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C^*-ALGEBRAS OF FUNCTIONS ON DIRECT PRODUCTS OF SEMIGROUPS

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ABSTRACT. Berglund and Milnes, generalizing results of deLeeuw and Glicksberg, have shown that if S_1 and S_2 are semitopological semigroups with right and left identities respectively, then the almost periodic (AP) compactification of the direct product $S_1 \times S_2$ is the direct product of the AP compactifications of S_1 and S_2—in symbols $(S_1 \times S_2)^{AP} = S_1^{AP} \times S_2^{AP}$. They also showed that the analog of this result holds in the weakly almost periodic (WAP) case if S_1 is a compact topological group. In this paper we extend these results, first by replacing the spaces AP and WAP by more general C^*-algebras of functions, and second by considering direct products of arbitrarily many semigroups. Several general theorems are proved from which the following corollaries may be derived: If S_1 is a dense subsemigroup of a compact topological group G then $(S_1 \times S_2)^{WAP} = G \times S_2^{WAP}$ and $(S_1 \times S_2)^{LUC} = G \times S_2^{LUC}$. If $\{S_i: i \in I\}$ is a family of semitopological semigroups with identities and $S = \Pi\{S_i: i \in I\}$, then $S^{AP} = \Pi\{S_i^{AP}: i \in I\}$ and $S^{SAP} = \Pi\{S_i^{SAP}: i \in I\}$.

1. **Introduction.** Let S_1 and S_2 be semitopological semigroups, $S = S_1 \times S_2$ their direct product, and F a sub-C^*-algebra of $C(S)$. Define

(1) $$F_1 = \{f(\cdot, s_2): f \in F, s_2 \in S_2\}, F_2 = \{f(s_1, \cdot): f \in F, s_1 \in S_1\}.$$

Suppose that S has a right topological F-compactification S^F (as defined below) and that S_i has a right topological F_i-compactification $S_i^{F_i}$ $(i = 1, 2)$. We wish to determine conditions under which S^F is (canonically isomorphic to) the direct product of $S_1^{F_1}$ and $S_2^{F_2}$—in symbols,

(2) $$S^F = S_1^{F_1} \times S_2^{F_2}.$$

Special cases of (2) have been verified by several authors. In **[6]** it was shown that (2) holds for the case $F = AP(S)$ (hence $F_i = AP(S_i)$) when S_i is a commutative topological semigroup with identity $(i = 1, 2)$. The restrictions of commutativity and joint continuity of multiplication were later removed in **[8]** using the device of tensor products. In **[4]** it was shown that S_1 and S_2 need only have right and left identities respectively.

The situation is less well-behaved in the weakly almost periodic case. For example, if $S_1 = S_2$ is any commutative topological semigroup with

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identity for which $WAP(S_1) \neq AP(S_1)$ (a non-compact locally compact abelian topological group, say) then (2) fails for $F = WAP(S)$ [**8**]. On the other hand Berglund and Milnes have shown that (2) holds in the WAP case if S_1 is a compact topological group and S_2 is any semitopological semigroup with left identity [**4**]; and Milnes has given an example of a locally compact non-compact topological group G such that (2) obtains in the WAP case if $S_1 = S_2 = G$ [**9**].

In this paper we prove a general result which gives necessary and sufficient conditions on F for (2) to hold, and from this we derive extensions of and complements to the results mentioned above. The analog of (2) for the case of an infinite direct product is also proved.

