# ČECH THEORY: ITS PAST, PRESENT, AND FUTURE

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ABSTRACT. We survey the development of Čech Theory with an eye towards certain recent developments in the homotopy theory of pro-spaces.

## CONTENTS

1. **Introduction.** The main problem in topology is the classification of topological spaces up to homeomorphism. Recall that two topological spaces $X$ and $Y$ are said to be homeomorphic if there exist continuous maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. This problem is very difficult. As is usual in mathematics, when one is faced by a very difficult problem, one first tries to solve an easier problem. Hence arises homotopy theory. The main problem in homotopy theory is the classification of topological spaces up to homotopy equivalence. Recall that two continuous maps $f, g: X \to Y$ are said to be homotopic, denoted $f \simeq g$, if there exists a continuous map $F: X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Two topological spaces $X$ and $Y$ are now said to be homotopy equivalent (or to have the same homotopy type) if there exists continuous maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. This is still too hard a problem in general. So one first attacks the problem for a particularly nice class of topological spaces, namely compact polyhedra. Poincaré was the first person to systematically study the classification problem for polyhedra. In [136], Poincaré associated certain numerical invariants (namely, the Betti and torsion numbers) to a complex. These invariants are homotopy type invariants and can often be used to show that two complexes do not have the same homotopy type. During the late 1920's, under the influence of E. Noether, Poincare's constructions were reinterpreted as defining the integral homol-

ogy groups of a complex. Now that finite complexes were reasonably well understood, the stage was set for the attack to "approximate" bad spaces by nice spaces, for example, complexes. In §2, we will describe the early (1925–1945) development of these ideas by Alexandroff, Lefschetz and Christie. In §4, we will describe the recent (1968–) rebirth and further development of these earlier ideas initiated by Borsuk and called by him shape theory. In §3, we will describe how very similar ideas were developed by Algebraic Geometers during the period 1950–1970 in their attempts to prove the Weil conjectures. In §5, we will describe a more sophisticated approach to the homotopy theory of pro-spaces [70] called Steenrod Homotopy Theory. §6 consists of a list of open problems.

2. **Čech Theory.** During the late 1920's, P. Alexandroff [7] showed that any compact metric space can be approximated by an inverse sequence of finite complexes. More precisely, Alexandroff defined a *projection spectrum* to be an inverse sequence of simplicial complexes and onto simplicial maps,

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \ldots,$$

and showed that every compact metric space was the topological inverse limit of some projection spectrum (the topological inverse limit of a sequence

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \ldots,$$

is the subspace of the product $\Pi_{i=0}^{\infty} X_i$ consisting of points $(x_0, x_1, \ldots)$ such that $f_i(x_{i+1}) = x_i$). Alexandroff also showed that the projection spectrum can be chosen in the following way. An *open covering* of a topological space $X$ is a family $\{V_\alpha\}_{\alpha \in A}$ of open subsets of $X$ such that $\bigcup_{\alpha \in A} V_\alpha = X$. The *nerve* of $\{V_\alpha\}$, $N\{V_\alpha\}_{\alpha \in A}$, is an abstract simplicial complex having as vertices the elements of $A$ and as typical $n$-simplex an $(n + 1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ of elements of $A$ such that the intersection $\bigcap_{i=0}^{n} V_{\alpha_i}$ is nonempty. Figure (2.1) below illustrates the notion of the nerve of a covering.

$$V_{1,4} \cup V_{3,6} \cup V_{5,2} = S^1$$

$$(V_{1,4}, V_{5,2}) \qquad (V_{1,4}, V_{3,6}) = N\{V_{1,4}, V_{3,6}, V_{5,2}\}.$$

$$(V_{5,2}, V_{3,6})$$
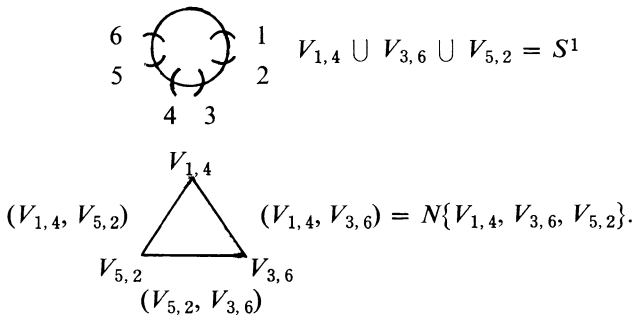
Figure (2.1)

A covering $\{W_\beta\}_{\beta \in B}$ is said to refine $\{V_\alpha\}_{\alpha \in A}$ if there exists a *refining map* $\nu: B \to A$ such that $W_\beta \subset V_{\nu(\beta)}$. A refining map $\nu$ induces a simplicial map $\nu_*: N\{W_\beta\} \to N\{V_\alpha\}$ in the obvious manner. Alexandroff showed that every compact metric space $X$ admits a sequence of finite open coverings $\{V_\alpha^n\}_{\alpha \in A_n}$ and refining maps $\nu_n: A_{n+1} \to A_n$ such that the sequence of nerves

$$N\{V_\alpha^0\} \xleftarrow{\nu_{0_*}} N\{V_\beta^1\} \longleftarrow \cdots$$

is a projection spectrum with inverse limit homeomorphic to $X$.

