alpha)(h^{-1}, \alpha^{n-1}) = (h^{-1}k^{-1}, \alpha^n).$$

Thus if k is an element such that $k \neq k^{-1}$ and $h = e$, the inverse property is not satisfied, and hence the loop constructed is not Moufang.

The smallest member of the family is of order 12 (when $H = C_3$) and its multiplication table is given in table V.

If $H = \{e, \beta, \beta^2\}$, then the element (β^i, α^j) is denoted by $3j + i$.

(c) Let G be C_{2n+1}, $n \geqq 2$, generated by α. We list below those elements on which L, R and T are non-trivial;

$$\begin{aligned}
L\colon\ & (0, n+r)\ 1 \leqq r \leqq n,\ (n, 2n),\ (n, n+2),\\
& (n+r+1, n-r)\ 0 \leqq r \leqq n,\ (n+2, 2n),\ (2n, n+2).\\
R\colon\ & (r, 0)\ 1 \leqq r \leqq n,\ (2n, n),\ (n+2, n),\\
& (n+r+1, n-r)\ 0 \leqq r \leqq n,\ (n+2, 2n),\ (2n, n+2).\\
T\colon\ & (n+r+1, n-r)\ 0 \leqq r \leqq n,\ (n+2, 2n),\ (2n, n+2).
\end{aligned}$$

(here again we have abbreviated (α^i, α^j) to (i, j)).

TABLE V

1	2	3	4	5	6	7	8	9	10	11	12
2	3	1	5	6	4	9	7	8	12	10	11
3	1	2	6	4	5	8	9	7	11	12	10
4	6	5	7	8	9	10	11	12	1	2	3
5	4	6	8	9	7	11	12	10	2	3	1
6	5	4	9	7	8	12	10	11	3	1	2
7	9	8	10	11	12	1	3	2	4	5	6
8	7	9	11	12	10	3	2	1	6	4	5
9	8	7	12	10	11	2	1	3	5	6	4
10	11	12	1	3	2	4	6	5	7	9	8
11	12	10	3	2	1	5	4	6	9	8	7
12	10	11	2	1	3	6	5	4	8	7	9

$L(g, g^{-1}) = R(g, g^{-1})$ is then satisfied, and we again demonstrate that any corresponding extension is WIP by table VI.

TABLE VI

	$(g_2, g_1) = \beta$	$(g_1, (g_2g_1)^{-1}) = \gamma$	$L(\beta)$	$R(\beta)$	$T(\beta)$	$L(\gamma)$	$R(\gamma)$	$T(\gamma)$
$1 \leqq r \leqq n$	$(0, n + r)$	$(n + r, n - r + 1)$	ρ	i	i	ρ	ρ	τ
	$(n, 2n)$	$(2n, n + 2)$	ρ	i	i	ρ	ρ	τ
	$(1, n + 2)$	$(n + 2, 2n)$	ρ	i	i	ρ	ρ	τ
$0 \leqq r \leqq n$	$(n + r + 1, n - r)$	$(n - r, 0)$	ρ	ρ	τ	i	ρ	i
	$(n + 2, 2n)$	$(2n, n)$	ρ	ρ	τ	i	ρ	i
	$(2n, n + 2)$	$(n + 2, 2n)$	ρ	ρ	τ	i	ρ	i
$1 \leqq r \leqq n$	$(r, 0)$	$(0, 2n + 1 - r)$	i	ρ	i	ρ	i	i
	$(2n, n)$	$(n, n + 2)$	i	ρ	i	ρ	i	i

On all other pairs the corresponding values for L, R and T are all i.

Again, we take H to be of exponent greater than 2, and take the corresponding extension. For arbitrary h, k in H,

$$[(h, \alpha^{n+2})(k, \alpha^{2n})]^{-1} = ((hk)^{-1}, \alpha^{n+1})^{-1} = (hk, \alpha^{n}),$$
$$(k, \alpha^{2n})^{-1}(h, \alpha^{n+2})^{-1} = (k^{-1}, \alpha)(h^{-1}, \alpha^{n-1}) = (k^{-1}h^{-1}, \alpha^{n}).$$

Thus, if h is the element e, and k is an element of H such that $k \neq k^{-1}$, the inverse property is not satisfied, which implies that the loop is not Moufang.

The smallest example of this family occurs when $H = C_3$, $G = C_5$, and its multiplication table is given in table VII. If H is generated by β, then the element (β^i, α^j) is denoted by $3i + j$.

It may be noted that the above constructions apply also when H is an

TABLE VII

1	2	3	4	5	6	7	8	9	10	11	x2	13	14	15
2	3	1	5	6	4	8	9	7	12	10	11	15	13	14
3	1	2	6	4	5	9	7	8	11	12	10	14	15	13
4	6	5	7	8	9	10	11	12	13	14	15	1	2	3
5	4	6	8	9	7	11	12	10	14	15	13	2	3	1
6	5	4	9	7	8	x2	10	11	15	13	14	3	1	2
7	9	8	10	11	12	13	14	15	1	2	3	4	5	6
8	7	9	11	x2	10	14	15	13	2	3	1	6	4	5
9	8	7	12	10	11	15	13	14	3	1	2	5	6	4
10	11	x2	13	14	15	1	3	2	4	5	6	7	8	9
11	12	10	14	15	13	3	2	1	5	6	4	8	9	7
12	10	11	15	13	14	2	1	3	6	4	5	9	7	8
13	14	15	1	3	2	4	6	5	7	8	9	10	12	11
14	15	13	3	2	1	5	4	6	8	9	7	12	11	10
15	13	14	2	1	3	6	5	4	9	7	8	11	10	12

arbitrary inverse property loop. Further non-commutative WIP loops may be obtained by the methods of [3], where central extensions of an abelian group A by an abelian group G were constructed. If A is the centre of a non-abelian group H, then as in group extension theory, a loop extension of A by G corresponds to a loop extension of H by G, and if H is not commutative the resulting extension will not be commutative. These extensions are rather different from those in families (a), (b) and (c) above, in that all the latter are split in the sense that the loop E has a normal subloop $H_1 = \{(h, I),\ h \in H\}$ isomorphic to H, and a normal subloop $G_1 = \{(h, I), h \in G\}$ isomorphic to G, such that $H_1 \cap G_1 = (e, I)$. This is not so for those extensions constructed by the methods of [3].

4. **Summary and conclusion.** We have thus constructed non-commutative WIP loops for all composite orders except 8 and $2p$, where p is an odd prime. It is not known whether non-commutative WIP loops exist of order a prime (in [3] commutative WIP loops were constructed for all primes of the form $3n + 1$). In addition, a general method has been obtained to construct certain types of loop extensions, and further applications of this method will be given elsewhere. It seems to be of interest to investigate further the group of G-actions.

REFERENCES

1. A.A. Albert, *Quasigroups* II, Trans. Amer. Math. Soc. **55** (1944), 401–419.
2. R. Baer, *Nets and groups* II, Trans. Amer. Math. Soc. **47** (1960), 435–439.
3. K.W. Johnson and B.L. Sharma, *Constructions of commutative weak inverse property loops of most orders.* To appear.

4. J.L. Osborn, *Loops with the weak inverse property*, Pacific J. Math. **10** (1960), 295–304.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE WEST INDIES, MONA, KINGSTON 7, JAMAICA.
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALLAHABAD, ALLAHABAD (UP), INDIA.