ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 3, Summer 1980

STRONGLY EXPOSED POINTS IN $L^p(\mu, E)$

JERRY JOHNSON

ABSTRACT. A sufficient condition is given for a function to be a strongly exposed point of the unit ball of $L^p(\mu, E)$ for any Banach space E, $1 < p < \infty$. It is then shown that the unit ball of $L^p(\mu, E)$ is the closed convex hull of the "simple strongly exposed points" if E has the Radon-Nikodym property.

Sundaresan **[3]** (see also Turett and Uhl **[6]**) showed that if E is a Banach space with the Radon-Nikodym property (RNP) then the space $L^p(\Omega, \Sigma, \mu, E) \equiv L^p(\mu, E)$ $(1 < p < \infty)$ also has RNP. One corollary of this result is that the unit ball of $L^p(\mu, E)$ is the closed convex hull of its strongly exposed points. For this reason it was suggested by J. J. Uhl that it would be useful to have available a characterization of these functions.

In **[1, 4** and **5]** the problem of characterizing the extreme points of the unit ball of $L^p(\mu, E)$ was considered and, with modest restrictions on E and (Ω, Σ, μ), it was shown that f is an extreme point if and only if $\|f\|_p = 1$ and for almost all $t \in \{t \mid f(t) \neq 0\}$, $f(t)/\|f(t)\|$ is an extreme point of the unit ball of E. This suggests a similar characterization for strongly exposed points; Theorem 1 gives a sufficient condition for f to be strongly exposed. We were unable to obtain the necessity, but got something a little better in a way (Theorem 2); namely, that the unit ball of $L^p(\mu, E)$ is the closed convex hull of the "simple strongly exposed points" if E has RNP. We assume throughout that (Ω, Σ, μ) is a finite measure space, U denotes the unit ball of E and E^* the dual of E. If f: $\Omega \to E$, $|f|(t) = \|f(t)\|$.

A point $x \in U$ is said to be strongly exposed by $x^* \in E^*$ if $x^*(x) = \|x^*\| = 1$, and any sequence $\{x_n\} \subset U$ for which $x^*(x_n) \to 1$ satisfies $\|x_n - x\| \to 0$. We state the following simple modification of the definition for later reference and omit its easy proof:

LEMMA 1. *Let $x \in E$ and $x^* \in E^*$ be such that $x^*(x) = \|x\| = \|x^*\| = 1$. Suppose every sequence $\{x_n\} \subset U$ with $x^*(x_n - x) \to 0$ has a subsequence converging to x. Then x^* strongly exposes x.*

For any unfamiliar notation or terminology we refer the reader to **[0]**.

THEOREM 1. *Let $f \in L^p(\mu, E)$, $1 < p < \infty$, and $\|f\|_p = 1$. Put $S =$*

Received by the editors on February 7, 1978.

$\{t \in \Omega | f(t) \neq 0\}$ *and suppose there is a (strongly) measurable function* $g_0: \Omega \to E^*$ *such that for almost all* $t \in S$, $g_0(t)$ *has norm* 1 *and strongly exposes* $f(t)/\|f(t)\|$. *Then f is strongly exposed by* $g = |f|^{p-1}g_0$.

PROOF. $g \in L^q(\mu, E^*)$ and $\|g\|_q = 1$ $(1/p + 1/q = 1)$. Suppose $\|h_n\|_p \leqq 1$ and $\int\langle h_n, g\rangle \to 1$. By Lemma 1 it is enough to find a subsequence of $\{h_n\}$ converging to f. First $\int\langle h_n, g\rangle = \int_S |f|^{p-1}\langle h_n, g_0\rangle \to 1$, $|f|^{p-1}$ is an L^q function of norm one, and $\langle h_n, g_0\rangle$ is an L^p function of norm $\leqq 1$. Hence, $\langle h_n, g_0\rangle$ converges to the function in L^p that is strongly exposed by $|f|^{p-1}$, namely $|f|$. Also $1 \geqq \int|h_n||g| \geqq \int\langle h_n, g\rangle \to 1$ so $|h_n|$ converges in L^p to the function strongly exposed by $|g| = |f|^{p-1}$, which is $|f|$ again. We conclude that both $\langle h_n, g_0\rangle$ and $|h_n|$ converge in L^p to $|f|$. Thus there is a subsequence $\{h_{n_k}\}$ so that $\langle h_{n_k}, g_0\rangle$ and $|h_{n_k}|$ converge a.e. to $|f|$. Now, put $\varphi_k(s) = h_{n_k}(s)/\|h_{n_k}(s)\|$ if $h_{n_k}(s) \neq 0$ and $\varphi_k = 0$ otherwise. $\langle\varphi_k(s), g_0(s)\rangle = \langle h_{n_k}(s), g_0(s)\rangle/\|h_{n_k}(s)\| \to 1$ for $s \in S$. Since $g_0(s)$ strongly exposes $f(s)/\|f(s)\|$ a.e. in S, we have $\|\varphi_k(s) - f(s)/\|f(s)\|\| \to 0$ a.e. on S. Since $|h_{n_k}| \to |f|$ a.e., we get $\|h_{n_k}(s) - f(s)\| \to 0$ a.e. on Ω. Since $|h_{n_k} - f| \to 0$ a.e. and $|h_{n_k}| \to |f|$ in L^p, the dominated convergence theorem gives $\int_\Omega \|h_{n_k}(s) - f(s)\|^p d\mu(s) \to 0$. This completes the proof.