In [40] Čech extended Alexandroff's ideas to more general topological spaces by associating to any topological space its inverse system (indexed by a partially ordered directed set) of nerves of finite open coverings. This definition is only useful for compact spaces; for general spaces one should use the inverse system of nerves of all open coverings. Another problem is that the refining maps are not unique (though they are unique up to homotopy since any two are contiguous). One way around the nonuniqueness problem is to follow Dowker [55] and to define a second type of nerve associated to any covering $\{U_\alpha\}$ of a space $X$. Define the Vietoris nerve of $\{U_\alpha\}$, $VN\{U_\alpha\}$, to be the abstract simplicial complex whose vertices are the points of $X$ and whose typical $n$-simplex is an $(n + 1)$-tuple of points $(x_0, x_1, \ldots, x_n)$ of $X$ such that for some $\alpha$, $\{x_0, \ldots, x_n\} \subset U_\alpha$. If $\{W_\beta\}$ refines $\{U_\alpha\}$, then the identity map on $X$ determines a canonical simplicial map $VN\{W_\beta\} \to VN\{U_\alpha\}$. Unlike Alexandroff's nerve, the Vietoris nerve is not easy to visualize. This is the price one must pay in order to "rigidify" the Čech construction. On the other hand Dowker [55] showed that $N\{U_\alpha\}$ and $VN\{U_\alpha\}$ have the same homotopy type.

In [114] Lefschetz penetrated deeply into the problems involved in the study of locally compact infinite complexes and of compact metric spaces. We refer the reader to [114, p. 295–297, 325–328, 334]. In Chapter VI of [115] Lefschetz studies inverse systems of complexes which he calls nets. A systematic study of the homotopy theory of nets was given by Lefschetz's student D. Christie in his Princeton doctorial dissertation (see [50]).

3. **Étale Homotopy Theory.** In [178] Weil made several conjectures which have had a profound effect on the development of algebraic geometry in the past twenty-five years. It became clear that what was needed in order to attack the Weil conjectures was a suitable theory of varieties in characteristic $p$ and a good cohomology theory such that non-singular varieties satisfied Poincaré duality and a Lefschetz fixed point theorem. Grothendieck undertook the development of such a theory. We now describe some of the ideas he introduced.

DEFINITION 3.1. A Grothendieck topology $\tau$ on a category $C$ consists of

a category $C = \mathrm{Cat}\,(\tau)$ and a set $\mathrm{Cov}\,(\tau)$ of families $\{U_i \to^{\phi_i} U\}_{i \in I}$ of maps in Cat $(\tau)$ called coverings (where in each covering the range $U$ of the maps $\phi_i$ is fixed) satisfying:

 1. If $\phi$ is an isomorphism, then $\cdot\{\phi\} \in \mathrm{Cov}\,(\tau)$.
 2. If $\{U_i \to U\} \in \mathrm{Cov}\,(\tau)$ and $\{V_{ij} \to U_i\} \in \mathrm{Cov}\,(\tau)$ for each $i$, then the family $\{V_{ij} \to U\}$ obtained by composition is in Cov $(\tau)$.
 3. If $\{U_i \to U\} \in \mathrm{Cov}\,(\tau)$ and $V \to U \in \mathrm{Cat}\,(\tau)$ is arbitrary, then the fiber product $U_i \times_U V$ exists and $\{U_i \times_U V \to V\} \in \mathrm{Cov}\,(\tau)$.
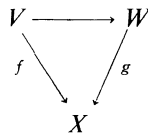
3.2. **The Classical Grothendieck Topology.** Let $X$ be a topological space and let $C_X$ be the category whose objects are the open subsets of $X$ and whose morphisms are given by

$$C_X(U, V) = \begin{cases} \phi, \text{ if } U \not\subset V \\ \text{the inclusion map, if } U \subset V. \end{cases}$$

Define $\mathrm{Cov}\,(\tau_X)$ by: $\{U_\alpha \to U\} \in \mathrm{Cov}\,(\tau_X)$ if and only if $\bigcup_\alpha U_\alpha = U$. In this case, condition 3 of (3.1) just states that the restriction of a covering is a covering.

3.3. **The Étale Topology of Schemes.** If $X$ is a scheme, then there are serveral natural Grothendieck topologies one can associate with $X$. In particular, besides the Zariski topology $Z_X$, one also has the étale topology $E_X$. One takes as objects of the étale topology on $X$, not only the Zariski open subsets $U$ of $X$, but also surjective étale mappings $V \to U$ (an étale map should be thought of as a finite covering space).

If $V \to^f X$ and $W \to^g X$ are objects of $E_X$, then $E_X(f, g)$ is the set of morphisms from $V$ to $W$ over $X$, i.e., the set of commutative diagrams

$$V \longrightarrow W$$
$$f \searrow \quad \swarrow g$$
$$X$$

Unlike the classical case (3.2), $E_X(f, g)$ may have more than one element. $\{V_\alpha \to^{f_\alpha} U_\alpha\}$ is a covering in $E_X$ if $\bigcup_\alpha f(V_\alpha) = U$. (See [11] or [12] for more details concerning Grothendieck topologies and the étale topology.)

