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ON ABSOLUTE EULER-KNOPP AND DE LA VALLÉE-POUSSIN SUMMABILITY

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In memory of Professor Richard F. DeMar.

1. **Introduction.** The de la Vallée-Poussin means of a series $\sum a_n$ are defined by

$$(1.1) \qquad V_n = \sum_{k=0}^{n} \frac{(n!)^2}{(n-k)!\,(n+k)!}\, a_k.$$

The Euler-Knopp means $E_n(x)$ of $\sum a_n$ are defined for $0 < x \leqq 1$ by

$$(1.2) \qquad E_n(x) = \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} x^j (1-x)^{n-j} a_k, \quad 0 < x < 1;\ E_n(1) = \sum_{k=0}^{n} a_k.$$

The series $\sum a_n$ is said to be V-summable if the sequence $\{V_n\}$ converges and $E(x)$-summable if $\{E_n(x)\}$ converges for a given $x \in (0, 1)$. The $E(x)$, V, and (C, λ) methods belong to a general class of summability methods defined by T. H. Gronwall [**2**]. These methods involve an identity of the form

$$(1.3) \qquad (1-w)^{-\lambda-1} \sum_{n=0}^{\infty} a_n [f(w)]^n = \sum_{n=0}^{\infty} \binom{n+\lambda}{n} U_n w^n.$$

(Gronwall's definition is slightly more general.) The function $f(w)$ in (1.3) is assumed to be analytic and univalent in $|w| < 1$ with $f(0) = 0$, $f(1) = 1$. Near $w = 1$ the inverse function $w = f^{-1}(z)$ is assumed to have the form

$$w = 1 - (1-z)^{\mu}(b_0 + b_1 z + \cdots)$$

with $\mu \geqq 1$ and $b_0 > 0$.

When $\lambda = 0$ in (1.3) and $f(w)$ is replaced by

$$(1.4) \qquad f_1(w) = \frac{xw}{1-(1-x)w}, \quad 0 < x \leqq 1,$$

then $U_n = E_n(x)$. When $\lambda = -1/2$ in (1.3) and $f(w)$ is replaced by

$$(1.5) \qquad f_2(w) = \frac{1-(1-w)^{1/2}}{1+(1-w)^{1/2}},$$

then $U_n = V_n$. When $f(w) = w$ in (1.3), then, as is well known, $U_n = \sigma_n(\lambda)$, the (C, λ) mean of $\sum a_n$.

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For a given sequence g_n we will write $\Delta g_n = g_n - g_{n-1}$, $n = 0, 1, 2, \ldots$; $g_{-1} = 0$. The series $\sum a_n$ is said to be summable $|V|$, $|E(x)|$, or $|(C, \lambda)|$ respectively if $\sum |\Delta V_n| < \infty$, $\sum |\Delta E_n(x)| < \infty$, or $\sum |\Delta \sigma_n(\lambda)| < \infty$. It follows from a fundamental theorem of Gronwall (see Theorem 1 in [1] for a corrected version) that (C, λ) summability with $\lambda \geqq 0$ implies V summability and that $E(x)$ summability with $(\sqrt{2} - 1)/\sqrt{2} < x \leqq 1$ implies V summability. B. Kwee [5] proved that $|(C, \lambda)|$ summability with $\lambda \geqq 0$ implies $|V|$ summability. Here we will prove the following corresponding result for $|E(x)|$ and $|V|$.

THEOREM. *If* $\sum |\Delta E_n(x)| < \infty$ *and* $(\sqrt{2} - 1)/\sqrt{2} < x \leqq 1$, *then* $\sum |\Delta V_n| < \infty$.

2. **Preliminaries.** It is known [4] that in order for a series-to-series transformation $t_n = \sum_k \alpha_{nk} x_k$ to be such that $\sum |x_k| < \infty$ implies $\sum |t_n| < \infty$, it is necessary and sufficient that

$$\sup_k \sum_n |\alpha_{nk}| < \infty. \tag{2.1}$$

The proof of the theorem stated in the introduction depends on determining α_{nk} such that

$$\Delta V_n = \sum_{k=0}^{n} \alpha_{nk} \, \Delta E_k(x), \tag{2.2}$$

and then showing that the α_{nk} satisfy (2.1). The explicit representations (1.1) and (1.2) for V_n and $E_n(x)$ will not be used. Instead we will use the relations of the form (1.3) satisfied by $\{V_n\}$ and $\{E_n(x)\}$. This section will be devoted to proving the following preliminary result.

