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SINGULAR NONLINEAR EVOLUTION EQUATIONS

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ABSTRACT. Sufficient conditions are given to obtain existence and uniqueness of strong solutions to $u'(t) + A(u(t)) \ni f(t)$ on $(-\infty, 0)$ where A is a maximal monotone operator in Hilbert space. Applications to certain nonlinear problems for partial differential equations are described.

1. **Introduction.** We shall consider nonlinear evolution equations of the form

$$\frac{du(t)}{dt} + \mu u(t) + A(u(t)) \ni f(t), \quad -\infty < t < 0, \tag{1.1}$$

in a Hilbert space $\mathbf{H}$, where $\mu \in \mathbf{R}$, the real numbers, and A is a maximal monotone operator in $\mathbf{H}$ [2]. The solution will be obtained in the Hilbert space $\mathscr{H}_\omega$ of functions $u: (-\infty, 0) \to \mathbf{H}$ which are square-summable with the measure $e^{-2\omega t}\,dt$ for appropriate $\omega \in \mathbf{R}$. That is, $u \in W^{1,2}_w((-\infty, 0), \mathbf{H})$, the class of functions u in $\mathscr{H}_\omega$ whose (strong) derivatives u' belong to $\mathscr{H}_\omega$.

We first show that the linear operator "$(d/dt) + \mu$" is maximal monotone on $\mathscr{H}_\omega$ when $\mu + \omega \geqq 0$. Then we obtain

THEOREM 1. *Let A be maximal monotone in $\mathbf{H}$, $A(0) \ni 0$ and $\omega + \mu > 0$. For each $f \in W^{1,2}_\omega((-\infty, 0), \mathbf{H})$ there exists a unique solution $u \in W^{1,2}_\omega((-\infty, 0), \mathbf{H})$ of (1.1).*

For a restricted class of maximal montone operators, the subdifferentials, we obtain a corresponding result. Let $\varphi: \mathbf{H} \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. The operator on $\mathbf{H}$ defined by

$$\partial\varphi(u) \equiv \{f \in \mathbf{H}: (f, v - u)_H \leqq \varphi(v) - \varphi(u) \text{ for all } v \in \mathbf{H}\}$$

is a maximal monotone $\partial\varphi$ called the subdifferential of φ [2].

THEOREM 2. *Let $\varphi: \mathbf{H} \to [0, +\infty]$ be convex and lower semicontinuous with $\varphi(u_0) = 0$ for some $u_0 \in \mathbf{H}$. The operator "$(d/dt) + \mu + \partial\varphi$" is maximal monotone on $\mathscr{H}_\omega$ in each of the following situations:* (a) $\mu \geqq 0$, $2\omega + \mu \geqq 0$ *and one of* $\mu = 0$ *or* $\varphi(0) = 0$ *or* $\omega < 0$; (b) $\mu < 0$, *there is a* $p \geqq 2$ *such that* $\varphi(\lambda u) \leqq \lambda^p \varphi(u)$ *for all* $\lambda \geqq 1$, *and* $2\omega + p\mu > 0$. *If, in*

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addition, $\omega\mu \neq 0$, *then for each* $f \in \mathscr{H}_\omega$ *there exists a unique solution* $u \in W^{1,2}_\omega((-\infty, 0), \mathbf{H})$ *of* (1.1).

Both of these Theorems will be obtained from known results (e.g., from **[2]**) on the perturbation of maximal monotone operators in Hilbert space. Related results for (1.1) were given in **[1]**, [7], **[8**, pp. 505–511] and **[10]**.

The preceding results apply immediately to the singular equation

$$\frac{1}{b(s)}\frac{dv(s)}{ds} + \mu v(s) + A(v(s)) \ni g(s), \quad -\infty \leqq s_0 < s < s_1, \tag{1.2}$$

where b is a locally-integrable positive function on the interval (s_0, s_1). The effect of the singularity at s_0 and s_1 can be determined by the change-of-variable $t = B(s)$ where B is an absolutely continuous primitive of b on (s_0, s_1). Thus, v is a solution of (1.2) if and only if the function $u \equiv v \circ B^{-1}$ satisfies (1.1) on the interval $(B(s_0), B(s_1))$. If $B(s_0) > -\infty$, then the Cauchy problem is appropriate for (1.2). If $B(s_0) = -\infty$, then (1.2) is uniquely resolved (without initial data) by applying Theorem 1 or Theorem 2. Also see Theorem 6 of **[5]**.