DEFINITION 3.4. If $\tau$ is a Grothendieck topology, then a contravariant functor from Cat $(\tau)$ to the category of abelian groups, Ab, is called a *presheaf* of abelian group on $\tau$. A presheaf $F$ on $\tau$ is a *sheaf* if it satisfies the following extra condition:

For each $U \in \mathrm{Cat}\,(\tau)$, and each $\{U_i \to^{f_i} U\} \in \mathrm{Cov}\,(\tau)$, and each family $\{S_i | S_i \in F(U_i)$ with $F(\pi_1)(S_i) = F(\pi_2)(S_j)$ in $F(U_i \times_U U_j)$, there is a *unique* $S \in F(U)$ such that $F(f_i)(S) = S_i$ in $F(U_i)$. Here $\pi_1$ and $\pi_2$ are the

natural projections on $U_i \times_U U_j$. In fancier language, we require that the sequence of sets

$$F(U) \xrightarrow{\amalg_i F(f_i)} \prod_i F(U_i) \underset{F(\pi_2)}{\overset{F(\pi_1)}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

be exact.

Let $P(\tau)$ denote the category of presheaves on $\tau$ and let $S(\tau)$ denote the full sub-category of $P(\tau)$ consisting of sheaves on $\tau$. (See [120] for basic categorical language.) The natural inclusion $S(\tau) \to P(\tau)$ has an adjoint $\#: P(\tau) \to S(\tau)$. If $F \in P(\tau)$, then $\#(F) = F^\#$ is called the sheaf associated to $F$. $P(\tau)$ and $S(\tau)$ are abelian categories (abelian categories are categories having many of the formal properties of the category of abelian groups; see [120]).

If $C$ is an abelian category, then an object $I$ of $C$ is said to be *injective* if the contravariant functor $A \mapsto C(A, I)$ is an exact functor from $C$ to Ab (equivalently, whenever $A \to B$ is a monomorphism in $C$, the induced map $C(B, I) \to C(A, I)$ is surjective). The category $C$ is said *to have enough injectives* if every object in $C$ can be embedded in an injective object. The categories $P(\tau)$ and $S(\tau)$ have enough injectives. If $C$ has enough injectives, then every $A \in C$ has an *injective resolution*, i.e., there exists a long exact sequence

(3.5)                    $0 \longrightarrow A \xrightarrow{i} I^0 \xrightarrow{i^0} I^1 \longrightarrow \dots$

with each $I^k$ injective. If $T$ is a covariant additive left exact functor from $C$ to Ab (or, more generally, to any abelian category), then applying $T$ to (3.5) yields a (non-exact) chain complex

(3.6)                    $0 \longrightarrow T(I^0) \xrightarrow{T(i^0)} T(I^1) \xrightarrow{T(i^1)} \dots$

The *cohomology groups* of (3.6),

$$R^p T(A) \equiv H^p(A, T) \equiv \text{Ker } T(i^p) \, / \, \text{Im } T(i^{p-1})$$

depend (up to isomorphism) only on $T$ and $A$ and not on the choice of injective resolution (3.5). $R^p T$ is called the $p$th right *derived functor* of $T$. $R^0 T = T$ and, if $A$ is injective, $R^p T(A) = 0$ for $p > 0$.

If $\tau$ is a Grothendieck topology and $U \in \text{Cat}(\tau)$, then $U$ determines covariant functors $\Gamma^p_U: p(\tau) \to$ Ab and $\Gamma^S_U: S(\tau) \to$ Ab and their derived functors $R^p \Gamma^p_U$ and $R^p \Gamma^S_U$. If $X$ is a scheme and $E_X$ is the étale topology for $X$ and $F$ is an étale sheaf on $X$ (i.e., on $E_X$), then $R^p \Gamma^S_X(F)$ is called the étale cohomology of $X$ with coefficients in $F$ and will be denoted by $H^p_{et}(X; F)$. $R^p \Gamma^p_X(F)$ is called the Čech cohomology of $X$ with coefficients in $F$ and will be denoted by $\check{H}^p(X; F)$. $H^*_{et}$ is the 'good' cohomology theory which was needed in order to attack the Weil conjectures—an attack which has now been successfully completed by Deligne [52]. In general,

the Čech and étale cohomology theories differ; but there is always a spectral sequence connecting them.

The above difinition is pretty, but it is difficult to compute with. The following is a more traditional definition of $\check{H}$. Let $\{V_i \to^{\phi_i} X\}$ be an étale covering of $X$ and

$$V = \coprod_i V_i \xrightarrow{\phi = \amalg \phi_i} X,$$

where '$\amalg$' denotes coproduct, i.e., disjoint union. Form the 'simplicial object' (see [126])

$$(3.7) \qquad V \rightleftarrows V \times_X V \underset{\displaystyle\rightleftarrows}{\rightleftarrows} V \times_X V \times_X V \cdots.$$

Applying $F$ to (3.7) yields the cosimplicial abelian group

$$(3.8) \qquad F(V) \rightrightarrows F(V \times_X V) \underset{\displaystyle\rightrightarrows}{\rightrightarrows} \cdots.$$

