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FIELDS WITH FREE MULTIPLICATIVE GROUPS MODULO TORSION

WARREN MAY

Fields whose multiplicative groups are free modulo roots of unity occur in several familar settings. Not only are finite algebraic number fields of this type, but so is every field that is finitely generated over its prime subfield. In addition, there are interesting examples which are infinitely generated. A theorem of Schenkman [6] shows that if K is generated over the rational numbers by a family of algebraic elements of bounded degree, then the multiplicative group K^* is free modulo torsion. Likewise, it follows from [5] that the same conclusion holds if K is the maximal cyclotomic extension of a finite algebraic number field.

The main purpose of this paper is to generalize two theorems in [5] from countable fields to arbitrary fields. These theorems were based on the last two examples mentioned above, and were shown by utilizing Pontryagin's theorem on countable torsion-free abelian groups. We can now avoid this utilization by a more thorough consideration of the field structures involved. We shall prove the following generalization of the theorem of Schenkman.

THEOREM 1. *Assume that F is a field such that for every finite extension field E, E^* is free modulo torsion. Let K be any field generated over F by algebraic elements whose degrees over F are bounded. Then K^* is free modulo torsion.*

In the second result, K is taken to be an (infinite) Galois extension of F with abelian Galois group.

THEOREM 2. *Assume that F is a field such that for every finite extension field E, E^* is free modulo torsion and E contains only finitely many roots of unity. Let K be any abelian extension field of F. Then K^* is free modulo torsion.*

We point out that it is not enough in either theorem to assume that F^* alone is free modulo torsion (see Example 1 in [5]). Furthermore, it is not enough in Theorem 2 to assume that F alone contains only finitely many roots of unity. To see this, let C be the maximal cyclotomic extension of the rational numbers, let F be the maximal real subfield of C, and let K

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be generated over C by $\{\alpha_p | p \text{ prime}\}$, where $\alpha_p^p = (2+\sqrt{-1})(2-\sqrt{-1})^{-1}$ for every p. The only roots of unity in F are ± 1, and a suitable application of Theorem 2 shows that E^* is free modulo torsion for every finite extension E of F. Moreover, K is easily shown to be an abelian extension field of F. However, K^* is clearly not free modulo torsion.

1. **Some Results on E^*/F^*.** Let F be a field and let E be an extension field of F. Various aspects of the quotient group E^*/F^* have been studied, for example, by Kneser [4], Siegel [7], Brandis [1] and Kaplansky [3]. We shall devote this section to proving several facts pertaining to the torsion subgroup of this quotient.

Let $T = T(E/F)$ denote the subgroup of E^* containing F^* such that T/F^* is the torsion subgroup of E^*/F^*. Therefore $T = \{x \in E^* | x^m \in F^*$ for some $m \geqq 1\}$. Let W denote the subgroup of E^* generated by F^* and by all the roots of unity that lie in E^*. Thus $E^* \supseteq T \supseteq W \supseteq F^*$. It is possible for W/F^* to be infinite even when E is a finite extension of F, consequently we shall restrict our considerations to T/W. For each prime number p, let T_p (respectively, W_p) denote the subgroup of E^* containing F^* such that T_p/F^* (respectively, W_p/F^*) is the p-component of T/F^* (respectively, W/F^*).

LEMMA 1. *Let p be a prime different from the characteristic of F, and let η_0 be a primitive p^{th} root of unity if p is odd, or a primitive 4^{th} root of unity if $p = 2$. Put $E = F(\eta_0)$. Then $T_p = W_p$ except when $p = 2$, $\eta_0 \notin F$, $\sqrt{-2} \notin F$, and E contains only finitely many roots of unity whose orders are powers of 2. In the exceptional case, we have $(T_2 : W_2) = 2$.*

PROOF. Let p be a prime and let $n = [E: F]$. We claim that T_p/W_p has exponent n. Let $\alpha \in T_p$. Then $\alpha^{p^r} = \gamma$ for some $r \geqq 1$ and $\gamma \in F$. If N denotes the norm mapping from E^* to F^*, then $N(\alpha)^{p^r} = \gamma^n$. Thus $(\alpha^n N(\alpha)^{-1})^{p^r} = 1$. It follows that $\alpha^n \in W_p$. In particular, if p is an odd prime, then n divides $p - 1$, hence we can conclude that $T_p = W_p$ in this case.

