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GLOBAL PROPERTIES OF SPACES OF AR's

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ABSTRACT. We study the hyperspace (denoted AR_h^X) of compact absolute retract subsets of certain finite-dimensional compacta X. The topology of AR_h^X is induced by Borsuk's homotopy metric. We show AR_h^X is contractible if X is pseudoisotopically contractible. We show AR_h^X is simply-connected if X is a sphere of dimension greater than 1.

1. **Introduction.** Let X be a finite-dimensional compactum and let 2_h^X be the space of nonempty compact ANR subsets of X introduced by Borsuk [2]. If d is a metric for X, the topology of 2_h^X is induced by the *homotopy metric* d_h, which may be described as follows: $d_h(A_i, A) \to 0$ if and only if

a) $d_s(A_i, A) \to 0$, where d_s is the well-known Hausdorff metric, and

b) for every $\varepsilon > 0$ there is a $\delta > 0$ such that every A_i-subset of diameter less than δ contracts to a point in an A_i-subset of diameter less than ε.

We let AR_h^X be the subspace of 2_h^X consisting of the members of 2_h^X that are absolute retracts (AR's). Since AR_h^X is open and closed in 2_h^X ([2], p. 200), AR_h^X is a union of components of 2_h^X.

Let I denote the interval [0, 1]. We will use the following lemmas.

LEMMA 1.1. ([1], 4.2, p. 43). *If $A \in 2_h^X$ and $f\colon A \times I \to X$ is an isotopy, then the function $g\colon I \to 2_h^X$ defined by $g(t) = f_t(A)$ is continuous.*

LEMMA 1.2. ([4], 2.1). *Let U be open in X. Then $\{A \in 2_h^X | A \subset U\}$ is open in 2_h^X.*

2. We will denote by $s(A, \delta, \varepsilon)$ the words "every A-subset of diameter less than δ contracts to a point in an A-subset of diameter less than ε." We prove the following lemma.

LEMMA 2.1. *Let X and Y be finite-dimensional compacta with $X \subset Y$. Let $f\colon X \times I \to Y$ be an isotopy. Then the induced function $f_*\colon 2_h^X \times I \to 2_h^Y$ defined by $f_*(A, t) = f_t(A)$ is continuous.*

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PROOF. Let d be a metric for Y and let $\varepsilon > 0$. Since $X \times I$ is compact there is a $\delta > 0$ such that

(1) $d(x_0, x_1) < \delta$ and $|t_0 - t_1| < \delta$ implies $d(f(x_0, t_0), f(x_1, t_1)) < \varepsilon/2$.

Let $(A_i, t_i) \to (A_0, t_0)$ in $2_h^X \times I$. There is a positive integer k such that $i > k$ implies $d_s(A_i, A_0) < \delta$ and $|t_i - t_0| < \delta$. Let $B_i = f_*(A_i, t_i)$, $B_0 = f_*(A_0, t_0)$. It follows from (1) that $i > k$ implies $d_s(B_i, B_0) < \varepsilon/2$. We conclude $d_s(B_i, B_0) \to 0$.

There exists $r > 0$ such that for all i,

(2) $$s(A_i, r, \delta).$$

There exists $u > 0$ such that

(3) $$\text{if } y, z \in f(X \times \{t_0\}) \text{ and } d(y, z) < 3u, \text{ then } d(f_{t_0}^{-1}(y), f_{t_0}^{-1}(z)) < r/2.$$

There exists a positive integer m such that

(4) $$i > m \text{ implies } d(f(x, t_i), f(x, t_0)) < u \text{ for all } x \in X.$$

Let $y_i, y_i' \in f(X \times \{t_i\})$ be such that $d(y_i, y_i') < u$. There exist $x_i, x_i' \in X$, $z_i, z_i' \in f(X \times \{t_0\})$ such that $y_i = f(x_i, t_i)$, $y_i' = f(x_i', t_i)$, $z_i = f(x_i, t_0)$, $z_i' = f(x_i', t_0)$. Using (4), for $i > m$ we have

$$d(z_i, z_i') \leqq d(z_i, y_i) + d(y_i, y_i') + d(y_i', z_i') < u + u + u = 3u.$$

It follows from (3) that $d(x_i, x_i') < r/2$.

Let $C_i \subset B_i$ satisfy diam $C_i < u$. The above implies

$$\operatorname{diam} f_{t_i}^{-1}(C_i) < 2r/2 = r.$$

By (2), there is a deformation $h_i: f_{t_i}^{-1}(C_i) \times I \to A_i$ of $f_{t_i}^{-1}(C_i)$ to a point such that the image of h_i has diameter less than δ. The map $g_i: C_i \times I \to B_i$ defined by $g_i(y, t) = f_{t_i}(h_i(f_{t_i}^{-1}(y), t))$ contracts C_i to a point in B_i, and by (1) follows that diam $g_i(C_i \times I) < 2\varepsilon/2 = \varepsilon$. Thus $s(B_i, u, \varepsilon)$ for all $i > m$. It follows that $d_h(B_i, B_0) \to 0$.