Taking alternating sums of the coface maps in (3.8) yields the cochain complex

$$(3.9) \qquad F(U) \xrightarrow{d^0} F(V \times_X V) \xrightarrow{d_1} \cdots.$$

The cohomology groups of (3.9) will be denoted $H^k(V \to X; F)$. $W \to X$ is said to refine $V \to X$ if there exists a commutative diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\;\nu\;} & V \\
 & \searrow \quad \swarrow & \\
 & X &
\end{array}
$$
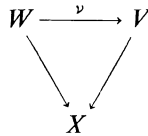
Such a $\nu$, called a refining map, induces a homomorphism $H^k(V \to X; F) \to^{\nu^*} H^k(W \to X; f)$, which turns out to be independent of the choice of refining map. One thus obtains a direct system of abelian groups $\{H^k(V \to X; F)\}$ indexed by the directed set of étale coverings of $X$ ordered by refinement. One now defines

$$\check{H}^k(X; F) = \operatorname*{colim}_{\text{(coverings of } X)} \{H^k(V \to X; F)\}.$$

The reason that one obtains Čech cohomology instead of étale cohomology by this construction is that there are not enough étale coverings of $X$. We only used the coverings in order to obtain the simplicial objects (3.7). A *hypercovering* of $X$ is a simplicial object

$$(3.10) \qquad V. = (V^0 \rightleftarrows V^1 \underset{\displaystyle\rightleftarrows}{\rightleftarrows} V^2 \cdots)$$

in $E_X$ such that the canonical maps $V^0 \to X$ and $V^{n+1} \to (\operatorname{Cosk}_n V.)_{n+1}$ are coverings (see [13]). Simplicial objects of the form (3.7) are *not*, in

general, cofinal in the collection of all hypercoverings of $X$. The homotopy category of hypercoverings of $X$, $HR(X)$, forms a filtering category (a concept more general than directed set; in particular, instead of at most a unique morphism $i' \to i$, one is merely required to have for any pair of morphisms $i' \rightrightarrows i$, a morphism $i'' \to i' \rightrightarrows i$ such that the compositions $i'' \rightrightarrows i$ are equal). For a hypercovering $V.$, we can form, as above, coho-mology groups $H^k(V.; F)$. One now obtains

$$H^k_{et}(X; F) = \operatorname*{colim}_{\text{(hypercoverings of } X)} \{H^k(V.; F)\}.$$

While the above construction is adequate for obtaining the étale cohomology of schemes, it does not yield other homotopy type invariants. What one would really like is a Čech-like construction which associates an inverse system of complexes to any scheme. Here is such a construction [13, p. 111] (see [116] for an alternative construction). An object $V$ in a Grothendieck topology $\tau$ is called *connected* if $V$ is not the initial object $\phi$, and $V$ has no non-trivial coproduct decomposition, i.e., $V = V_1 \amalg V_2$ implies $V_i = \phi$ for exactly one $i$. $\tau$ is called *locally connected* if every object of $\tau$ is a coproduct of connected objects. Associating to an object its set of connected components yields a functor $\pi: \tau \to$ Sets. Applying $\pi$ to the hypercovering $V.$ of (3.10) yields a simplicial set

$$(3.11) \qquad \pi(V.) = \left(\pi(V^0) \rightleftarrows \pi(V^1) \rightleftarrows \pi(V^2) \cdots\right).$$

One thus associates to every locally connected scheme $X$ the inverse system

$$(3.12) \qquad X_{et} = \prod E_X = \{\pi(V.)\}_{V. \in HR(X)}$$

in the homotopy category of simplicial sets, $\mathrm{Ho}(SS)$, indexed by the filter-ing category $HR(X)$. One would like this association to be functorial. For this purpose one must first define an appropriate category of inverse systems. Such a definition is implicit in [50] and was first made explicit in [88]. To any category $C$ there is an associated category pro-$C$ whose objects are inverse systems $\{A_i\}_{i \in I}$ in $C$ indexed by filtering categories $I$ and whose morphism sets are defined by

$$\text{pro-}C(\{A_i\}_I, \{B_j\}_J) \equiv \lim_J \operatorname{colim}_I \{C(A_i, B_j)\}.$$

In pro-$C$ cofinal objects are isomorphic; hence, the isomorphism class of $\{A_i\}$ in pro-$C$ is its 'germ at $\infty$'. (See the appendix of [13] for a systematic discussion of pro-theory.) Now let $\mathrm{Schemes}_{lC}$ denote the category of locally connected schemes. Then the étale construction defines a functor

$$(3.13) \qquad \mathrm{Schemes}_{lC} \xrightarrow{\ E\ } \text{pro-}\mathrm{Ho}(SS).$$

Similarly, the ordinary Čech construction defines a functor

(3.14)                    Top $\xrightarrow{C}$ pro-Ho($SS$).

Étale (Čech) homotopy theory is the study of the invariants one can associate to schemes (topological spaces) using the functor $E(C)$. The first step in this study is a systematic investigation of the algebraic topology of pro-Ho($SS$) (the algebraic topology of Ho($SS$) is developed in [127]). This was largely accomplished by Artin and Mazur in [13].