We may now assume that $p = 2$ and that $\eta_0 \notin F$. Let U denote the 2-component of the group of roots of unity that lie in E. Since $[E: F] = 2$, it follows from the above that if $\alpha \in T_2$, then $\alpha^2 = \zeta\gamma$ for some $\zeta \in U$ and $\gamma \in F$. We claim that if $\zeta = \zeta_1^2$ for some $\zeta_1 \in U$, then $\alpha \in W_2$. Observe that $(\alpha\zeta_1^{-1})^2 \in F$, hence $\alpha\zeta_1^{-1} = \eta_0^i\gamma_1$ for some integer i and $\gamma_1 \in F$. Thus $\alpha \in W_2$, as claimed. In particular, if U is infinite, then $T_2 = W_2$.

We may now further assume that U is finite, say $U = \langle\eta\rangle$. Let σ denote the nontrivial automorphism of E over F. First suppose that $\sqrt{-2} \in F$. Then it is easy to show that $\sigma(\eta) = -\eta^{-1}$ since $\eta_0 \notin F$ and $\sqrt{-2} \in F$. If $\alpha \in T_2$, then $\alpha^2 = \eta^i\gamma$ for some integer i and $\gamma \in F$. Thus $\sigma(\alpha)^2 = -\eta^{-i}\gamma$, and hence $N(\alpha)^2 = (-1)^i\gamma^2$. Consequently, $(-1)^i$ is a square in F. Since

$\eta_0 \notin F$, we conclude that i is even. Hence, $\alpha \in W_2$ by the claim in the previous paragraph. Therefore, $T_2 = W_2$ in this case. Finally, we now suppose that $\sqrt{-2} \notin F$. In this situation, one has $\sigma(\eta) = \eta^{-1}$. If we put $\alpha_0 = 1 + \eta$, then $\alpha_0^2 = \eta\gamma_0$, where $\gamma_0 = \eta + \eta^{-1} + 2 \in F$. Thus $\alpha_0 \in T_2$. We claim that $\alpha_0 \notin W_2$. Suppose that $\alpha_0 = \eta^i\gamma$ for some integer i and $\gamma \in F$. Then $1 + \eta = \eta^i\gamma$ Applying σ, we obtain $1 + \eta^{-1} = \eta^{-i}\gamma$. Thus $\eta = (1 + \eta)(1 + \eta^{-1})^{-1} = \eta^{2i}$. This is a contradiction since η has even order, thus $\alpha_0 \notin W_2$. To show that α_0 generates T_2 modulo W_2, let $\alpha \in T_2$. Then $\alpha^2 = \eta^i\gamma$ for some integer i and $\gamma \in F$. Therefore $(\alpha_0^i\alpha^{-1})^2 = \gamma_0^i\gamma^{-1}$. By the claim in the previous paragraph, we conclude that $\alpha_0^i\alpha^{-1} \in W_2$. Hence, $(T_2\colon W_2) = 2$ in this case.

We now combine Lemma 1 with a theorem of Kneser to obtain

PROPOSITION 1. *Let E be a finite separable extension field of F. Then $(T\colon W)$ divides $[E\colon F]$.*

PROOF. Let p be a prime. It suffices to show that $(T_p\colon W_p)$ divides $[E\colon F]$. Let η_0 be taken as in the statement of the lemma, and let $F_0 = F$ if $\eta_0 \notin E$, or $F_0 = F(\eta_0)$ if $\eta_0 \in E$. Then F_0 satisfies the hypothesis of the theorem of Kneser [4] applied to $T_pF_0^*$. Consequently, $(T_pF_0^*\colon F_0^*)$ divides $[E\colon F_0]$. Put $S = T_p \cap F_0^*$. Then $(T_p\colon S)$ divides $[E\colon F_0]$. Note that $S = T_p(F_0/F)$ and $S \cap W_p = W_p(F_0/F)$; thus, Lemma 1 implies that $(S\colon S \cap W_p)$ divides $[F_0\colon F]$. It follows that $(T_p\colon W_p)$ divides $[E\colon F]$.