Related results for periodic solutions of (1.1) follow from the above when $\mu > 0$. For example, if $f \in L^2((-T, 0), \mathbf{H})$, then its T-periodic extension belongs to $\mathscr{H}_\omega$ for every $\omega < 0$. Choosing $\omega = -\mu/2$, we apply Theorem 2 to obtain a solution of (1.1) which is T-periodic since f is T-periodic. Theorem 1 similarly applies when $f \in W^{1,2}((-T, 0), \mathbf{H})$ and $f(-T) = f(0)$. See **[2]** for related results.

2. **The Singular Problem.** Let L_ω be the operator on $\mathscr{H}_\omega$ defined by $L_\omega(u) \equiv du/dt - \omega u$ with domain $D(L_\omega) \equiv W^{1,2}_\omega((-\infty, 0), \mathbf{H})$.

PROPOSITION. L_ω *is maximal monotone on* $\mathscr{H}_\omega$ *and for each* $u \in D(L_\omega)$ *we have*

$$\lim_{t \to -\infty} e^{-\omega t} u(t) = 0$$

in $\mathbf{H}$. *Finally, for each* $\lambda > 0$ *we have* $(\lambda + L_\omega)u = f$ *if and only if*

$$u(t) = \int_{-\infty}^{t} f(s) e^{(\lambda-\omega)(s-t)}\, ds, \quad -\infty < t < 0. \tag{2.1}$$

PROOF. We first consider the case of $\omega = 0$. Suppose $\lambda > 0$ and $(\lambda + L_0)u = f$. Then

$$e^{\lambda t}u(t) - e^{\lambda s}u(s) = \int_s^t f(r)e^{\lambda r}dr, \quad s < t \leqq 0,$$

$$\left|e^{\lambda t}u(t) - e^{\lambda s}u(s)\right|_H^2 \leqq \int_s^t |f(r)|^2\, dr\, \frac{1}{2\lambda}(e^{2\lambda t} - e^{2\lambda s}),$$

so $\lim_{t\to-\infty} e^{\lambda t}u(t) \equiv c$ exists in $\mathbf{H}$ and we have

$$u(t) = ce^{-\lambda t} + \int_{-\infty}^{t} f(s)e^{\lambda(s-t)}\,ds, \quad t \leqq 0.$$

Below we show the second term is in $\mathscr{H}_0$; the first term belongs to $\mathscr{H}_0$ only if $c = 0$ so we obtain (2.1) for $u \in D(L_0)$.

Now let $f \in \mathscr{H}_0$, $\lambda > 0$, and define u by (2.1) with $\omega = 0$. The indicated integral converges because $f \in \mathscr{H}_0$ and $e^{2\lambda s}$ is integrable on $(-\infty, t)$. Note that

$$\lambda \int_{-\infty}^{t} e^{\lambda(s-t)}\,ds = 1, \quad \lambda u(t) = \int_{-\infty}^{t} f(s)\{\lambda e^{\lambda(s-t)}\}\,ds,$$

so the convexity of $\xi \mapsto |\xi|_H^2$ implies

$$\lambda^2|u(t)|_H^2 \leqq \int_{-\infty}^{t} |f(s)|_H^2\{\lambda e^{\lambda(s-t)}\}\,ds. \tag{2.2}$$

This gives by Fubini's theorem

$$\lambda^2\|u\|_{\mathscr{H}_0}^2 \leqq \int_{-\infty}^{0}\int_{s}^{0} |f(s)|_H^2\,\lambda e^{\lambda(s-t)}\,dtds \leqq \|f\|_{\mathscr{H}_0}^2,$$

so $u \in D(L_0)$ and $\lambda\|u\|_{\mathscr{H}_0} \leqq \|f\|_{\mathscr{H}_0}$. This shows $(\lambda + L_0)^{-1}$ is bounded by $1/\lambda$ for each $\lambda > 0$, hence, L_0 is maximal monotone on $\mathscr{H}_0$. Finally, from (2.2) we obtain

$$\lambda|u(t)|_H^2 \leqq \int_{-\infty}^{t} |f(s)|_H^2\,ds,\ t \leqq 0, \tag{2.3}$$

so $u(-\infty) = 0$. This finishes the case $\omega = 0$.