Basic to much of the work in [13] is the extension of the functorial Postnikoff decomposition from $SS$ to pro-$SS$. If $X \in SS$, then one has a simplicial set, the $n$th coskeleton of $X$, $\text{Cosk}_n X$ obtained from $X$ by "killing homotopy" in dimensions greater than or equal to $n$. $\text{Cosk}_n$ can be made into a functor $\text{Cosk}_n: SS \to SS$ (see [13]). Combining the $\text{Cosk}_n$ for various $n$ yields a functor $\natural: SS \to \text{pro-}SS$ defined by $\natural (X) = X^\natural = \{\text{Cosk}_n X\}$. $\natural$ extends to a functor $\natural: \text{pro-}SS \to \text{pro-}SS$, where $\{X_\alpha\}^\natural = \{\text{Cosk}_n X_\alpha\}$. From now on we work with the pointed categories $SS_*$, Ho($SS_*$), etc. For $X = \{X_\alpha\} \in \text{pro-}SS_*$, we define the homotopy and homology pro-groups of $X$ as

$$\pi_n(X) = \{\pi_n(X_\alpha)\} \in \text{pro-Groups}$$

and

$$H_n(X; A) = \{H_n(X_\alpha; A)\} \in \text{pro-Ab, where } A \in \text{Ab.}$$

With the above notation, the "Fundamental Theorem of Covering Spaces" and the "Hurewicz Theorem" are easily extended to pro-$SS_*$ (see [13]; also compare with [76], [117], [113]). On the other hand, the "Whitehead Theorem" is much more subtle. Let $f: X \to Y$ be a morphism in pro-$SS_0$ ($SS_0$ is the category of pointed, *connected* simplicial sets) such that $f_*: \pi_n(X) \to \pi_n(Y)$ is an isomorphism of pro-groups for all $n$. Then, Artin and Mazur [13; p. 37] showed that $f$ always induces a homotopy equivalence of Postnikoff systems (i.e., $f^\natural: X^\natural \to Y^\natural$ is invertible in pro-Ho($SS_0$)), but may not itself be a homotopy equivalence. A simple counter-example is the unique map from $\{\bigvee_{k>n} S^k\}_{n>0}$ to a point. Under appropriate finite dimensional or "movable" assumptions, it is possible to show that $f$ itself is a homotopy equivalence (see [13, Theorem (12.5)], [133], [122], [64], [67], [132], [61], [80], [70, §5]). In [70] a fibration $F \to E \to {}^f B$ in pro-$SS_0$ is constructed (using an interesting map of Adams [3]) such that $F$ is contractible (i.e., homotopy equivalent to a point) but $f$ is *not* a homotopy equivalence.

In [13; §12] Artin and Mazur considered the problem of comparing the classical homotopy type, $X_{cl}$, of a scheme $X$ over the complex numbers with its étale homotopy type $X_{et}$. They showed [13; Corollary 12.10, p. 143] that, under reasonable hypotheses, $X_{et}$ is canonically isomorphic (in pro-Ho($CW_0$)) to the profinite completion of $X_{cl}$. The Artin-Mazur

definition of the profinite completion of a $CW$-complex is the epitome of the basic philosophy of Čech theory: namely, to approximate a bad object by an inverse system of nice objects.

DEFINITION 3.15. A *complete class* $C$ of groups is a full subcategory of Groups satisfying:

a. $0 \in C$;

b. A subgroup of a $C$-group is in $C$. Moreover, if $0 \to A \to B \to D \to 0$ is an exact sequence of groups, then $B \in C$ if and only if $A \in C$ and $D \in C$,

c. If $A, B \in C$, then the product $A^B$ of $A$ with itself indexed by $B$ is in $C$.

EXAMPLES 3.16. The class of finite groups, or of finite groups whose orders are products of primes coming from a given set of primes, are complete classes of groups.

Now let $\mathrm{Cho}(CW_0)$ denote the full subcategory of $\mathrm{Ho}(CW_0)$ consisting of pointed $CW$-complexes whose homotopy groups are all in $C$. (Note: We pass freely back and forth between the equivalent categories $\mathrm{Ho}(SS_0)$ and $\mathrm{Ho}(CW_0)$ by using the singular and geometric realization functors (see [127]).) For $X \in CW_0$, let $(X \downarrow C)$ denote the homotopy category of $C$-complexes under $X$, i.e., the objects of $(X \downarrow C)$ are homotopy classes of maps $X \to W$, with $W \in \mathrm{CHo}(CW_0)$, and a morphism in $(X \downarrow C)$ is a commutative diagram (in $\mathrm{Ho}(CW_0)$)

$$
\begin{array}{ccc}
 & X & \\
 \swarrow & & \searrow \\
W & \longrightarrow & W'.
\end{array}
$$
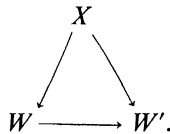
$(X \downarrow C)$ is a filtering category. Hence, the functor $(X \to W) \mapsto W$ determines a pro-object, $\hat{X}$, in pro-$\mathrm{CHo}(CW_0)$, called the $C$-completion of $X$. In fact, the $C$-completion determines a functor $\hat{} : \mathrm{pro\text{-}Ho}(CW_0) \to \mathrm{pro\text{-}CHo}(CW_0)$ which is adjoint to the natural inclusion $\mathrm{pro\text{-}CHo}(CW_0) \to \mathrm{pro\text{-}Ho}(CW_0)$. When $C$ is the class of finite groups, $\hat{X}$ is called the profinite completion of $X$.

