ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 3, Summer 1980

ISOMETRY AND FIXED POINT THEOREMS FOR ASYMPTOTICALLY EXPANSIVE MAPPINGS IN COMPACT SPACES

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ABSTRACT. An isometry theorem in a compact metric space that generalizes a result of Freudenthal and Hurewicz is proved. A common fixed point theorem for a family of asymptotically expansive mappings of a compact set in a Banach space is then deduced.

1. **Introduction.** Let M be a metric space. A mapping $T: M \to M$ is called nonexpansive if $d(Tx, Ty) \leqq d(x, y)$ for all $x, y \in M$. In 1963, DeMarr proved the following theorem.

THEOREM A. *Let B be a Banach space and let X be a nonempty compact convex subset of B. If F is a nonempty commutative family of nonexpansive mappings of X into itself, then the family F has a common fixed point in X.*

A mapping T is said to be asymptotically nonexpansive if for each $x, y \in M$,

$$d(T^i x, T^i y) \leqq k_i d(x, y), \qquad i = 1, 2, 3, \dots ,$$

where k_i is a fixed sequence of positive numbers such that $k_i \to 1$ as $i \to \infty$. The concept of "asymptotic nonexpansiveness" was introduced by Goebel and Kirk in [5]. In 1974, Goebel, Kirk and Thele [6] showed that the conclusion of Theorem A is still valid if F is merely assumed to be a commutative family of asymptotically nonexpansive mappings. They also proved (see [6]) that an asymptotically nonexpansive mapping of a compact metric space onto itself must be an isometry. Since the inverse of a nonexpansive (asymptotically nonexpansive) mapping is expansive (asymptotically expansive), one is naturally led to the study of expansive and asymptotically expansive mappings whose precise definitions are given below.

DEFINITION. Let M be a metric space. A mapping $T: M \to M$ is said to be expansive if $d(Tx, Ty) \geqq d(x, y)$ for all $x, y \in M$. And it is said to be asymptotically expansive if for each $x, y \in M$,

$$d(T^i x, T^i y) \geqq k_i d(x, y) \qquad i = 1, 2, 3, \dots,$$

Received by the editors on April 24, 1978

where k_i is a fixed sequence of positive numbers such that $k_i \to 1$ as $i \to \infty$.

It is the purpose of this paper to study the above mentioned results if the nonexpansiveness (asymptotically nonexpansiveness) of the mappings is replaced by expansiveness (asymptotically expansiveness). Some results on continuous expansive mappings have been obtained by Borges in [**1**].

2. **Main Results.** One of the aims of this paper is to investigate the valdity of Theorem A if the nonexpansiveness of the mappings is replaced by expansiveness. It will be shown that the conclusion of Theorem A is still valid if F is merely assumed to be a nonempty family of asymptotically expansive mappings. We will first prove an interesting fact, namely Theorem 1 below, that is an essential tool in establishing Theorem 3 which generalizes a result of Freudenthal and Hurewicz [**4**].

THEOREM 1. *Let M be a compact metric space and T an asymptotically expansive mapping of M into itself. Then for each pair of points $x, y \in M$, the two sequences of iterates $\{T^m x\}$, $\{T^m y\}$ have convergent subsequences of the same indices, say $\{T^{m_j} x\}$, $\{T^{m_j} y\}$ such that $T^{m_j} x \to x$, $T^{m_j} y \to y$ as $j \to \infty$.*

PROOF. Since T: $M \to M$ is asymptotically expansive, there exists a sequence $\{\beta_i\}$ of positive numbers with $\beta_i \to 1$ as $i \to \infty$, and such that $d(T^i x, T^i y) \geqq \beta_i d(x, y)$ for all $x, y \in M$. Fix $x_0, y_0 \in M$ and define $\{x_n\}$, $\{y_n\}$ by $Tx_{n-1} = x_n$, $Ty_{n-1} = y_n$ for $n = 1, 2, \dots$. Since M is compact, we may assume without loss of generality that the two sequences $\{x_n\}$, $\{y_n\}$ have convergent subquences of the same indices, say $\{x_{n_k}\}$, $\{y_{n_k}\}$ and two point x_∞, y_∞ such that $x_{n_k} \to x_\infty, y_{n_k} \to y_\infty$ as $k \to \infty$.

Let $\varepsilon > 0$ be given. We may choose N such that if $k \geqq N$, we have

$$d(x_{n_k}, x_{n_{k+i}}) < \varepsilon,\ d(y_{n_k}, y_{n_{k+i}}) < \varepsilon \text{ for } i = 1, 2, \dots .$$

Fix a $k \geqq N$ and define $P_i = n_{k+i} - n_k$ for $i = 1, 2, \dots$, obviously $P_1 < P_2 < P_3 \dots$ and $P_i \to \infty$ as $i \to \infty$. And

$$d(x_{n_k}, x_{n_{k+i}}) = d(x_{n_k}, x_{n_k+p_i}) = d(T^{n_k}x_0, T^{n_k}x_{p_i}) \geqq \beta_{n_k} d(x_0, x_{p_i}),$$

hence $d(x_0, x_{p_i}) \leqq (\beta_{n_k})^{-1} d(x_{n_k}, x_{n_k+p_i}) < \beta\varepsilon$ for $i = 1, 2, \dots,$ where $\beta = \sup\, (\beta_n)^{-1}$. Similarly,

$$d(y_0, y_{p_i}) \leqq (\beta_{n_k})^{-1} d(y_{n_k}, y_{n_k+p_i}) \leqq \beta d(y_{n_k}, y_{n_k+p_i}) < \beta\varepsilon,\ \forall p_i.$$

