ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 3, Summer 1980

# FUSION FREE REPRESENTATIONS OF FINITE GROUPS

MARC J. LIPMAN

I. INTRODUCTION. Let $G$ be a finite group. Let $Q$ denote the field of rational numbers. If $\sigma \in G$, $\langle\sigma\rangle$ is the group generated by $\sigma$, $|\sigma|$ is the order of $\langle\sigma\rangle$, and if $S$ and $S' \subset G$, $S \sim S'$ means $S$ and $S'$ are conjugate subsets of $G$. It is well known that the following definitions are equivalent:

DEFINITION. $G$ is a $Q$-group if every complex character of $G$ is $Q$-valued.

DEFINITION. $G$ has *cyclic conjugacy* if for every $\sigma$ and $\tau \in G$, $\langle\sigma\rangle \sim \langle\tau\rangle$ iff $\sigma \sim \tau$. Equivalently, $G$ has cyclic conjugacy if for every $\sigma$ and $\tau \in G$ such that $\langle\sigma\rangle = \langle\tau\rangle$, then $\sigma \sim \tau$.

This paper presents two other criteria for $Q$-groups, one in terms of permutation representations, one in terms of rational representations. The essential concept is the "fusion free representation".

DEFINITION. Let $f$: $G \to H$ be a homomorphism of groups. $f$ is *fusion free* if for every $\sigma$ and $\tau \in G$: $\sigma \sim \tau$ in $G$ iff $f(\sigma) \sim f(\tau)$ in $H$. This is denoted by $G \sqsubset H$. If $f$ is fusion free then $f$ is $1 - 1$. Thus we consider $G$ to be a subgroup of $H$, justifying the notation. A *fusion free representation* of $G$ is any representation of $G$ consisting of a fusion free homomorphism.

The main results of this paper are:

THEOREM 2.5. *Let $G$ be a finite group. $G$ is a $Q$-group iff for some $n \geqq 1$, $G \sqsubset S_n$.*

COROLLARY 4.3. *Let $G$ be a finite group. $G$ is a $Q$-group iff for some $n \geqq 1$, $G \sqsubset \mathrm{GL}(n, Q)$.*

Let $\sigma \in S_n$. The "type" of $\sigma$ is an $n$-tuple $(c_1, \ldots, c_n)$, where $c_i$ is the number of cycles of length $i$ in $\sigma$. For $\sigma$ and $\tau \in S_n$, $\sigma \sim \tau$ iff $\sigma$ and $\tau$ have the same type. Further, if $\langle\sigma\rangle = \langle\tau\rangle$ then $\sigma$ and $\tau$ have the same cycle structure, that is, $\sigma \sim \tau$. Thus for all $n \geqq 1$, $S_n$ is a $Q$-group.

Let $k$ be a field with char$(k) \nmid n!$. ($n!$ is the order of $S_n$.) Choose an ordered basis for an $n$-dimensional vector space over $k$. Define the natural mapping nat: $S_n \to \mathrm{GL}(n, k)$ by assigning to each $\sigma \in S_n$ the permutation matrix in $\mathrm{GL}(n, k)$ associated to $\sigma$. In Section III the following theorem is proved:

Received by the editors on March 7, 1978.

THEOREM 3.4. *Let $k$ be a field with* $\operatorname{char}(k) \nmid n!$. *Then, using the natural embedding,* $S_n \sqsubset \mathrm{GL}(n, k)$.

II. FUSION FREE PERMUTATION REPRESENTATIONS.

PROPOSITION 2.1. *If $G \sqsubset S_n$, then $G$ is a $Q$-group.*

PROOF. Suppose $\langle \sigma \rangle = \langle \tau \rangle$ for $\sigma$ and $\tau \in G$. Then $\sigma \sim \tau$ in $S_n$ since $S_n$ is a $Q$-group. $G \sqsubset S_n$ implies $\sigma \sim \tau$ in $G$.

The goal for the remainder of this section is to prove the converse of Proposition 2.1. Let $G$ be a $Q$-group. Let $\sigma_1, \ldots, \sigma_s$ be a full set of representatives for the conjugacy classes of $G$, ordered so that $|\sigma_i| \leqq |\sigma_{i+1}|$ for $i = 1, \ldots, s-1$.

Consider the characters $\alpha_i = 1_{\langle \sigma_i \rangle}^G$ for $i = 1, \ldots, s$. $\alpha_i$ is the character of $G$ of the permutation representation $(G, G/\langle \sigma_i \rangle)$. That is, $\alpha_i$ is the character of the representation of $G$ acting by left multiplication on the left cosets of $\langle \sigma_i \rangle$ in $G$.