2. **Preliminaries.** Let S be a semitopological semigroup (as defined in [**1**]), $B(S)$ the C^*-algebra of bounded complex-valued functions on S, and $C(S)$ the sub-C^*-algebra of continuous functions. For $s \in S$ let R_s and L_s denote respectively the right and left translation operators on $B(S)$ defined by

$$(R_s f)(t) = f(ts),\ (L_s f)(t) = f(st)\ (f \in B(S),\ t \in S).$$

Denote by βS the spectrum (= space of non-trivial continuous complex homomorphisms) of $B(S)$. A subset F of $B(S)$ is *right* (resp. *left*) *translation invariant* if $R_s F \subset F$ (resp. $L_s F \subset F$) for all $s \in S$; *translation invariant* if it is both right and left translation invariant; and *left m-introverted* if F contains all functions of the form $s \to z(L_s f)$, where $f \in F$ and $z \in \beta S$. If F is left m-introverted then it is also right translation invariant, as can be seen by taking z to be evaluation at points of S. The converse holds if F is a weakly closed subset of $WAP(S)$ (see definition below).

Let F be a sub-C^*-algebra of $C(S)$ containing the constant functions. An *F-compactification of S* is a pair (X, u), where X is a compact (Hausdorff) topological space and $u: S \to X$ is a continuous mapping with range dense in X such that $u^*C(X) = F$. Here $u^*: C(X) \to C(S)$ denotes the dual mapping $f \to f \circ u$. The Gelfand theory of commutative C^*-algebras shows that the pair consisting of the spectrum of F (with the relativized weak* topology) and the evaluation mapping is an F-compactification of S. We shall call this the *canonical F-compactification of S*. Furthermore, F-compactifications are unique up to isomorphism in the following sense: If (X, u) and (Y, v) are F-compactifications of S then there exists a homomorphism $w: X \to Y$ such that $v = w \circ u$.

An F-compactification (X, u) of S will be called *right topological* (abbreviated r.t.) if it enjoys the following properties:

(a) X is a semigroup such that the right multiplication mapping $x \to xy$: $X \to X$ is continuous for each $y \in Y$; and

(b) u is a homomorphism, and the left multiplication mapping $x \to u(s)x$: $X \to X$ is continuous for each $s \in S$.

If a r.t. F-compactification of S exists then F must be translation invariant and left m-introverted. The left translation invariance is a consequence of (b) and the identity $L_s f = u^* L_{u(s)} Uf$ $(s \in S, f \in F)$, where $U\colon F \to C(X)$ denotes the inverse of u^*. To see that F is left m-introverted, observe that the spectrum of F (which is the restriction to F of βS) consists of all mappings of the form $h_x\colon f \to (Uf)(x)$ $(f \in F,\ x \in X)$, and that the mapping $s \to h_x(L_s f)$ is just $u^*(R_x Uf)$.

Conversely, if F is translation invariant and left m-introverted then any F-compactification (X, u) of S is r.t. In fact, if (X, u) is canonical and if $T_y\colon F \to F$ is the algebra homomorphism defined by $(T_y f)(s) = y(L_s f)$ $(s \in S, f \in F)$, then $xy = x \circ T_y$ defines a multiplication on X which satisfies the requirements of (a) and (b). (For details about these and related facts, and for further references, see [**2**, **3**].)

We shall frequently use the notation S^F to denote an F-compactification of S, the mapping u being understood. Thus equation (2) in the introduction simply asserts that $(X_1 \times X_2, u_1 \times u_2)$ is an F-compactification of S, where (X_i, u_i) denctes an F_i-compactification of S_i, and $u_1 \times u_2\colon S \to X_1 \times X_2$ is the product mapping.

For convenience we shall call a sub-C^*-algebra F of $C(S)$ *admissible* if F is translation invariant, left m-introverted, and contains the constant functions. The following are standard examples of admissible sub-C^*-algebras of $C(S)$:

$AP(S) = \{f \in C(S)\colon R_S f \text{ is relatively norm compact}\}$,
$WAP(S) = \{f \in C(S)\colon R_S f \text{ is relatively weakly compact}\}$,
$LUC(S) = \{f \in C(S)\colon s \to L_s f \text{ is norm continuous}\}$,
$SAP(S)$ = closed linear span of the coefficients of all continuous finite dimensional unitary representations of S.