REMARK 3.17. One can also obtain a functor $M: \mathrm{Top} \to \mathrm{pro\text{-}Ho}(CW)$ by associating to $X \in \mathrm{Top}$ the pro-object of complexes under $X$. $M$ is naturally equivalent to the Čech functor, $C_n: \mathrm{Top} \to \mathrm{pro\text{-}Ho}(CW)$, based upon "numerable coverings" (compare [123]).

The tools described above were developed by Grothendieck, Artin and others with applications to algebraic geometry and number theory in mind. But they have also turned out to have important applications to algebraic topology. In his work on the Riemann-Roch-Hirzebruch Theorem, Grothendieck was led to associate to a scheme $X$ the abelian group $K(X)$ of vector bundles over $X$ (see [119]). In [17] Atiyah and Hirzebruch trans-

lated Grothendieck's $K$-theory into topological $K$-theory. This $K$-theory has led to the solution of many important topological problems; the most spectacular being Adams' solution of the vector field problem on spheres [5] and Atiyah and Singer's solution of the index problem for elliptic operators [18]. In [16] Atiyah introduced another important closely related group $J(X)$, In attempting to compute $J(X)$, Adams, made a conjecture. In [140] Quillen presented a program for proving Adams' conjecture. He showed that a characteristic $p$ version of the Adams' conjecture was immediately provable and that one could lift the characteristic $p$ Adams' conjecture back to characteristic 0 by using certain comparison theorems of Artin and Mazur (see [13]; p. 143–146). Quillen's program was completed by Friedlander [77] (see also [78], [141]). In [161] Sullivan showed that the Adams' conjecture was a special case of the much more general phenomena of "Galois symmetry" in the étale homotopy theory of algebraic varieties (the Galois group of the algebraic numbers over the rationals acts on any rational variety $V$ (continuously in the étale topology, but *discontinuously* in the classical topology) and, hence, acts on its étale homotopy type $V_{et}$). Sullivan's work was motivated by his interest in the structure of manifolds and by his previous work on the Hauptvermutung [163]. He was led to the study of the homotopy types of certain classifying spaces which naturally arise in geometric topology (such as $G/PL$, see [30] for a readable survey). He quickly discovered that the prime 2 and the odd primes had to be treated differently. This led him to introduce the idea of fracturing homotopy theory into mod-$p$ components. These techniques have become very important in recent work in algebraic topology (see for example [25] and [96], or, for an elementary survey, [97].

4. **Shape Theory.** In the late 1960's Borsuk sparked an avalanche of interest in the study of the global homotopy properties of compacta (see [124] for a survey of the early development (through 1971) of shape theory). Borsuk's original formulation of the shape theory of compact subsets of Hilbert space [24] lacked the flexibility of Christie's formulation [50], but it had the advantage of being more geometric. This added geometry was quickly capitalized upon by Chapman in [41].
   Let

$$s = \prod_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) \subset Q = \prod_{n=1}^{\infty} \left[ -\frac{1}{n}, \frac{1}{n} \right].$$

(Recall that Anderson (see [10]) has shown that $s$ is homeomorphic to the Hilbert space $l^2$). Following [24], [124] and [41], one defines the fundamental category or shape category, Sh, as follows. The objects of Sh are compact subsets of $s$. If $X$ and $Y$ are compact subsets of $s$, then a fundamental sequence $\underline{f}: X \to Y$ is defined as a sequence of maps $f_n: Q \to Q$

with the property that for every neighborhood $V$ of $Y$ in $Q$ there exists a neighborhood $U$ of $X$ in $Q$ and an integer $n_0$ such that for $n$, $n' \geqq n_0$, the restrictions $f_{n|U}$ and $f_{n'|U}$ are homotopic in $V$. Note that $f_n(X)$ does not have to be contained in $Y$; it only has to be near $Y$. Two fundamental sequences $f$, $f' : X \to Y$ are considered homotopic, $f \simeq f'$, provided that for every neighborhood $V$ of $Y$ in $Q$ there exists a neighborhood $U$ of $X$ in $Q$ and an integer $n_0$ such that for $n \geqq n_0$, $f_{n|U}$ and $f_{n'|U}$ are homotopic in $V$. The morphisms in Sh are now taken to be homotopy equivalence classes of fundamental sequences. These definitions should be compared with those of Christie and Grothendieck. Two compacta $X$ and $Y$ contained in $S$ are said to have the same shape if they are isomorphic in Sh,

In [41] Chapman proved the following beautiful theorem.

THEOREM 4.1 (CHAPMAN COMPLEMENT). *If $X$ and $Y$ are compacta in $s$, then $X$ and $Y$ have the same shape if and only if their complements $Q\backslash X$ and $Q\backslash Y$ are homeomorphic.*

Following earilier work of Chapman [43], Geoghegan and Summerhill [82], Hollingsworth and Rushing [98], Venema [173] proved the following finite dimensional version of Chapman's complement theorem.