LEMMA. *Define* $B_{nk} = B_{nk}(x)$ *by*

$$\sum_{n=k}^{\infty} B_{nk} w^n = w(1 - (1 - w)^{1/2})^{k-1} [1 + (2x - 1)(1 - w)^{1/2}]^{-k-1}. \tag{2.3}$$

Then the α_{nk} *in* (2.2) *are given by* $\alpha_{nk} = kxB_{nk}/n\binom{n-1/2}{n}$.

PROOF. First, we will develop some formulas for ΔU_n (U_n as defined in (1.3)) and then specialize to V_n and $E_n(x)$. Define b_{nk} for $n, k = 0, 1, 2, \ldots$ by

$$(1 - w)^{-\lambda-1}[f(w)]^k = \sum_{n=k}^{\infty} b_{nk} w^n, \quad k = 0, 1, 2, \ldots \tag{2.4}$$

and $b_{nk} = 0, n < k$. From (2.4) and (1.3) we find

$$\binom{n + \lambda}{n} U_n = \sum_{k=0}^{n} b_{nk} a_k, \quad n = 0, 1, 2, \ldots . \tag{2.5}$$

Now for $n = 1, 2, \ldots$, we have after a minor computation

$$n\binom{n+\lambda}{n}\Delta U_n = \sum_{k=0}^{n} [nb_{nk} - (n+\lambda)b_{n-1,k}]a_k. \tag{2.6}$$

If $\lambda \neq 0$, then define γ_{nk} by

$$\gamma_{nk} = nb_{nk} - (n+\lambda)b_{n-1,k};\ n = 1, 2, \ldots, k = 0, 1, 2, \ldots .$$

Then from (2.4) follows

$$\sum_{n=k}^{\infty} \gamma_{nk} w^{n-1} = k(1-w)^{-\lambda} [f(w)]^{k-1} f'(w),\ k = 1, 2, \ldots . \tag{2.7}$$

In particular, for V summability with $\lambda = -1/2$ and $f(w) = f_1(w)$ we obtain

$$n\binom{n-1/2}{n}\Delta V_n = \sum_{k=0}^{n} \gamma_{nk} a_k,\ n = 1, 2, \ldots . \tag{2.8}$$

$$\sum_{n=k}^{\infty} \gamma_{nk} w^{n-1} = k(1-(1-w)^{1/2})^{k-1}(1+(1-w)^{1/2})^{-k-1}, \quad k = 1, 2, \ldots . \tag{2.9}$$

When $\lambda = 0$ in (2.6), setting $\beta_{nk} = b_{nk} - b_{n-1,k}$ yields

$$\Delta U_n = \sum_{k=0}^{n} \beta_{nk} a_k,\ n = 0, 1, 2, \ldots , \tag{2.10}$$

$$\sum_{n=k}^{\infty} \beta_{nk} w^n = [f(w)]^k,\ k = 1, 2, \ldots . \tag{2.11}$$

Consequently, we have for the Euler-Knopp case

$$\Delta E_n(x) = \sum_{k=0}^{n} \beta_{nk} a_k,\ n = 0, 1, 2, \ldots \tag{2.12}$$

$$\sum_{n=k}^{\infty} \beta_{nk} w^n = (xw)^k[1-(1-x)w]^{-k},\ k = 0, 1, 2, \ldots . \tag{2.13}$$

Equations (2.12) and (2. 13) can be inverted to give

$$a_n = \sum_{k=0}^{n} \varepsilon_{nk} \Delta E_k(x),\ n = 0, 1, 2, \ldots , \tag{2.14}$$

$$\sum_{n=k}^{\infty} \varepsilon_{nk} w^n = w^k[x+(1-x)w]^{-k},\ k = 0, 1, 2, \ldots . \tag{2.15}$$

From (2.8) and (2.14) it follows that

(2.16) $$n\binom{n-1/2}{n}\Delta V_n = \sum_{k=0}^{n} C_{nk}\Delta E_k(x),$$

where $C_{nk} = \sum_{j=k}^{n} \gamma_{nj}\varepsilon_{jk}$. A computation using (2.9) and (2.15) shows that

$$\sum_{n=k}^{\infty} C_{nk}w^n = kxw(1-(1-w)^{1/2})^{k-1}[1+(2x-1)(1-w)^{1/2}]^{-k-1},$$

so that $C_{nk} = kxB_{nk}$ and the lemma is proved.