We shall occasionally suppress the letter S in the notation $AP(S)$, etc. The abbreviations AP, WAP, LUC, and SAP stand for "almost periodic", "weakly almost periodic", "left uniformly continuous", and "strongly almost periodic", respectively. It is well known that S^{AP} is a topological semigroup, S^{WAP} a semitopological semigroup, and S^{SAP} a topological group [**1**, **3**, **5**].

Each of the above examples of admissible algebras is stable under the dual of a continuous homomorphism. By this we mean that if F is one of these algebras and H is the corresponding algebra on another semitopological semigroup T, then for any continuous homomorphism $w\colon S \to T$, $w^*(H) \subset F$.

Note that $B(S) = C(S_d)$ is admissible, where S_d denotes S with the discrete topology. Thus the Stone-Čech compactification of S_d is a r.t. $B(S)$-compactification of S_d.

In the next section we shall make use of the following important property of admissible subalgcbras: If F is an admissible sub-C^*-algebra of

$C(S)$ and w: $T \to S$ is a continuous homomorphism from a semitopological semigroup T into S, then $w^*(F)$ is admissible. (That $w^*(F)$ is a C^*-algebra follows, for example, from [**12**; p. 43].)

3. **Compactifications of** $S_1 \times S_2$. Throughout this section S_1 and S_2 denote semitopological semigroups with right and left identities respectively (each denoted by 1), and $S = S_1 \times S_2$ their direct product. We shall let p_i: $S \to S_i$ denote the projection onto S_i, and q_i: $S_i \to S$ the injection mapping $(i = 1, 2)$ (e.g. $q_1(s_1) = (s_1, 1)$).

THEOREM 1. *Let F be a translation invariant sub-C^*-algebra of $C(S)$ containing the constant functions and satisfying $(q_iq_i)^*F \subset F$ $(i = 1, 2)$, and let F_1 and F_2 be the algebras defined in* (1). *Then F is admissible and $S^F = S_1^{F_1} \times S_2^{F_2}$ if and only if F_1 and F_2 are admissible, and for each $f \in F$ either $f(S_1, \cdot)$ is relatively norm compact in $C(S_2)$ or $f(\cdot, S_2)$ is relatively norm compact in $C(S_1)$.*

PROOF. The translation invariance of F implies that $F_i = q_i^*(F)$ $(i = 1, 2)$. Assume that F is admissible and that (2) holds. Then by the remark at the end of the last section, F_1 and F_2 are admissible. Let (X_i, u_i) be a r.t. F_i-compactification of S_i $(i = 1, 2)$. By hypothesis, each $f \in F$ is of the form $(u_1 \times u_2)^*(g)$ for some $g \in C(X_1 \times X_2)$. Since $g(X_1, \cdot)$ is obviously norm compact in $C(X_2)$, $f(S_1, \cdot)$ must be relatively norm compact in $C(S_2)$. Similarly, $f(\cdot, S_2)$ is relatively norm compact in $C(S_1)$.

Conversely, suppose that F_1 and F_2 are admissible and that the compactness criterion holds. Then for each $f \in F$, both $f(\cdot, S_2)$ and $f(S_1, \cdot)$ are relatively norm compact. (This follows from argument $6^{\circ\circ} \Rightarrow 5^\circ$ on p. 577 of [**11**].) Let (X, u) denote the canonical F-compactification of S, (Z, v) the canonical r.t. $B(S)$-compactification of S_d, and (X_i, u_i) the canonical r.t. F_i-compactificantion of S_i. To show that F is admissible and $(X_1 \times X_2, u_1 \times u_2)$ is an F-compactification of S it suffices to construct a homeomorphism w: $X_1 \times X_2 \to X$ which satisfies $w \circ (u_1 \times u_2) = u$.