We shall not need Proposition 1 in proving Theorems 1 and 2. However, we will make use of the following result, which shows that the subgroup T is a direct factor of E^* when E satisfies a hypothesis relevant to the theorems.

PROPOSITION 2. *Let E be a finite extension field of F, and assume that E^* is free modulo torsion. Then $E^* = A \times T$ for some free abelian subgroup A of E^*.*

PROOF. Let N denote the norm mapping from E^* to F^*, let H denote the kernel of N, and let $n = [E\colon F]$. If $\alpha \in E^*$, then $N(\alpha^n N(\alpha)^{-1}) = 1$. It follows that $(E^*)^n \subseteq HF^*$. Let H_0 denote the torsion subgroup of H. If $\alpha \in H \cap T$, then $\alpha^m \in F^*$ for some positive integer m, hence $1 = N(\alpha^m) = \alpha^{mn}$. Thus, we see that $H \cap T = H_0$. Now let $f\colon E^* \to E^*/T$ be the natural map. Since E^* is free modulo torsion, it follows that H/H_0 is free. But $f(H) \cong H/H_0$, and what we have shown above implies that $(E^*/T)^n \subseteq f(H)$. Since E^*/T is torsion-free, we conclude that E^*/T is free. The proof is thus complete.

If the hypothesis that E^* is free modulo torsion is omitted, then T may no longer be a direct factor of E^*. We remark that one can give an example in which E is a quadratic extension of F, F^* is free modulo torsion, but

T/F^* is not even a direct factor of E^*/F^*.

We shall now give an example to show that the proposition can also fail for infinite algebraic extensions. Let $\{t_p | p$ prime or $p = 0\}$ be algebraically independent transcendentals over the rational numbers Q, let F be the field generated over Q by $\{t_p\}$, and let E be the field generated over Q by $\{\alpha_p\}$, where $\alpha_0^2 = t_0$ and $\alpha_p^p = t_p(\alpha_0 + 1)$ for p prime. Then $\alpha_0 + 1 \notin T$, but $\alpha_p^p \equiv \alpha_0 + 1$(modulo T) for every prime p. Hence E^*/T is not free. But E^* is free modulo torsion since $\{\alpha_p\}$ are algebraically independent transcendentals over Q. Thus T cannot have a complement in E^* since such a complement would necessarily be free.

2. **Proofs of the Theorems.** Let K be an extension filed of F, and let τ be an ordinal. We shall say that a family of subfields $\{E_\alpha | \alpha < \tau\}$ forms a continuous chain of finite extensions from F to K if $E_\beta \subseteq E_\alpha$ for $\beta \leqq \alpha < \tau$, $[E_{\alpha+1}: E_\alpha] < \aleph_0$ for $\alpha + 1 < \tau$, $E_\alpha = U_{\beta<\alpha} E_\beta$ for α a limit ordinal $< \tau$, $F = E_0$, and $K = U_{\alpha<\tau} E_\alpha$.

LEMMA 2. *Let $\{E_\alpha | \alpha < \tau\}$ be a continuous chain of finite extensions from F to K, and let $T_\alpha = T(K/E_\alpha)$ for $\alpha < \tau$. Assume that E_α^* is free modulo torsion and that T_α/E_α^* is bounded for every α. Then K^* is free modulo torsion.*