THEOREM 4.2 ([173]). *Let $X$, $Y \subset R^n$, $n \geq 6$, be compacta satisfying the inessential loops condition (an imbedding condition) whose shape dimensions are in the trivial range (i.e., $2 \operatorname{Dim}(X) + 2 \leqq n$). Then, $X$ and $Y$ have the same shape if and only if $R^n\backslash X$ and $R^n\backslash Y$ are homeomorphic.*

COROLLARY 4.3 ([82]). *Two compact submanifolds of $R^n$ whose dimensions are in the trivial range have the same homotopy type if and only if their complements are homeomorphic.*

These complement theorems suggest that there might be an intimate relationship between the shape theory of compacta in $s$ and the *proper homotopy theory* of complements in $Q$ of compacta in $s$ (e.g., given a Lefschetz fundamental complex $L$ for $X \subset s$ one can show that $L \times Q$ is homeomorphic to $Q\backslash X$, and hence that $L$ and $Q\backslash X$ have the same proper homotopy type). Following Chapman [41] we define a map $f : X \to Y$ to be *proper* if for each compactum $B \subset Y$ there exists a compactum $A \subset X$ such that $f(X\backslash A) \subset Y\backslash B$. Then proper maps $f$, $g : X \to Y$ are said to be *weakly properly homotopic* if for each compactum $B \subset Y$ there exists a compactum $A \subset X$ and a homotopy (dependent on $B$) $F = \{F_t\} : X \times [0, 1] \to Y$ such that $F_0 = f$, $F_1 = g$, and $F((X\backslash A) \times [0, 1]) \subset Y\backslash B$. If, in fact, there exists a proper map $f : X \times [0, 1] \to Y$ which satisfies $F_0 = f$ and $F_1 = g$, then we say that $f$ and $g$ are *properly homotopic*. We thus obtain two categories, Ho($P$) and w-Ho($P$), whose objects are complements in $Q$ of compacta in $s$ and whose morphisms are proper homotopy

equivalence classes of proper maps and weak proper homotopy equivalence classes of proper maps, respectively.

THEOREM 4.4 ([41]). *There is a category isomorphism T from* Sh *to* w-Ho($P$) *such that for each object* $X \in$ Sh, $T(X) = Q \backslash X$.

REMARK 4.5. The reason that Sh is isomorphic to w-Ho($P$), and not to Ho($P$), is because Borsuk's shape theory corresponds to Christie's theory of weak net—net homotopy classes of maps and not to a possible theory of strong net—net homotopy classes of maps (see §5).

A natural question to ask is whether every weak proper homotopy equivalence is in fact a proper homotopy equivalence. We do not know the answer to this question; but we can show [71] that every weak proper homotopy equivalence is weakly properly homotopic to a proper homotopy equivalence. See also Problem 2 of §6.

Chapman and Siebenmann [41] [48] have considered the problem of finding a boundary for an open $Q$-manifold (a $Q$-manifold is a metric space which is locally homeomorphic to $Q$: see [9] for a survey of infinite dimensional topology through 1970). If $W$ is a $Q$-manifold, then $W$ is said to admit a boundary if there exists a compact $Q$-manifold $M$ and a compact $Z$-set $X \subset M$ such that $W$ is homeomorphic to $M \backslash X$ ($X \subset Y$ is a $Z$-set in $Y$ if for each open subset $U$ in $Y$, the inclusion $U \backslash X \to U$ is a homotopy equivalence [8]; every compactum in $s$ or in $Q \backslash s$ is a $Z$-set in $Q$; likewise if $X$ has infinite codimension or if $X$ is a subset of $\partial M^k$ where $M^k$ is a finite dimensional manifold). The finite dimensional version of this problem has been studied in [31], [150], [98] and [171]. The infinite dimensional problem studied by Chapman and Siebenmann is closest in spirit to the work of Tucker [171].

An obvious necessary condition for $W$ to admit a boundary is that its end, $E(W) = \{W \backslash C | C$ is a compactum in $W\}$, be isomorphic in pro-Ho(ANR) to the shape of some compactum $X \subset s$, Sh($X$) = $\{U | U$ is open in $Q$ and $X \subset U\}$ (this is an alternative approach to shape theory; see [76]). The end of $W$ is said to be *tame* if it has the shape of a compactum (one actually has to work in a pointed setting, but for simplicity of exposition we suppress this fine point). In [48] Chapman and Siebenmann develop and obstruction theory for tame ends such that: 1. $W$ admits a boundary if and only if the obstructions associated to $E(W)$ vanish; 2. If $W$ admits a boundary, then for any compactum $X$ such that Sh($X$) $\simeq$ $E(W)$, there exist a compact $Q$-manifold $M$ with $X$ a $Z$-set in $M$ such that $M \backslash X$ is homeomorphic to $W$, i.e., any compactum in the shape class of $E(W)$ can be stuck on at $\infty$ to $W$; 3. All possible obstructions are realized.

REMARKS 4.6. For the finite dimensional problem studied by Sieben-

mann is [150], the natural definition of tame end would be that the end of $W$ have the shape of a (pointed) closed manifold. But with this definition not all obstructions are realized. Thus, Siebenmann was led to define a tame end to be one which is dominated in pro-Ho($ANR_0$) by a finite complex. He was then able to obtain a good obstruction theory in dimensions $\geq 6$. (This viewpoint will probably appear in the revised version of Siebenmann's thesis if the revised version ever appears).