COROLLARY 1. If $f = \sum_{j=1}^n x_j\chi_{Aj}$, $\|f\|_p = 1$ and $x_j/\|x_j\|$ is strongly exposed by $x_j^* \in E^*$ for each j, then f is strongly exposed by

$$\sum_{j=1}^n \|x_j\|^{p-1}x_j^*\chi_{Aj}.$$

REMARK. For $\alpha \in (0, 1)$, the "slice map": $x^* \to \{x \in U | x^*(x) \geqq 1 - \alpha\}$ is continuous from the set of strongly exposing functionals in E^* to the closed convex subsets of U with the Hausdorff metric. What seems to be needed for a converse to theorem 1 is a judicious application of, say, the Michael selection theorem to this set-valued map. So far I haven't found it.

THEOREM 2. *Assume that E has* RNP. *Let S denote the set of all functions* $f \in L^p(\mu, E)$, $1 < p < \infty$, *such that* $f = \sum_{j=1}^n x_j\chi_{Aj}$, $\|f\|_p = 1$ and $x_j/\|x_j\|$ is strongly exposed in U. Then the unit ball of $L^p(\mu, E)$ is the closed convex hull of S.

PROOF. Let φ be a continuous linear functional on $L^p(\mu, E)$. We will show that $\sup\{\varphi(f) | f \in S\} = \|\varphi\|$. The conclusion of the theorem then follows by a standard application of the separation theorem. If one knew that the dual of $L^p(\mu, E)$ happened to be $L^q(\mu, E^*)$ in the canonical way, the proof would be rather immediate. However, this is true if and only if E^* has RNP (see [0]). Thus, a slightly different approach is necessary. Let $\|\varphi\| = 1$ and $\varepsilon > 0$. There is a simple function $g = \sum_{i=1}^n y_i\chi_{Ai}$ so that $\|g\|_p = 1$ and $\varphi(g) > 1 - \varepsilon/3$. Let B be the n-fold product of E with

$$\|(x_1, \ldots, x_n)\| = \Big\| \sum_{i=1}^{n} x_i \chi_{Ai} \Big\|_p = \Big(\sum_{i=1}^{n} \|x_i\|^p \mu A_i \Big)^{1/p}.$$

Let $\varphi_0 \in B^*$ be given by

$$\varphi_0(x_1, \ldots, x_n) = \varphi\Big(\sum_{j=1}^{n} x_j \chi_{Aj} \Big)$$

Now, B has RNP because E does, so the strongly exposing functionals are dense in B^*. (This is implicit in [**2**, lemmas 5 and 6].) Hence there is a strongly exposing functional $\psi_0 \in B^*$ with $\|\psi_0 - \varphi_0\| < \varepsilon/3$. Let $\mathbf{z} = (z_1, \ldots, z_n)$ be the point in B of norm one strongly exposed by ψ_0. We claim $f = \sum_{j=1}^{n} z_j \chi_{Aj} \in S$ and $\varphi(f) > 1 - \varepsilon$. First, $\|f\|_p$ is the norm of $\bar{\mathbf{z}}$ in B which is 1. Also, $\varphi(f) = \varphi_0(\mathbf{z}) \geqq \psi_0(\mathbf{z}) - \varepsilon/3 = \|\psi_0\| - \varepsilon/3 \geqq \|\varphi_0\| - 2\varepsilon/3 \geqq \varphi_0(y_1, \ldots, y_n) - 2\varepsilon/3 = \varphi(g) - 2\varepsilon/3 > 1 - \varepsilon$. Now, observe that z_j is strongly exposed in $\{z | \, \|z\| \leqq \|z_j\|\}$ as follows: Let $\psi_0 = (e_1^*, \ldots, e_n^*)$. Suppose $\|w_k\| \leqq \|z_j\|$, $k = 1, 2, \ldots$ and $\lim_k e_j^*(w_k - z_j) = 0$. If $\bar{\mathbf{w}}_k$ is $\bar{\mathbf{z}}$ with z_j replaced by w_k, then $\|\bar{\mathbf{w}}_k\| \leqq \|\bar{\mathbf{z}}\|$ and $\psi_0(\bar{\mathbf{z}} - \bar{\mathbf{w}}_k) = e_j^*(z_j - w_k) \to 0$. Since ψ_0 strongly exposes $\bar{\mathbf{z}}$, $\|w_k - z_j\| = \|\bar{\mathbf{w}}_k - \bar{\mathbf{z}}\| / \mu Aj \to 0$ as $k \to \infty$. This completes the proof.

We close with the following question: If E has the KreinMil-man property, does $L^p(\mu, E)$, $1 < \mathrm{p} < \infty$?

References

0. J. Diestel and J. J. Uhl, *Vector measures*, A.M.S. Surveys, No. 15. 1977.

1. J. A. Johnson, *Extreme measurable selections*, Proc. A.M.S., (**44**) 1974, 107–111.

2. R. R. Phelps, *Dentability and extreme points in Banach spaces*, J. of Funct. Anal., **16** (1974), 78–90.

3. K. Sundaresan, *The Radon-Nikodym Theorem for Lebesgue Bochner function spaces*, (preprint)

4. ———, Extreme points of the unit cell in Lebesgue-Bochner function spaces. I, Proc. A.M.S. 23 (1969), 179–184.

5. ———, Extreme points of the unit cell in Lebesgue-Bochner function spaces, Colloq. Math. 22 (1970), 111–119.

6. B. Turret and J. J. Uhl, Lp(μ, X) ($1 < p < \infty$) *has the Radon-Nikodym property if X does by martingales*, Proc. A.M.S., 61 (1976), 347–350.

Department of Mathematics, Oklahoma State University, Stillwater OK 74074.