For the general case note that $f \in \mathscr{H}_\omega$ if and only if $fe^{-\omega t} \in \mathscr{H}_0$ and from above we have for $\lambda > 0$, $(\lambda + L_\omega)u = ((\lambda - \omega) + L_0)u = f$ if and only if

$$u(t) = \int_{-\infty}^{t} f(s)e^{(\lambda-\omega)(s-t)}\,ds,$$

hence

$$e^{-\omega t}\,u(t) = \int_{-\infty}^{t} \{f(s)e^{-\omega s}\}e^{\lambda(s-t)}ds.$$

Thus $(\lambda + L_\omega)^{-1}$ is bounded on $\mathscr{H}_\omega$ by $1/\lambda$ for each $\lambda > 0$, thereby showing L_ω is maximal monotone, and $\lim_{t\to-\infty} e^{-\omega t}u(t) = 0$ as above.

REMARKS. From (2.3) we obtain the uniform bound

$$\sup_{t\leq 0}\{e^{-\omega t}|u(t)|_H\} \leqq \|f\|_{\mathscr{H}_\omega}/\lambda^{1/2},\ \lambda > 0,$$

where $u = (\lambda + L_\omega)^{-1}f$. (See **[5, 7, 8]** for related results).

For each $\mu \in \mathbf{R}$ with $\omega + \mu > 0$ we have $L_{-\mu} \equiv L_0 + \mu$ strongly monotone on $\mathscr{H}_\omega$:

$$((L_0 + \mu)u, u)_{\mathscr{H}_\omega} \geqq (\omega + \mu)\|u\|^2_{\mathscr{H}_\omega}, \quad u \in D(L_0).$$

This implies the uniqueness claims in both of the theorems.

PROOF OF THEOREM 1. Define the maximal monotone operator $\mathscr{A}$ on $\mathscr{H}_\omega$ by $v \in \mathscr{A}(u)$ if $v(t) \in A(u(t))$ for a.e. $t < 0$. Then (1.1) is of the form

$$L_\omega(u) + \mathscr{A}(u) + (\omega + \mu)u \ni f \tag{2.4}$$

in $\mathscr{H}_\omega$. We approximate $\mathscr{A}$ by the Lipschitz monotone $\mathscr{A}_\lambda$ and consider the family of problems

$$L_\omega(u_\lambda) + \mathscr{A}_\lambda(u_\lambda) + (\omega + \mu)u_\lambda = f, \ \lambda > 0. \tag{2.5}$$

From Theorem 2.4 of **[2]** it follows that (2.4) has a solution if and only if $\{\mathscr{A}_\lambda(u_\lambda)\}_{\lambda>0}$ is bounded. But Lemma 2.5 of **[2]** shows $\{u_\lambda\}_{\lambda>0}$ is bounded so it suffices to show $\{L_\omega u_\lambda\}_{\lambda>0}$ is bounded in $\mathscr{H}_\omega$. To this end, let $h > 0$ and extend f and hence, the solution of (2.5) to $(-\infty, h)$ appropriately. Then we have (by monotonicity of A_λ)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_\lambda(t + h) - u_\lambda(t)|^2_H + \mu|u_\lambda(t + h) - u_\lambda(t)|^2_H \\
&\quad \leqq (f(t + h) - f(t), u_\lambda(t + h) - u_\lambda(t))_H \\
&\quad \leqq \frac{1}{2}\left\{\frac{1}{\mu + \omega}|f(t + h) - f(t)|^2_H + (\mu + \omega)|u(t + h) - u(t)|^2_H\right\}, \\
&\quad t \leqq 0.
\end{aligned}$$

From this follow the estimates

$$\frac{d}{dt}(|u_\lambda(t + h) - u_\lambda(t)|^2_H e^{(\mu-\omega)t}) \leqq \frac{1}{\mu + \omega}|f(t + h) - f(t)|^2_H e^{(\mu-\omega)t},$$

$$\begin{aligned}
|u_\lambda(t + h) - u_\lambda(t)|^2_H e^{(\mu-\omega)t} &\leqq |u_\lambda(\tau + h) - u_\lambda(\tau)|^2_H e^{(\mu-\omega)\tau} \\
&\quad + \frac{1}{\mu+\omega}\int_\tau^t |f(s + h) - f(s)|^2_H e^{(\mu-\omega)s}\, ds, \ t < 0.
\end{aligned}$$