We may thus choose such a p_i and call it m_1. We have then

$$d(x_0, T^{m_1}x_0) < \beta\varepsilon, \qquad d(y_0, T^{m_1}y_0) < \beta\varepsilon.$$

We can show by the same procedure that if $\varepsilon/2 > 0$ is given, there exists $\{T^{q_i}x_0\}$, $\{T^{q_i} y_0\}$ such that

$$d(x_0, T^{q_i}x_0) < \beta(\varepsilon/2), d(y_0, T^{q_i}y_0) < \beta(\varepsilon/2) \text{ for } i = 1, 2, \ldots,$$

and $q_i \to \infty$ is $i \to \infty$.
Hence we may choose a $q_i < m_1$ and call it m_2. Inductively, we may obtain $\{T^{m_j}x_0\}$ and $\{T^{m_j}y_0\}$ with

$$d(x_0, T^{m_j}x_0) < \beta(\varepsilon/j), \qquad d(y_0, T^{m_j}y_0) < \beta(\varepsilon/j),$$

where $m_1 < m_2 < m_3 \ldots$. Hence $T^{m_j}x_0 \to x_0$, $T^{m_j}y_0 \to y_0$.

An interesting consequence that follows immediately from Theorem 1 is the following.

THEOREM 2. *Let M be a compact metric space. Suppose T: $M \to M$ is asymptotically expansive. Then each isolated point in M is a periodic point of T.*

PROOF. Suppose x_0 is an isolated point in M! Then there exists an open neighborhood $N(x_0)$ such that $[N(x_0)\backslash\{x_0\}] \cap M = \phi$. According to Theorem 1, we know that $\{T^n x_0\}$ has a convergent subsequence $\{T^{n_k}x_0\}$ such that $T^{n_k}x_0 \to x_0$ as $k \to \infty$. Consequently, there exists a positive integer $N > 0$ such that $T^{n_k}x_0 \in N(x_0) \cap M$ whenever $n_k \geqq N$, that is $T^{n_k}x_0 = x_0$.

Another easy consequence of Theorem 1 is the following result that generalizes a theorem established by Freudenthal and Hurewicz.

THEOREM 3. *Let M be a compact metric space. Suppose T: $M \to M$ is asymptotically expansive. Then T is an isometry of M onto M.*

PROOF. Let x and y be two arbitrary points in M. By Theorem 1, $\{T^n x\}$, $\{T^n y\}$ have convergent subsequences of the same indices $\{T^{n_j}x\}$, $\{T^{n_j}y\}$ such that $T^{n_j}x \to x$, $T^{n_j}y \to y$ as $j \to \infty$. By asymptotic expansiveness of T, we know $d(T^{n_j}x, T^{n_j}y) \geqq (\beta_{n_j-1})d(Tx, Ty)$, letting $j \to \infty$, we get

$$d(x, y) \geqq d(Tx, Ty). \tag{1}$$

Also, by applying Theorem 1 to the pair Tx, Ty, we know $\{T^n(Tx)\}$ $\{T^n(Ty)\}$ have convergent subsequences of the same indices, say $\{T^{n_j}(Tx)\}$, $\{T^{n_j}(Ty)\}$ such that $T^{n_j}(Tx) \to Tx$, $T^{n_j}(Ty) \to Ty$. But $d(T^n(Tx), T^n(Ty)) = d(T^{n+1}x, T^{n+1}y) \geqq \beta_{n+1}d(x, y)$, Hence $d(T^{n_j}(Tx), T^{n_j}(Ty)) \geqq \beta_{n_j+1}d(x, y)$. Letting $j \to \infty$, we get

$$d(Tx, Ty) \geqq d(x, y). \tag{2}$$

From (1) and (2), $d(x, y) = d(Tx, Ty)$ showing that T is an isometry. Surjectivity of T follows from the compactness of the space.

Observing that an expansive mapping is certainly asymptotically expansive, we get:

COROLLARY 1 ([**4**]). *Let M be a compact metric space. Suppose $T: M \to M$ is expansive, then T is an isometry.*

As an application of Theorem 3, we have:

THEOREM 4. *Let B be a Banach space and let X be a nonempty compact convex subset of B. If F is a nonempty family of asymptotically expansive mappings of X into itself, then the family F has a common fixed point in X.*

PROOF. By Theorem 3, each $f \in F$ is an isometry of X onto X. It follows then from a theorem of Brodskii and Milman [**2**] that the family F has a common fixed point in X.

REMARK 1. Demarr's result (Theorem A) has been generalized by the author in [7]. Extension of another theorem of Freudenthal and Hurewicz (see [4]) and their local versions can be found in [**8**].

REMARK 2. Goebel and Kirk have shown in [**5**] that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings. In noncompact metric space, it is even easier to give an example of an asymptotically expansive mapping that is not expansive.

EXAMPLE. Let $x_1 = 3/2$, $x_2 = 5/2$, $x_n = n$ if $n \geqq 3$. Suppose that $X = \{x_1, x_2, x_3, \dots\}$ is equipped with the usual metric and $T: (X, d) \to (X, d)$ is defined by $Tx_n = x_{n+1}$. It is easy to see that T is not expansive. However T is asymptotically expansive.

ACKNOWLEDGEMENT. This paper is part of the author's Ph. D. dissertation at the University of Iowa, written under the supervision of Professor W.A. Kirk.

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