PROPOSITION 2.2. *Let $G$ be a $Q$-group. With the above notation, if $j > i$ then $\alpha_i(\sigma_j) = 0$.*

PROOF. $\alpha_i(\sigma_j) \neq 0$ iff for some $\tau \in G$, $\sigma_j \tau \langle \sigma_i \rangle = \tau \langle \sigma_i \rangle$
iff $\sigma_j \sim \sigma_i^k$ for some integer $k$.
If $j > i$, $|\sigma_j| \geqq |\sigma_i|$ and $\langle \sigma_j \rangle \nsim \langle \sigma_i \rangle$ since $G$ is a $Q$-group. Thus $\sigma_j \nsim \sigma_i^k$ for any $k$. Thus $\alpha_i(\sigma_j) = 0$.

Note that $\alpha_i(\sigma_i) = [N_G(\langle \sigma_i \rangle) : \langle \sigma_i \rangle] =$ the index of $\langle \sigma_i \rangle$ in its normalizer in $G \neq 0$.

THEOREM 2.3. *Let $G$ be a $Q$-group. Use the above notation. There exists a proper permutation character $\chi$ of $G$ so that if $i \neq j$ then $\chi(\sigma_i) \neq \chi(\sigma_j)$.*

PROOF. Let $\chi = \sum_{l=1}^{s} a_l \alpha_l$ with the $a_l$ chosen as follows: Let $a_s = 1$. For $j = s-1, \ldots, 1$ let $a_j = 1 + \sum_{k=j+1}^{s} a_k \alpha_k(\sigma_{j+1})$. Then for all $i > j$, $\sum_{k=j}^{s} a_k \alpha_k(\sigma_j) > \sum_{k=i}^{s} a_k \alpha_k(\sigma_i)$. By Proposition 2.2 $\chi(\sigma_i) = \sum_{k=i}^{s} a_k \alpha_k(\sigma_i)$, so that if $i > j$ $\chi(\sigma_j) > \chi(\sigma_i)$.

THEOREM 2.4. *Let $G$ be a $Q$-group. For some $n \geqq 1$, $G \sqsubset S_n$.*

PROOF. Choose $\chi$ as in Theorem 2.3. Let $n = \chi(1)$. Let $X: G \to S_n$ be the permutation representation afforded by $\chi$. Then via $X$, $G \sqsubset S_n$.

PROPOSITION 2.1 and Theorem 2.4 immediately give

THEOREM 2.5. *Let $G$ be a finite group. $G$ is a $Q$-group iff for some $n > 1$, $G \sqsubset S_n$.*

III. FUSION FREE REPRESENTATION OF $S_n$. Let $k$ be an algebraically closed field with char($k$) $\nmid n!$ Consider the natural embedding nat: $S_n \to \mathrm{GL}(n, k)$ described in Section I. Let $\sigma \in S_n$ and suppose $\sigma$ has type $(c_1, \ldots, c_n)$. Considering $\sigma$ as a permutation matrix in GL($n$, $k$) define $w_j$ to be the multiplicity of any primitive $j^{\text{th}}$ root of unity as an eigenvalue of $\sigma$.

LEMMA 3.1: *$w_j$ is well defined. For all $j = 1, \ldots, n$ $w_j = c_j + c_{2j} + c_{3j} + \cdots = \sum_{i; j|i} c_i$.*

PROOF. $\sigma$ is similar to a permutation matrix that is the direct sum of matrices of cycles. Thus the characteristic polynomial of $\sigma$ is the product of the characteristic polynomials of the cycles of $\sigma$. A cycle of length $i$ has characteristic polynomial $X^i - 1$. If $\zeta$ is a primitive $j^{\text{th}}$ root of unity, then $\zeta$ is a root of $X^i - 1$ iff $j|i$. Further, if $j|i$ then $\zeta$ is a root of $X^i - 1$ of multiplicity exactly 1. Hence $w_j$ = the number of cycles of length a multiple of $j$, giving the result.

LEMMA 3.2. *For all $j = 1, \ldots, n$, $c_j = \sum_i \mu(i) w_{ij}$, where $\mu$ is the classical Möbius function.*

PROOF. Möbius inversion on the partially ordered set of the integers with the dual division ordering: $i \leqq j$ iff $j|i$. (See [**1**], p. 83.)

THEOREM 3.3. *Using the natural embedding, $S_n \sqsubset \mathrm{GL}(n, k)$.*

PROOF. Choose $\sigma$ and $\tau \in S_n$. If $\sigma \sim \tau$ in GL($n$, $k$) then $\sigma$ and $\tau$ have the same eigenvalue structure. By Lemma 3.2 this eigenvalue structure determines a unique type. Thus $\sigma \sim \tau$ in $S_n$.