Since $\mu - \omega \geqq -2\omega$ we have by the Proposition

$$\lim_{\tau\to-\infty} |u_\lambda(\tau)|^2_H e^{(\mu-\omega)\tau} \leqq \lim_{\tau\to-\infty} |u_\lambda(\tau)|^2_H e^{-2\omega\tau} = 0$$

so from above we have

$$\begin{aligned}
&|u_\lambda(t + h) - u_\lambda(t)|^2_H e^{-2\omega t} \\
&\quad \leqq \frac{1}{\mu+\omega}\int_{-\infty}^t |f(s + h) - f(s)|^2_H e^{-2\omega s} e^{(\mu+\omega)(t-s)}\, ds,
\end{aligned}$$

$$\int_{-\infty}^0 |u_\lambda(t + h) - u_\lambda(t)|^2_H e^{-2\omega t}\, dt$$

$$\leqq \frac{1}{\mu+\omega}\int_{-\infty}^{0} |f(s+h)-f(s)|_H^2\, e^{-2\omega s}\int_s^0 e^{(\mu+\omega)(s-t)}\,dt\,ds$$

$$\leqq \frac{1}{(\mu+\omega)^2}\int_{-\infty}^{0} |f(s+h)-f(s)|_H^2\, e^{-2\omega s}\,ds.$$

Dividing by h^2 and letting $h \to 0$ we obtain

$$\left\| \frac{du_\lambda}{dt} \right\|_{\mathscr{H}_\omega} \leqq \frac{1}{\mu+\omega}\left\| \frac{df}{dt} \right\|_{\mathscr{H}_\omega}, \quad \lambda > 0, \tag{2.6}$$

and this yields the desired result.

PROOF OF THEOREM 2. Define the convex lower semicontinuous $\Phi\colon \mathscr{H}_\omega \to [0, \infty]$ by

$$\Phi(u) = \int_{-\infty}^{0} \varphi(u(t))e^{-2\omega t}\,dt.$$

Since $\Phi(u_0) = 0$, it is proper. Note $\Phi(0) < \infty$ if $\omega < 0$ or $\varphi(0) = 0$. We shall apply Proposition 2.17 of [2] to show $L_0 + \mu + \partial\Phi$ is maximal monotone; thus, if $\lambda > 0$ and $[I + \lambda(L_0 + \mu)]u_\lambda = u$, it suffices to prove there is a constant C such that $\Phi(u_\lambda) \leqq \Phi(u) + \lambda C$ for all sufficiently small $\lambda > 0$. From our Proposition above we have the representation

$$u_\lambda(t) = \int_{-\infty}^{0} u(s)\exp\left(\frac{1+\lambda\mu}{\lambda}(s-t)\right)\lambda^{-1}\,ds. \tag{2.7}$$

Since

$$\int_{-\infty}^{0} \exp\left(\frac{1+\lambda\mu}{\lambda}(s-t)\right)\lambda^{-1}\,ds = (1+\lambda\mu)^{-1},$$

we multiply (2.7) by $1 + \lambda\mu$ and apply the convex φ to obtain

$$e^{-2\omega t}\varphi(1+\lambda\mu)u_\lambda(t))$$
$$\leqq \int_{-\infty}^{t} \varphi(u(s))\exp\left(\frac{1+\lambda\mu+2\omega\lambda}{\lambda}(s-t)\right)(1+\lambda\mu/\lambda)e^{-2\omega s}\,ds.$$

From Fubini's Theorem follows

$$\frac{1+\lambda\mu+2\omega\lambda}{1+\lambda\mu}\Phi((1+\lambda\mu)u_\lambda)$$
$$\leqq \int_{-\infty}^{0} \varphi(u(s))e^{-2\omega s}\int_s^0 \exp\left(\frac{1+\lambda\mu+2\omega\lambda}{\lambda}(s-t)\right)(1+\lambda\mu+2\omega\lambda/\lambda)\,dt\,ds$$
$$\leqq \Phi(u),$$

where $\lambda > 0$ is so small that all coefficients are positive, so a rescaling of u, u_λ gives

$$\Phi(u_\lambda) \leqq \frac{1 + \mu\lambda}{1 + \lambda\mu + \lambda 2\omega} \Phi\left(\frac{1}{1 + \lambda\mu} u\right). \tag{2.8}$$