Given $(x_1, x_2) \in X_1 \times X_2$ there exist nets $\{s_{1m}\}$ in S_1, $\{s_{2n}\}$ in S_2 and members z_1, z_2 of Z such that $u_1(s_{1m}) \to x_1$, $u_2(s_{2n}) \to x_2$, $v(s_{1m}, 1) \to z_1$ and $v(1, s_{2n}) \to z_2$. Define $w(x_1, x_2) = (z_1z_2)|_F$, where z_1z_2 is the product of z_1 and z_2 in Z. To see that $w(x_1, x_2)$ is well-defined, let $\{s'_{1k}\}$, $\{s'_{2h}\}$, z'_1 and z'_2 correspond to (x_1, x_2) in the analogous manner. For any $f \in F$, $z_1(f) = \lim_m f(s_{1m}, 1) = x_1(q_1{}^*f) = z'_1(f)$, and similarly $z_2(f) = z'_2(f)$. Hence if $s_2 \in S_2$,

$$\begin{aligned}\lim_m v(1, s_2)(L_{(s_{1m},1)} f) &= \lim_m f(s_{1m}, s_2) = z_1(R_{(1,s_2)} f) = z'_1(R_{(1,s_2)} f)\\ &= \lim_k v(1, s_2)(L_{(s'_{1k},1)} f).\end{aligned}$$

By our compactness criterion this convergence is uniform in s_2, hence

$$\lim_m z_2(L_{(s_{1m},1)}f) = \lim_k z_2(L_{(s'_{1k},1)}f).$$

But the left side of this equation is $z_1z_2(f)$, and the right side $z'_1z'_2(f)$.

That w is one-one follows from the equations

$$(3) \qquad z_1z_2(p_i{}^*f_i) = z_i(p_i{}^*f_i) = x_i(f_i),$$

$(f_i \in F_i;\ i = 1, 2)$. To show that w is onto, let $x \in X$ and let $\{u(s_{1n}, s_{2n})\}$ be a net in $u(S)$ converging to x. We may assume w.l.o.g. that the nets $\{v(s_{1n}, s_{2n})\}$, $\{v(s_{1n}, 1)\}$ and $\{v(1, s_{2n})\}$ converge in Z to, say, z, z_1, and z_2 respectively. For each $f \in F$,

$$[v(s_1, 1)z_2](f) = \lim_n f(s_1, s_{2n}),$$

and since this convergence is uniform in $s_1 \in S_1$,

$$[v(s_{1n}, 1)z_2](f) - v(s_{1n}, s_{2n})(f) \to 0.$$

Therefore $(z_1z_2)(f) = z(f) = x(f)$, and setting $x_i = z_i \circ (p_i^*|_{F_i})$, we have $w(x_1, x_2) = x$.

Equations (3) imply that the inverse of w is continuous, hence w is a homeomorphism. Finally, the identity $w \circ (u_1 \times u_2) = u$ is an immediate consequence of the definition of w.

REMARKS. The sufficiency part of the proof may be simplified somewhat if we know that F is admissible, for then the definition of $w(x_1, x_2)$ reduces to y_1y_2, where $y_i = x_i \circ (q_i^*|_F)$, mulitiplication now taking place in X.

The following simple observation will be useful in the proofs of the corollaries which follow: If F is a translation invariant subalgebra of $C(S)$ and $H_i \subset C(S_i)$ $(i = 1, 2)$ are given subalgebras which satisfy

$$(4) \qquad p_i^* H_i \subset F \text{ and } q_i^* F \subset H_i$$

then $H_i = F_i$ and $(q_ip_i)^*F \subset F$. In particular, this is the case if H_1, H_2, and F belong to a class of algebras which is stable under the duals of continuous homomorphisms in the sense described at the end of section 2.

COROLLARY 1. *Let H_i be an admissible sub-C^*-algebra of $C(S_i)$, and let F be the C^*-algebra of all functions $f \in C(S)$ such that $f(S_1, \cdot) \subset H_2$, $f(\cdot, S_2) \subset H_1$ and $f(S_1, \cdot)$ is relatively norm compact in $C(S_2)$. Then F is admissible and $S^F = S_1^{H_1} \times S_2^{H_2}$.*

PROOF. F is clearly translation invariant, and the containments in (4) obviously hold. Therefore the conclusion follows from Theorem 1 and the preceding remark.