THEOREM 2.2. *Let X be a pseudoisotopically contractible finite dimensional compactum. Then AR_h^X is contractible.*

PROOF. Let $f: X \times I \to X$ be a pseudoisotopy such that $f(X \times \{1\}) = \{p\}$ for some $p \in X$. Let $f_*: AR_h^X \times I \to AR_h^X$ be defined by $f_*(A, t) = f_t(A)$. By 2.1, f_* is continuous for $0 \leqq t < 1$.

Let $\varepsilon > 0$. There is a $t_0 \in I \backslash \{1\}$ such that

(1) $t > t_0$ implies $d(f(x, t), p) < \varepsilon/2$ for all $x \in X$.

Let (A_i, t_i) be a sequence in $AR_h^X \times I$ with $t_i \to 1$. There is an integer n such that $i > n$ implies $t_i > t_0$. For such i, it follows from (1) that $d_s(f_*(A_i, t_i), \{p\}) < \varepsilon$. We conclude that $d_s(f_*(A_i, t_i), \{p\}) \to 0$. Further, since $f_*(A_i, t_i) \in AR_h^X$, it follows from (1) that for $i > n$ we have $s(f_*(A_i, t_i), \varepsilon, \varepsilon)$. Thus $d_h(f_*(A_i, t_i), \{p\}) \to 0$, so f_* is continuous.

Let $S = S^n$ denote the unit sphere in Fuclidean $(n + 1)$-space, $n > 0$.

THEOREM 2.3. *AR_h^S is a path-component of 2_h^S.*

PROOF. Let $A_i \in AR_h^S$, $i = 0, 1$. There is a closed neighborhood B_i of A_i in S such that B_i is homeomorphic to the cube I^n. It follows from 2.2 that there are points $p_i \in S$ and paths in AR_h^S from A_i to $\{p_i\}$. Let $f\colon I \to S$ be a map such that $f(0) = p_0, f(1) = p_1$. It follows from 1.1 that there is a path in AR_h^S from $\{p_0\}$ to $\{p_1\}$. Hence there is a path in AR_h^S from A_0 to A_1.

We remark that (depending on the Poincare conjecture) a fake cube contains an AR that is a subset of no ball in the fake cube. Thus the proof of 2.3 would not work if we were to replace S by a fake cube.

LEMMA 2.4. *Let $f\colon I \to AR_h^S$ be a map. Let $u \in S \backslash f(0)$, $v \in S \backslash f(1)$. Then there is a map $g\colon (I, 0, 1) \to (S, u, v)$ such that for all $t \in I$, $g(t) \notin f(t)$.*

PROOF. From 1.2 it follows that there exist $0 = t_0 < t_1 < \cdots < t_m = 1$, points u_j and open sets $B_j \subset S \backslash f(t_j)$ such that $u_j \in B_j$ $(u_0 = u,\ u_m = v)$, and connected neighborhoods U_j of t_j in I with $U_j \cap U_{j+1} \neq \emptyset$ for $j < m$ such that $t \in U_j$ implies $B_j \subset S \backslash f(t)$. Let $s_j \in U_j \cap U_{j+1}$ and $Y_j = f(s_j)$.

Since $S \backslash Y_j$ is a connected open set ([3], 2.21, p. 103), there is an arc $A_j \subset S \backslash Y_j$ from u_j to u_{j+1}. Since A_j is compact, 1.2 implies there is a neighborhood $\mathscr{V}_j$ of Y_j in AR_h^S such that $Y \in \mathscr{V}_j$ implies $A_j \subset S \backslash Y$. There exist r_j, R_j such that $t_j < r_j < s_j < R_j < t_{j+1}$ and $f([r_j, R_j]) \subset \mathscr{V}_j \cap f(U_j \cap U_{j+1})$.

Let $p_j \in A_j \cap B_j$, $q_j \in A_j \cap B_{j+1}$ be such that the subarcs $\overline{u_j p_j}$ and $\overline{q_j u_{j+1}}$ of A_j lie in B_j and B_{j+1}, respectively. We define the map g as follows:

$g|[t_j, r_j]$ maps $([t_j, r_j], t_j, r_j)$ onto $(\overline{u_j p_j}, u_j, p_j)$;

$g|[r_j, R_j]$ maps $([r_j, R_j], r_j, R_j)$ onto $(\overline{p_j q_j}, p_j, q_j)$;

$g|[R_j, t_{j+1}]$ maps $([R_j, t_{j+1}], R_j, t_{j+1})$ onto $(\overline{q_j u_{j+1}}, q_j, u_{j+1})$.

Since U_j is connected, our choices of r_j and p_j imply that if $t_j \leqq t \leqq r_j$, then $g(t) \notin f(t)$. Similarly, $g(t) \notin f(t)$ if $R_j \leqq t \leqq t_{j+1}$. Our choices of $\mathscr{V}_j$, r_j, and R_j imply $g(t) \notin f(t)$ if $r_j \leqq t \leqq R_j$. Thus $g\colon (I, 0, 1) \to (S, u, v)$ satisfies for all $t \in I$, $g(t) \notin f(t)$.