If $\mu = 0$, the desired estimate holds (with $C = 0$) for any $\omega \geqq 0$. If $\mu > 0$, we use the convexity of Φ at 0 to obtain

$$\begin{aligned} \Phi(u_\lambda) &\leqq \frac{1 + \mu\lambda}{1 + \lambda\mu + \lambda 2\omega} \Phi\left(\frac{1}{1 + \lambda\mu} u + \frac{\lambda\mu}{1 + \lambda\mu} 0\right) \\ &\leqq \frac{1}{1 + \lambda(\mu + 2\omega)} \{\Phi(u) + \lambda\mu\Phi(0)\} \end{aligned}$$

so the desired estimate holds if $\mu + 2\omega \geqq 0$. Finally, if $\mu < 0$, then $(1 + \lambda\mu)^{-1} > 1$ so from (2.8) follows

$$\Phi(u_\lambda) \leqq [1/(1 + \lambda(\mu + 2\omega))(1 + \lambda\mu)^{p-1}]\Phi(u)$$

and the coefficient will be less than one for all λ sufficiently small. (Use the binominal expansion and the hypothesis $p\mu + 2\omega > 0$.) The preceding shows $L_0 + \mu + \partial\Phi$ is maximal monotone. With the above hypotheses, if $\omega\mu \neq 0$, then $\omega + \mu > 0$. So the operator is strongly monotone on $\mathscr{H}_\omega$, and, hence, surjective.

3. **Applications to PDE.** We briefly describe certain applications of the preceding results to the existence theory of nonlinear problems in partial differential equations. Such problems have motivated many developments in the theory of maximal monotone operators, so applications like those below are naturally anticipated. These examples were presented in [3], to which we refer for detailed proofs and supplementary material, and they are not intended to be best possible in any sense but serve only to suggest the types of results that can be so obtained.

Let Ω be an open set in $\mathbf{R}^n$ with boundary Γ an $(n - 1)$-dimensional manifold and Ω on one side of Γ. $H^m(\Omega)$ denotes the usual Sobolev space of (equivalence classes of) functions $u \in L^2(\Omega)$ such that each derivative $D^\alpha u \in L^2(\Omega)$ when order $|\alpha| \leqq m$. $H_0^m(\Omega)$ consists of functions $u \in H^m(\Omega)$ for which the trace on Γ of each derivative up to order $m - 1$ vanishes a.e. on Γ, and $H^{-m}(\Omega)$ is the dual of $H_0^m(\Omega)$. Finally ν is the unit exterior normal on Γ and $\partial/\partial\nu$ is the corresponding normal derivative (trace) on Γ.

EXAMPLE 1. Let j_1 and j_2 be functions $\mathbf{R} \to [0, \infty]$ which are convex, proper and lower semicontinuous with $j_k(0) = 0$, $k = 1, 2$. Denote their respective subdifferentials by $\beta_k, = \partial j_k$; thus, β_1, β_2 are general maximal monotone operators in $\mathbf{R}$ with $\beta_k(0) \ni 0$, $k = 1, 2$. Let $H = L^2(\Omega)$, and define

$$\varphi(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \varepsilon|u|^2 + 2j_1(u))\,dx + \int_\Gamma j_2(u)\,ds \tag{3.1}$$

with effective domain

$$D(\varphi) = \{u \in H^1(\Omega) : j_1(u) \in L^1(\Omega) \text{ and } j_2(u) \in L^1(\Gamma)\}.$$

The number $\varepsilon > 0$ will be prescribed below. Then φ is convex, proper and lower semicontinuous on H and its subdifferential is characterized by $f \in \partial\varphi(u)$ if and only if $u \in H^2(\Omega)$ and

$$\text{(3.2)} \qquad \begin{aligned} -\Delta u + \varepsilon u + \beta_1(u) &\ni f \quad \text{a.e. on } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta_2(u) &\ni 0 \quad \text{a.e. on } \Gamma, \end{aligned}$$

where Δ denotes the Laplacian in Ω.

COROLLARY 1. *Let $\omega > 0$. Then for each $f \in W_\omega^{1,2}((-\infty, 0), L^2(\Omega))$ there exists a unique $u \in W_\omega^{1,2}((-\infty, 0), L^2(\Omega))$ such that for a.e. $t < 0$ we have $u(\cdot, t) \in H^2(\Omega)$ and*

$$\text{(3.3)} \qquad \begin{aligned} \frac{\partial u}{\partial t} - \Delta u + \beta_1(u) &\ni f \quad \text{a.e. on } \Omega \times (-\infty, 0), \\ \frac{\partial u}{\partial \nu} + \beta_2(u) &\ni 0 \quad \text{a.e. on } \Gamma \times (-\infty, 0). \end{aligned}$$

Assume in addition there is a $p \geqq 2$ such that

$$\text{(3.4)} \qquad j_k(\lambda r) \leqq \lambda^p j_k(r) \textit{ for all } \lambda \geqq 1, r \in \mathbf{R}.$$

Then the above holds for each $f \in \mathscr{H}_\omega(L^2(\Omega))$.