COROLLARY 2. *If F is an admissible sub-C^*-algebra of $AP(S)$, then $S^F = S_1^{F_1} \times S_2^{F_2}$. In particular,*

(a) [4, 6, 8] $S^{AP} = S_1^{AP} \times S_2^{AP}$, *and*

(b) $S^{SAP} = S_1^{SAP} \times S_2^{SAP}$.

COROLLARY 3. *If S_1 is compact and F is an admissible sub-C^*-algebra of $LUC(S)$, then $S^F = S_1^{F_1} \times S_2^{F_2}$. In particular, if S_1 is a compact topological semigroup, then $S^{LUC} = S_1 \times S_2^{LUC}$.*

PROOF. The hypotheses imply that $f(S_1, \cdot)$ is relatively norm compact in $C(S_2)$ for each $f \in F$, hence the first statement follows from Theorem 1. The second part follows from the first by noting that under the stated conditions $LUC(S_1) = C(S_1)$.

COROLLARY 4. Let H_2 be an admissible sub-C^*-algebra of $LUC(S_2)$, and define

$$F = \{f \in LUC(S) : f(S_1, \cdot) \subset H_2, f(\cdot, S_2) \subset SAP(S_1)\}.$$

Suppose that S_1 is a dense subsemigroup of a compact topological semigroup T such that

$$(5) \qquad (tT) \cap S_1 \neq \emptyset \qquad (t \in T).$$

Then $S^F = S_1^{SAP} \times S_2^{H_2}$. In particular, if S_1 is a dense subsemigroup of a compact topological group G, then $S^{LUC} = G \times S^{LUC}$.

PROOF. F is obviously a translation invariant sub-C^*-algebra of $C(S)$. Furthermore, conditions (4) hold with $H_1 = SAP(S_1)$. Thus for the first part of the conclusion it remains to show that $f(S_1, \cdot)$ is relatively norm compact in $C(S_2)$ ($f \in F$). Let $\{s_m\}$ be a net in S_1 and $\{s_n\}$ a subnet converging to some member t of T. Choose any $r \in T$ such that $tr \in S_1$, and let $\{r_k\}$ be a net in S_1 converging to r. Then the net $\{s_n r_k\}$ converges to tr, hence given $\varepsilon > 0$ we may choose n' and k' such that for all $n \geqq n'$, $k \geqq k'$, and $(s_1, s_2) \in S$,

$$(6) \qquad |f(s_n r_k s_1, s_2) - f(tr\, s_1, s_2)| < \varepsilon.$$

Fix $s_2 \in S_2$ and set $g = f(\cdot, s_2)$. Let (X, u) be the canonical SAP-compactification of S_1. From (6),

$$(7) \qquad |u(s_n r_k s_1)(g) - u(tr\, s_1)(g)| < \varepsilon,$$

for all $s_1 \in S_1$, $n \geqq n'$, $k \geqq k'$. We may assume w.l.o.g. that $\{u(r_k)^{-1}\}$ converges to some $x \in X$. It follows from (7) then that

$$|u(s_n)(g) - [u(tr)x](g)| \leqq \varepsilon \qquad (n \geqq n'),$$

hence

$$|f(s_n, s_2) - f(s_m, s_2)| \leqq 2\varepsilon \qquad (n, m \geqq n').$$

Since s_2 was arbitrary, $\{f(s_n, \cdot)\}$ is a Cauchy net in $C(S_2)$ and therefore converges in the norm topology.

To prove the second statement, let $f \in LUC(S)$, $s_2 \in S_2$. Then $f(\cdot, s_2)$ has a continuous extension to G[10] and is therefore a member of $SAP(S_1)$. The conclusion now follows from the first part if we observe that $S_1^{SAP} = G$.