PROOF. Using the obvious analogue of the definition above, we observe that $\{T_\alpha | \alpha < \tau\}$ forms a continuous chain of subgroups from T_0 to K^*. (We do not require $(T_{\alpha+1}: T_\alpha)$ to be finite.) Note that T_0 is free modulo torsion since T_0/E_0^* is bounded and since E_0^* is free modulo torsion. Let B_0 be a free complement in T_0 for the torsion subgroup of T_0 (which in fact equals the torsion subgroup of K^*). It will suffice to show that $T_{\alpha+1}/T_\alpha$ is free for every α such that $\alpha + 1 < \tau$, for then we can choose a free abelian group $B_{\alpha+1}$ such that $T_{\alpha+1} = B_{\alpha+1} \times T_\alpha$. The group $\sum_{\alpha<\tau} B_\alpha$ is then easily seen to be a free complement in K^* for the torsion subgroup of K^*. To show that $T_{\alpha+1}/T_\alpha$ is free, let $T_\alpha' = T(E_{\alpha+1}/E_\alpha)$. Then $T_\alpha \cap E_{\alpha+1}^* = T_\alpha'$. Proposition 2 implies that $E_{\alpha+1}^*/T_\alpha'$ is free, hence $T_\alpha E_{\alpha+1}^*/T_\alpha$ is free. But $T_{\alpha+1}/T_\alpha$ is torsion-free, and $T_{\alpha+1}/T_\alpha E_{\alpha+1}^*$ is bounded since $T_{\alpha+1}/E_{\alpha+1}^*$ is bounded by hypothesis. Thus $T_{\alpha+1}/T_\alpha$ is free, hence the lemma is proved.

We shall now prove Theorem 1 with the aid of Lemma 2 and the proof of Theorem 1 in [5]. Let K and F satisfy the hypotheses of Theorem 1. We shall use induction on $[K: F]$. Clearly we may assume that $[K: F] \geqq \aleph_0$. Let τ be the first ordinal with $|\tau| = [K: F]$. If K is generated over F by elements whose degrees are bounded by n, then we may choose a family $\{x_\alpha | \alpha < \tau\}$ of such generators. For each $\alpha < \tau$, let E_α be generated over F by $\{x_\beta | \beta < \alpha\}$. It is immediate that $\{E_\alpha | \alpha < \tau\}$ is a continuous chain of finite extensions from F to K. Since $[E_\alpha: F] < |\tau|$, it follows by induction that E_α^* is free modulo torsion. Thus, all we need to finish the proof is to show that T_α/E_α^* is bounded. We now refer to the proof of Theorem 1 in [5]. If we take E in that proof to be the present E_α, then inspection of the proof reveals that countability of K and finiteness of E over F are superfluous hypotheses for the purpose of showing that T_α/E_α^* is bounded. We note that the slight adjustment of the ground field F that occurs in the proof can also be carried out in the present situation without difficulty. Thus T_α/E_α^* is bounded, concluding the proof of Theorem 1.

Before considering Theorem 2, we must prove a lemma that will be useful in managing the hypothesis concerning roots of unity.

LEMMA 3. *Let $F_1 \subseteq F_2 \subseteq \ldots$ be subfields of a field K, and put $F_\infty = \bigcup_{1\leq i} F_i$. Then there exist subfields $K_1 \subseteq K_2 \subseteq \ldots$ such that $K_i \cap F_\infty = F_i$ for $i \geqq 1$, and $\bigcup_{1\leq i} K_i = K$.*