3. **Proof of the theorem.** It is known [4] that if $0 < x_1 < x_2 \leqq 1$, then $|E(x_2)|$ summability implies $|E(x_1)|$ summability. Also, $n\binom{n-1/2}{n} \sim (n/\pi)^{1/2}$, so that it suffices to prove that if $(\sqrt{2}-1)/\sqrt{2} < x < 1/2$, then

(3.1) $$\sup_k k\sum_{n=k}^{\infty} n^{-1/2}|B_{nk}| < \infty.$$

To show that (3.1) holds we will write B_{nk} as a contour integral and then estimate the integral. Write $t = 1 - 2x$ and set

$$g(w) = (1-(1-w)^{1/2})(1-t(1-w)^{1/2})^{-1},\ h(w) = (1-t(1-w)^{1/2})^{-2}.$$

We take the branch of $(1-w)^{1/2}$ that equals 1 when $w = 0$. From the lemma, for a suitable contour C_n,

(3.2) $$B_{nk} = \frac{1}{2\pi i}\int_{C_n} [g(w)]^{k-1}h(w)w^{-n}\,dw$$

The contour C_n will be constructed using the mapping properties of $g(w)$.

Note first that $g(w)$ is univalent in $|w| < 1$ and that $g(1) = 1$. Next observe that a fractional linear transformation $\phi(z) = (1-z)/(1-tz)$ with $t < 1$ satisfies $|\phi(z)| < 1$ if and only if $|z-(1+t)^{-1}| < (1+t)^{-1}$. Since $g(w) = \phi((1-w)^{1/2})$, we have that if $t < 1$ and if $(1-w)^{1/2}$ lies in the disc C_t: $|w-(1+t)^{-1}| < (1+t)^{-1}$, then $|g(w)| < 1$. Simple geometric considerations show that $(1-w)^{1/2}$ lies in C_t for $|w| < 1$ if $(\sqrt{2}-1)/\sqrt{2} < x < 1/2$. Hence, $|g(w)| < 1$ for $|w| < 1$ and $(\sqrt{2}-1)/\sqrt{2} < x < 1/2$. Indeed, for these values of x, $|g(w)| < 1$ for $|w| \leqq 1$, $w \neq 1$. Write $z = g(w)$. The inverse function $w = g^{-1}(z)$ satisfies $\arg(1-w) = 2\arg(1-z) + o(1)$ as $w \to 1$. Thus, near $z = 1$ the image of $|w| < 1$ by $g(w)$ is contained in the sector $z = 1 + \rho e^{i\phi}$, $\rho > 0$, $3\pi/4 < \phi < 5\pi/4$.

The mapping properties discussed above allow us to choose $\beta \in (0, \pi/2)$ and $R_1 \in (0, 1)$, so that $|g(w)| < 1$ for w in the sector $w = 1 + \rho^i e^{\theta}$, $0 < \rho \leqq R_1$, $\beta < \theta < 2\pi - \beta$. Also we may choose $R_2 > 1$ satisfying the following three conditions.

1) Noting that $g(w)$ has a singularity in $(-\infty, -1)$; we choose R_2 so that this singularity lies outside of $|w| = R_2$.

2) Choose R_2 so that $|g(w)| < 1$ for $|w| \leqq R_2$ and $|1 - w| \geqq R_1$.

3) We require that $|w| = R_2$ intersect the ray $w = 1 + \rho e^{i\beta}$ at a point $w_0 = 1 + \rho_0 e^{i\beta}$ with $\rho_0 < R_1$.