REMARK. Note that condition (5) holds if tT has a non-empty interior for each $t \in T$.

COROLLARY 5. *Let H_2 be an admissible sub-C^*-algebra of $WAP(S_2)$, and define*

$$F = \{f \in WAP(S) : f(S_1, \cdot) \subset H_2, f(\cdot, S_2) \subset SAP(S_1)\}.$$

Then $S^F = S_1^{SAP} \times S_2^{H_2}$. In particular if S_1 is a dense subsemigroup of a compact topological group G, then $S^{WAP} = G \times S_2^{WAP}$.

PROOF. Since F is a translation invariant sub-C^*-algebra of $WAP(S)$, F is admissible. Also, conditions (4) are satisfied with $H_1 = SAP(S_1)$. Let (X, u) denote the canonical r.t. F-compactification of S and (X_1, u_1) the canonical $SAP(S_1)$-compactification of S_1. Define $v: X_1 \to X$ by $v(x_1)(f) = x_1(q_1{}^*f)$ $(f \in F)$. Since X_1 is a compact topological group, the action $(x_1, x) \to v(x_1)x$ of X_1 on X, which is separately continuous because $F \subset WAP(S)$, must be jointly continuous by a well-known theorem of Ellis [7]. Therefore, if $g \in C(X)$, the set of functions $g(v(x_1)u(1, \cdot))$ $(x_1 \in X_1)$ is norm compact in $C(S_2)$, so $g(u(S_1, \cdot))$ is relatively norm compact. This proves the first part of the corollary. The second part follows easily from the first.

REMARK. The above results suggest alternate characterizations of almost periodicity of a function $f \in C(S)$. For example, if S_1 is a group then Corollary 5 implies that $f \in AP(S)$ if and only if $f \in WAP(S)$, $f(S_1, \cdot) \subset AP(S_2)$ and $f(\cdot, S_2) \subset AP(S_1)$. And from Corollary 4, if S_1 is a dense subsemigroup of a compact topological group, then $f \in AP(S)$ if and only if $f \in LUC(S)$ and $f(S_1, \cdot) \subset AP(S_2)$.

For later reference we state without proof the following extension of Theorem 1, which may be proved by a simple induction argument. A somewhat sharper version of this theorem holds, but the one we give here is sufficient for our purposes.

THEOREM 2. *Let $S = \Pi\{S_i: i = 1, \ldots, n\}$, where each S_i is a semitopological semigroup with identity. For each non-empty subset a of $\{1, 2, \ldots, n\}$ let $p_a: S \to S_a$ and $q_a: S_a \to S$ denote respectively the projection and injection mappings. Let F be a translation invariant sub-C^*-algebra of $C(S)$ containing the constant functions and satisfying $(q_a p_a)^*F \subset F$ for all a, and set $F_i = q_i{}^*F$ $(i = 1, 2, \ldots, n)$. Then F is admissible and $S^F = \Pi\{S_i^{F_i}: i = 1, \ldots, n\}$ if and only if each F_i is admissible, and for each $f \in F$ and $k = 2, 3, \ldots, n$, $f(\cdot, \cdot, \ldots, S_k, \ldots, \cdot)$ is relatively norm compact in $C(\Pi\{S_i: i \neq k\})$.*

4. **Compactifications of Infinite Direct Products.** In this section we

consider compactifications of direct products $S = \Pi\{S_i: i \in I\}$, where I is an infinite index set and each S_i is a semitopological semigroup with identity. Before stating the main result we introduce the following convenient notation: For each non-empty subset $b \subset I$ let S_b denote the direct product of the semigroups $S_i(i \in b)$, $p_b: S \to S_b$ the projection mapping, and $q_b: S_b \to S$ the injection mapping. Let A denote the collection of all non-empty finite subsets of I. An arbitrary point of S shall be denoted by $(s_i: i \in I)$. Finally, if $f \in C(S)$, $j \in J$ and $s_j \in S_j$, set $I_j = I\backslash\{j\}$ and define $f_{s_j} \in C(\Pi\{S_i: i \in I_j\})$ by $f_{s_j}((s_i: i \in I_j)) = f((s_i: i \in I))$.