PROOF. Choose μ such that $-\omega < \mu < 0$ and set $\varepsilon = -\mu$. The first part is immediate from Theorem 1 applied to the convex function (3.1); c.f. (3.2). The second part follows from Theorem 2(b) by choosing $-2\omega/p < \mu < 0$.

REMARKS. The solution of (3.3) satisfies

$$\lim_{t \to -\infty} e^{-\omega t} \| u(\cdot, t) \|_{L^2(\Omega)} = 0.$$

This follows from our proposition above.

In like manner we can discuss more general equations, specifically, those for which $\partial u/\partial t$ has coefficient $1/b(t)$ as in (1.2) and the nonlinear A is monotone in its principle part. The boundary condition in (3.3) contains the three classical types and is of interest even when β_2 or β_2^{-1} is a function.

Variational inequalities arise from (3.3) when β_1, β_2 are "multivalued"; [6, Ch. 1] contains a general discussion. We mention two cases in regard to (3.4). First, if $\beta(0) = (-\infty, 0]$ and $\beta(r) = 0$ for $r > 0$, then $v + \beta(u) \ni 0$ if and only if $u \geqq 0$, $v \geqq 0$ and $uv = 0$. The corresponding applications include problems of semi-permeable constraint [6], the one-phase Stefan free-boundary problem [9], and optimal stopping time problems [1]. Second, if $\beta(0) = (-\infty, 0]$, $\beta(r) = 0$ for $0 < r < 1$, and $\beta(1) = [0, \infty)$, then $v + \beta(u) \ni 0$ if and only if one of $u = 0$ and $v \geqq 0$, $0 < u < 1$ and $v = 0$, or $u = 1$ and $v \leqq 0$. This corresponds to certain problems of control ("climatization" in [6]). Note that the convex j, $\beta = \partial j$, satisfies (3.4) in the first case but not in the second.

EXAMPLE 2. Let j: $\mathbf{R} \to [0, \infty]$ be a convex, proper and lower semicontinuous function with $j(0) = 0$. Assume that the subdifferential $\beta = \partial j$ is onto $\mathbf{R}$. Choose $H = H^{-1}(\Omega)$ and define

$$\varphi(u) = \int_\Omega j(u)dx$$

with domain $D(\varphi) = \{u \in L^1(\Omega): j(u) \in L^1(\Omega)\}$. Then φ is convex, proper and lower semicontinuous on H and its subdifferential is characterized by $f \in \partial\varphi(u)$ if and only if $(-\Delta)^{-1} f \in \beta(u)$ a.e. on Ω. Recall $-\Delta$ is an isomorphism of $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Thus we have

$$\partial\varphi(u) = \{-\Delta v: v \in H_0^1(\Omega),\ v(x) \in \beta(u(x)) \text{ a.e. } x \in \Omega\}.$$

For $\varepsilon > 0$ (to be chosen below) we consider

$$\varphi_\varepsilon(u) = \varphi(u) + (\varepsilon/2)\,[u, u]_{H^{-1}(\Omega)},\ u \in D(\varphi),$$

where the scalar-product on $H^{-1}(\Omega)$ is given by $\langle u, (-\Delta)^{-1}u\rangle$. We similarly obtain

$$\partial\varphi_\varepsilon(u) = \{\varepsilon u - \Delta v: v \in H_0^1(\Omega),\ v(x) \in \beta(u(x)) \text{ a.e. } x \in \Omega\}.$$

The proof of Corollary 1 gives the following.

COROLLARY 2. *Let* $\omega > 0$. *For each* $f \in W_\omega^{1,2}((-\infty, 0), H^{-1}(\Omega))$ *there exists a unique pair* $u \in W_\omega^{1,2}((-\infty, 0), H^{-1}(\Omega))$ *and* $v \in \mathscr{H}_\omega(H_0^1(\Omega))$ *such that* $u(\cdot, t) \in L^1(\Omega)$ *for a.e.* $t < 0$ *and*

$$\frac{\partial u}{\partial t} - \Delta v = f \tag{3.5}$$

$$v \in \beta(u) \text{ a.e. on } \Omega \times (-\infty, 0).$$

If we assume in addition that j satisfies (3.4), *then the above holds for each* $f \in \mathscr{H}_\omega(H^{-1}(\Omega))$.

REMARKS. As before, we have the asymptotic behavior

$$\lim_{t\to-\infty} e^{-\omega t}\|u(\cdot, t)\|_{H^{-1}(\Omega)} = 0.$$

Applications of (3.5) include the porous media equation and two-phase Stefan problems [4, 8].

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