Finally we choose $R_3 = R_3(n) < \min(R_1, n^{-2})$ such that $|1 - w| \leqq R_3$ is interior to $|w| \leqq R_2$. Now define the closed contour $C_n = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ where Γ_1 is the circular arc $|w| = R_2$ exterior to the sector $w = 1 + \rho e^{i\theta}, \rho > 0, -\beta < \theta < \beta$, Γ_2 is the circular arc $w = 1 + R_3 e^{i\theta}$, $\beta < \theta < 2\pi - \beta$, Γ_3 is the line segment $w = 1 + \rho e^{i\beta}$, $R_3 \leqq \rho \leqq \rho_0$, and Γ_4 is the line segment $w = 1 + \rho e^{-i\beta}$, $R_3 \leqq \rho \leqq \rho_0$. Note that $|g(w)| < 1$ for $w \in C_n$. Write

$$I_j = \frac{1}{2\pi i} \int_{\Gamma_j} [g(w)]^{k-1} h(w) w^{-n} dw,$$

then $|B_{nk}| \leqq |I_1| + |I_2| + |I_3| + |I_4|$. Let $A = \sup\{|h(w)| : w \in C_n, n = k, k+1, \dots\}$. Then $|I_1| \leqq A R_2^{-n+1}$ and

$$k \sum_{n=k}^{\infty} n^{-1/2} |I_1| = o(1) \text{ as } k \to \infty.$$

On Γ_2 we have $w = 1 + R_3 e^{i\theta}$, $R_3 < n^{-2}$, so

$$|I_2| \leqq \frac{A}{2\pi n^2} \int_0^{2\pi} |1 + R_3 e^{i\theta}|^{-n} d\theta.$$

Since $|1 + R_3 e^{i\theta}|^{-n} \leqq (1 - n^{-2})^{-n} = O(1)$, we can conclude that

$$k \sum_{n=k}^{\infty} n^{-1/2} |I_2| = o(1) \text{ as } k \to \infty.$$

Turning to I_3, we have

$$\sum_{n=k}^{\infty} |I_3| n^{-1/2} \leqq \frac{A}{2\pi} \int_0^{\rho_0} |g(1 + \rho e^{i\beta})|^{k-1} \sum_{n=k}^{\infty} |1 + \rho e^{i\beta}|^{-n} n^{-1/2} d\rho.$$

There exists a constant B so that for $0 < \rho \leqq \rho_0$

$$\sum_{n=k}^{\infty} |1 + \rho e^{i\beta}|^{-n} n^{-1/2} \leqq B |1 + \rho e^{i\beta}|^{-k} \rho^{-1/2},$$

and hence

$$\text{(3.3)} \qquad \sum_{n=k}^{\infty} |I_3| n^{-1/2} \leqq \frac{A \cdot B}{2\pi} \int_0^{\rho_0} |g(1 + \rho e^{i\beta})|^{k-1} |1 + \rho e^{i\beta}|^{-k} \rho^{-1/2} d\rho.$$

Since $w = 1 + \rho e^{i\beta}$ on Γ_3, $(1 - w)^{1/2} = -i \rho^{1/2} e^{i\beta/2}$, and then we have

$$|g(1 + \rho e^{i\beta})|^{k} |1 + \rho e^{i\beta}|^{-k} = |1 + it\rho^{1/2} e^{i\beta/2}|^{-k} \, |1 - i\rho^{1/2} e^{i\beta/2}|^{-k}.$$

Since $|g(w)|^{-1}$ is bounded on Γ_3 uniformly in n, we have from (3.3) that for some constant Q,

$$\sum_{n=k}^{\infty} |I_3| n^{-1/2} \leqq Q \int_0^{\rho_0} \exp\left(-\frac{k}{2} H(\rho)\right) \rho^{-1/2}\, d\rho, \tag{3.4}$$

where $H(\rho) = \log|1 + it\rho^{1/2}e^{i\beta/2}|^2\,|1 - i\rho^{1/2}e^{i\beta/2}|^2$. We may then apply the Method of Laplace (c.f. [3], Theorem 7.1) to the integral in (3.4) and obtain

$$\int_0^{\rho_0} \exp\left(-\frac{k}{2} H(\rho)\right) \rho^{-1/2} d\rho \sim \frac{1}{kx \sin(\beta/2)}.$$

Thus $k \sum_{n=k}^{\infty} |I_3| n^{-1/2} = O(1)$. The integral I_4 is handled in exactly the same way, and the proof of (3.1) is complete.

References

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