PROOF. A simple application of Zorn's lemma shows that we may choose a maximal subfield K_1 of K such that $K_1 \cap F_\infty = F_1$ and $K_1F_2 \cap F_\infty = F_2$. A second application shows that we may choose a maximal subfield K_2 of K such that $K_2 \supseteq K_1$, $K_2 \cap F_\infty = F_2$ and $K_2F_3 \cap F_\infty = F_3$. We may continue by induction to construct $K_1 \subseteq K_2 \subseteq \ldots$ such that $K_i \cap F_\infty = F_i$, $K_iF_{i+1} \cap F_\infty = F_{i+1}$, and K_i is maximal with respect to these two properties. Let $\alpha \in K$. We must show that $\alpha \in \bigcup_{1\leq i} K_i$. We observe that α cannot be transcendental over $\bigcup_{1\leq i} K_i$, since otherwise $K_1(\alpha)$ is easily seen to contradict the maximality of K_1. Hence, there is some $i_0 \geqq 1$ such that α is algebraic over K_{i_0}. Let $i \geqq i_0$. We may suppose that $K_i(\alpha) \neq K_i$ since otherwise we are done. By the maximality of K_i, we must have either $K_i(\alpha) \cap F_\infty \supsetneqq F_i$ or $K_i(\alpha)F_{i+1} \cap F_\infty \supsetneqq F_{i+1}$. First suppose that $K_i(\alpha) \cap F_\infty \supsetneqq F_i$. Choose $\beta \in K_i(\alpha) \cap F_\infty$, $\beta \notin F_i$. Then $\beta \in K_j$ for some $j > i$. Note that $\beta \notin K_i$, therefore $[K_i(\alpha): K_i] > [K_i(\alpha): K_i(\beta)]$. Consequently $[K_i(\alpha): K_i] > [K_j(\alpha): K_j]$. Now suppose that $K_i(\alpha)F_{i+1} \cap F_\infty \supsetneqq F_{i+1}$. Then $K_{i+1}(\alpha) \cap F_\infty \supsetneqq F_{i+1}$ since $K_{i+1}(\alpha) \supseteq K_i(\alpha)F_{i+1}$. Therefore, the previous case shows that $[K_i(\alpha): K_i] \geqq [K_{i+1}(\alpha): K_{i+1}] > [K_j(\alpha): K_j]$ for some $j > i$. Thus, in either case, this reduction of the degree of α implies that $\alpha \in \bigcup_{1\leq i} K_i$. The lemma is proved.

We now prove Theorem 2. Assume that K and F satisfy the hypotheses of Theorem 2. We shall use induction on $[K: F]$, which we may assume is infinite. First we shall suppose that K contains only finitely many roots of unity. Let τ be the first ordinal with $|\tau| = [K: F]$, and choose a family of generators $\{x_\alpha | \alpha < \tau\}$ of K over F. For each $\alpha < \tau$, let E_α be generated over F by $\{x_\beta | \beta < \alpha\}$. Then $\{E_\alpha | \alpha < \tau\}$ is a continuous chain of finite extensions from F to K. Since E_α is an abelian extension of F, and since $[E_\alpha: E] < |\tau|$, it follows by induction that E_α^* is free modulo torsion. To show that T_α/E^*_α is bounded, we refer to the proof of Theorem 2 in [5]. As was the case in the proof of Theorem 1, we observe that countability of K is a superfluous hypothesis in showing that T_α/E^*_α is bounded. Moreover, the finiteness of E over F may be replaced by the assumption that E contains only finitely many roots of unity, a hypothesis which the current E_α satisfies. Thus T_α/E^*_α is bounded. Lemma 2 now implies that K^* is free modulo torsion.

We now suppose that K may contain infinitely many roots of unity. Let F_∞ be the field generated over F by the roots of unity that lie in K. The hypotheses on F guarantee that we may choose subfields $F \subseteq F_1 \subseteq F_2 \subseteq \ldots$ such that $\bigcup_{1 \leqq i} F_i = F_\infty$, and F_i contains only finitely many roots of unity for each i. Apply Lemma 3 to obtain subfields $K_1 \subseteq K_2 \subseteq \ldots$ such that $\bigcup_{1 \leqq i} K_i = K$, and $K_i \cap F_\infty = F_i$ for every $i \geqq 1$. Put $K_0 = F$. It is clear that we may choose a continuous chain of finite extensions from K_i to K_{i+1} for every $i \geqq 0$. Thus, we obtain a continuous chain $\{E_\alpha | \alpha < \tau\}$ (for some ordinal τ) of finite extensions from F to K such that given $\alpha < \tau$, there exists $i \geqq 1$ such that $E_\alpha \subseteq K_i$. Thus each E_α contains only finitely many roots of unity. Since E_α is an abelian extension of F with $[E_\alpha\colon F] \leqq [K\colon F]$, what we have proved in the previous paragraph implies that E^*_α is free modulo torsion. Also, as above, the proof of Theorem 2 in [5] shows that T_α/E^*_α is bounded. An application of Lemma 2 now completes the proof.

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Department of Mathematics University of Arizona Tucson, Arizona 85721