THEOREM 3.4. *Let $k$ be any field with* char($k$) $\nmid n!$ *Using the natural embedding, $S_n \sqsubset \mathrm{GL}(n, k)$.*

PROOF. Let $K$ be the algebraic closure of $k$. Then $S_n \sqsubset \mathrm{GL}(n, K)$ and $S_n \subseteq \mathrm{GL}(n, k) \subseteq \mathrm{GL}(n, K)$. If $\sigma$ and $\tau \in S_n$ and $\sigma \sim \tau$ in GL($n$, $k$), then $\sigma \sim \tau$ in GL($n$, $K$). Thus $\sigma \sim \tau$ in $S_n$ so $S_n \sqsubset \mathrm{GL}(n, k)$.

COROLLARY 3.5. *Let $G$ be a $Q$-group. Let $k$ be a field with* char($k$) = 0. *For some $n \geqq 1$, $G \sqsubset \mathrm{GL}(n, k)$.*

PROOF. *For some $n \geqq 1$, $G \sqsubset S_n$. Since* char($k$) = 0, $S_n \sqsubset \mathrm{GL}(n, k)$. *Thus $G \sqsubset \mathrm{GL}(n, k)$.*

IV. FUSION FREE RATIONAL REPRESENTATIONS. Denote the $s^{\text{th}}$ cyclotomic polynomial over $Q$ by $\varphi_s(X)$.

LEMMA 4.1. *Let $M$ and $N \in \mathrm{GL}(n, Q)$ be matrices of finite order $r$. Suppose $\langle M \rangle = \langle N \rangle$. Then $M \sim N$.*

PROOF. $N = M^a$ for some a with $(a, r) = 1$. $M$ satisfies $X^r - 1 = \Pi_{s|r}\varphi_s(X)$. Thus the minimal polynomial of $M$, $m(X) = \Pi_{s|r}(\varphi_s(X))^{e_s}$, where $e_s = 0$ or 1 for all $s$, since each $\varphi_s(X)$ is irreducible and all are distinct. The characteristic polynomial of $M$, $c(X)$, can be written as the product of the $\varphi_s(X)$ appearing in $m(X)$. That is:

$$c(X) = \prod_{(s|r)} (\varphi_s(X))^{f_s}, \ f_s \neq 0 \text{ iff } e_s = 1.$$

If $\zeta$ is an eigenvalue of $M$ of multiplicity $f$, then $\zeta^a$ is an eigenvalue of $M^a = N$ of multiplicity at least $f$. Since $\zeta$ is a root of $\varphi_s(X)$ for some $s|r$ and $(a, r) = 1$, it follows that $\zeta^a$ is a root of the same $\varphi_s(X)$. That is, if $d(X)$ is the characteristic polynomial of $N$,

$$d(X) = \prod_{(s|r)} (\varphi_s(X))^{g_s}, \ g_s \geqq f_s.$$

But degree $(c(X)) = n =$ degree $(d(X))$, so $c(X) = d(X)$.

Because $m(X)$ is the product of distinct irreducibles, the rational canonical form of $M$ is completely determined by the number of blocks due to each irreducible factor of $m(X)$. That is, the numbers $f_s$, $s|r$, completely determine the class of $M$ in $\mathrm{GL}(n, k)$. Since $N$ also satisfies $X^r - 1$, etc., $c(X) = d(X)$ gives $M \sim N$.

THEOREM 4.2. *Let $G$ be a finite group. If $G \sqsubset \mathrm{GL}(n, Q)$, then $G$ is a $Q$-group.*

PROOF. For $\sigma$ and $\tau \in G$, if $\langle\sigma\rangle = \langle\tau\rangle$ then by Lemma 4.1 $\sigma \sim \tau$ in $\mathrm{GL}(n, Q)$. Then $G \sqsubset \mathrm{GL}(n, Q)$ gives $\sigma \sim \tau$ in $G$.

COROLLARY 4.3. *Let $G$ be a finite group. $G$ is a $Q$-group iff for some $n \geqq 1$, $G \sqsubset \mathrm{GL}(n, Q)$.*

PROOF. In Corollary 3.5 let $k = Q$. Then Corollary 3.5 and Theorem 4.2 give the result.

REFERENCE

1. Berge, C. *Principles of Combinatories*, Academic Press, New York, (1971).

INDIANA UNIVERSITY-PURDUE UNIVERSITY, FORT WAYNE INDIANA