COROLLARY 2.5. *Let (P, p) be a pointed one-dimensional compact polyhedron. Let $x_0 \in S$ and let $f\colon (P, p) \to (AR_h^S, \{x_0\})$ be a map. Then there is a map $g\colon (P, p) \to (S, -x_0)$ such that for all $y \in P$, $g(y) \notin f(y)$.*

PROOF. We may assume P is connected and that p is a vertex of P. Let $g(p) = -x_0$. We proceed inductively on the one-simplexes of P. Let s be a one-simplex of P with endpoints a and b such that $u = g(a)$ has been defined. If $g(b) = v$ has already been defined, we apply 2.4 to obtain $g|s$; otherwise, choose $v = g(b) \in S \backslash f(b)$ and apply 2.4 to obtain $g|s$. Thus we obtain $g: (P, p) \to (S, -x_0)$ such that for all $y \in P$, $g(y) \notin f(y)$.

It is clear that the map $\lambda: S \to AR_h^S$ defined by $\lambda(x) = \{x\}$ is an embedding. We prove the following theorem.

THEOREM 2.6. *Let $f: (P, p) \to (AR_h^S, \{x_0\})$ be a map, where P is a one-dimensional compact polyhedron. Then f is homotopic rel p to a map whose image lies in $\lambda(S)$.*

PROOF. Let $g: (P, p) \to (S, -x_0)$ be as in 2.5. We define $H: (P, p) \times I \to (AR_h^S, \{x_0\})$ by

$$H(y, t) = \left\{ \frac{(1-t)x - tg(y)}{\|(1-t)x - tg(y)\|} \,\Big|\, x \in f(y) \right\}.$$

By choice of g, the denominator is never 0. Hence H is well-defined. We observe $H(y, 0) = f(y)$, $H(y, 1) = \{-g(y)\} \in \lambda(S)$, and $H(p, t) = \{x_0\}$ for all $t \in I$. Since for each fixed $y \in P$ the collection $\{H_t(y) | t \in I\}$ traces a pseudoisotopy of $f(y)$ in S, $H(P \times I) \subset AR_h^S$ as claimed. The continuity of H follows from that of f, g, and vector operations, by an argument similar to those of 2.1 and 2.2.

For pointed topological spaces (A, a) and (B, b), let $[(A, a), (B, b)]$ denote the collection of pointed homotopy classes of maps from (A, a) to (B, b). We have the following corollary.

COROLLARY 2.7. *If (P, p) and λ are as above, then the function $\lambda_*: [(P, p), (S, x_0)] \to [(P, p), (AR_h^S, \{x_0\})]$ given by $\lambda_*([f]) = [\lambda \circ f]$ is surjective.*

PROOF. This is an immediate consequence of 2.6.

THEOREM 2.8. *$\lambda_*: \Pi_1(S, x_0) \to \Pi_1(AR_h^S, \{x_0\})$ is a surjection. Hence for $S = S^n$, $n > 1$, AR_h^S is simply-connected.*

PROOF. Take $P = S^1$ in 2.7.

Indeed, for $n = 1$ or 2, stronger results may be obtained. It is known ([4], 4.7) that for $n = 2$, the map λ is a homotopy equivalence. For $n = 1$, we have the following theorem.

THEOREM 2.9. *Let $S = S^1$ be the unit circle in the complex plane. Then AR_h^S is homeomorphic to the Cartesian product of S^1 and a half-open interval.*

PROOF. Let $A \in AR_h^S$ have endpoints x and y. (If $A = \{z\}$ for some $z \in S$ then $x = y = z$.) If $x \neq y$, let z be the unique point of A lying on the perpendicular bisector of the line segment from x to y. There is a unique θ such that $0 \leqq \theta < \pi$ and $\{x, y\} = \{ze^{i\theta}, ze^{-i\theta}\}$. It is easily seen that the map sending A to (z, θ) is a homeomorphism of AR_h^S onto $S^1 \times [0, \pi)$.

3. **Questions.** Several questions arise concerning possible improvements of the results in the previous section. We pose them in descending order relative to the degree of improvement that would result from affirmative answers.

It is known (2.2; also, [4], 4.7) that there are some finite-dimensional compacta X for which X and AR_h^X have the same homotopy type. An example of Ball and Ford ([1], 4.8, p. 45) shows that it is possible for X to be an AR while AR_h^X is disconnected. Thus we ask the following questions.

QUESTION 3.1. *Let* $\dim X > 2$. *Is it true that X and AR_h^X have the same homotopy type if X is a manifold? If $X = S^n$, $n > 2$?*

QUESTION 3.2. *Is it true that X and AR_h^X have the same homotopy groups if X is a sphere?*

QUESTION 3.3. *Let $S = S^n$, $n > 2$, and let λ be as above. Does λ induce surjections of all homotopy groups?*

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