THEOREM 3. *Let F be a translation invariant sub-C^*-algebra of $LUC(S)$ containing the constant functions and satisfying $(q_a p_a)^* F \subset F$ for each $a \in A$, and set $F_i = q_i^* F (i \in I)$. Then F is admissible and $S^F = \Pi\{S_i^F i: i \in I\}$ if and only if for each $j \in I$ and $f \in F$, F_j is admissible and $\{f_{s_j}: s_j \in S_j\}$ is relatively norm compact in $C(\Pi\{S_i: i \in I_j\})$.*

PROOF. We omit the straightforward verification of the necessity. For the sufficiency let (X_i, u_i) denote the canonical r.t. F_i-compactification of S_i, X the direct product $\Pi\{X_i: i \in I\}$, and $u: S \to X$. the product of the mappings u_i $(i \in I)$. We must show that (X, u) is a r.t. F-compactification of S.

The pair (X, u) evidently satisfies properties (a) and (b) of the definition of r.t. compactification (section 2), and $u(S)$ is dense in X. It remains to show that $u^*C(X) = F$. For each non-empty subset $b \subset I$ set $w_b = q_b p_b$. We show first that $\bigcup\{w_a^*(F): a \in A\}$ is dense in F. Given $f \in F$ and $\varepsilon > 0$ there exists $a \in A$ and a neighborhood V of the identity in S_a such that $\|L_s f - f\| < \varepsilon$ for all $s \in U = p_a^{-1}(V)$. Let $b = I\backslash a$. Then for any $t \in S$, $s = w_b(t) \in U$ and $t = sw_a(t)$, hence

$$|f(t) - f(w_a(t))| = |L_s f(w_a(t)) - f(w_a(t))| < \varepsilon.$$

Since t was arbitrary, $\|f - w_a^*(f)\| < \varepsilon$.

For each $a \in A$ let X_a denote the direct product of the semigroups X_i $(i \in a)$, $u_a: S_a \to X_a$ the product of the mappings u_i $(i \in a)$, and $\bar{p}_a: X \to X_a$ the projection mapping. Since $F_a = q_a^* F$ satisfies the hypotheses of Theorem 2, (X_a, u_a) is an F_a-compactification of S_a. Since $\bar{p}_a u = u_a p_a$, it follows that $w_a^* F = p_a^* F_a = u^* \bar{p}_a^* C(X_a) \subset u^* C(X) (a \in A)$, and therefore $F \subset u^* C(X)$. The reverse inclusion will follow from the Stone-Weierstrass Theorem if we can show that $(u^*)^{-1}(F)$ separates points of X. Let x and y be distinct points of X. Then for some $a \in A$, $p_a(x) \neq p_a(y)$, so there exists $g \in C(X_a)$ such that $h = \bar{p}_a^*(g)$ separates x and y. The desired conclusion now follows from the observation that $u^* h = p_a^* u_a^* g \in F$.

Part (a) of the following corollary was obtained (for the commutative topological case) by deLeeuw and Glicksberg in **[6]**.

COROLLARY. *Let F be an admissible sub-C^*-algebra of $AP(S)$ such that $(q_a p_a)^* F \subset F$ for each $a \in A$. Then $S^F = \Pi\{S_i^{F_i}: i \in I\}$. In particular,*
(a) $S^{AP} = \Pi\{S_i^{AP}: i \in I\}$, and
(b) $S^{SAP} = \Pi\{S_i^{SAP}: i \in I\}$.

COROLLARY. *If each S_i is compact then $S^{LUC} = \Pi\{S_i^{LUC}: i \in I\}$.*

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