In [181] West has considered the problem of classifying principal compact Lie Group actions on $Q_0$ and $s_0$, the Hilbert Cube and the Hilbert space with a point deleted (note that $s_0$ is homeomorphic to $s$ but that $Q_0$ is definitely not homeomorphic to $Q$). West first shows that any compact Lie Group $G$ can act principally on $Q_0$. Let $C(G) \equiv G \times [0, 1]/G \times \{0\}$ be the *cone* on $G$. The natural left action of $G$ on itself extends to an action with unique fixed point (the cone point) on $C(G)$. The countable product $\Pi_1^\infty C(G)$ is homeomorphic to $Q$ [182] [183]. Hence we obtain a natural action of $G$ on $Q$ having a unique fixed point, the infinite vertex $v$. The restricted action of $G$ on $Q_0 = \Pi_1^\infty C(G)\backslash\{v\}$ is free and principal (in the sense of Cartan [39]). Crossing $Q_0$ with $s$ we obtain a principal action of $G$ on $s = s_0$ (the reason for working with $s_0$ instead of $s$ is that other natural ways of obtaining principal actions naturally occur on $s_0$, e.g., the natural $Z_2$-action on $s_0$ obtained by sending $x \mapsto -x$ on $s$ and then deleting the unique fixed point 0. The actions described above will be called the *standard* actions on $Q_0$ and $s_0$ and any action not conjugate to a standard action will be called *exotic*.

Let $\rho$ and $\sigma$ be two principal actions of $G$ (a compact Lie group) on $s$. We will say that the actions are *nice* if the quotient spaces $s_0/\rho$ and $s_0/\sigma$ are $s$-manifolds (e.g., if $G$ is finite, then this is always the case). In any case, since $s_0$ in contractible, $s_0/\rho$ and $s_0/\sigma$ are both classifying spaces for $G$. Let $f: s_0/\rho \to s_0/\sigma$ be a map such that the induced bundle $f^*(s_0, \sigma)$ over $s_0/\rho$ is isomorphic to the bundle $(s_0, \rho)$ (see [102]). Since $f$ is a homotopy equivalence, if $s_0/\sigma$ and $s_0/\sigma$ are both $s$-manifolds, then $f$ is homotopic to a homeomorphism $g$ (Henderson [95] has shown that any homotopy equivalence between $s$-manifolds is homotopic to a homeomorphism). One thus obtains the diagram (4.7) of principal $G$-bundle isomorphisms

$$(4.7) \quad \begin{array}{ccccc} s_0 & \xrightarrow{\phi} & s_0 & \xrightarrow{\psi} & s_0 \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle g^*(\sigma)} & & \downarrow{\scriptstyle\sigma} \\ s_0/\rho & \xrightarrow{1} & s_0/\rho & \xrightarrow{g} & s_0/\sigma \end{array}$$

$h = \psi \circ \phi$ is an equivariant homeomorphism from $(s_0, \rho)$ to $(s_0, \sigma)$; hence $\rho$ and $\sigma$ are equivalent actions of $G$ on $s_0$. Thus, all nice actions of $G$ on $s_0$ are standard.

The $Q_0$ case is much more subtle. The main reason for this is that it is
*not* true that a homotopy equivalence of $Q$-manifolds is always homotopic
to a homeomorphism. Instead, one has West's result [182] that every
$\infty$-simple homotopy equivalence of $Q$-manifolds is properly homotopic to
a homeomorphism. (One must first extend Siebenmann's $\infty$-simple homo-
topy [149] from locally finite simplicial complexes to $Q$-manifolds by
using either Chapman's triangulation theorem for $Q$-manifolds [44] (i.e.,
every $Q$-manifold $M$ is homeomorphic to $K \times Q$ for some simplicial
complex $K$) or else Chapman's recent extension of simple homotopy
theory to $ANR$'s [45] (which itself rests on West's result [184] that any
locally compact $ANR$ is a $CE$ image of a $Q$-manifold (a proper map
$f:X \to Y$ is $CE$ if for every $y \in Y, f^{-1}(Y)$ has trivial shape)). Note that the
converse of West's result (i.e., every map which is properly homotopic to
a homeomorphism is $\infty$-simple) has been proven by Chapman [46] and
implies the topological invariance of Whitehead Torsion (i.e., every
homeomorphism is $\infty$-simple).) Now let $\rho$ and $\sigma$ be two principal actions
of $G$ on $Q_0$. Then West [181] has shown that there is a unique simple
structure on $Q_0/\rho$ and on $Q_0/\sigma$. Hence, any proper homotopy equivalence
$f: Q_0/\rho \to Q_0/\sigma$ is in fact $\infty$-simple. The only thing holding up the com-
pletion of the proof that $\rho$ and $\sigma$ are equivalent is the fact that we do not
know that $Q_0/\rho$ and $Q_0/\sigma$ have the same *proper* homotopy type (they
certainly do have the same homotopy type since they are both $BG$'s).
West originally expected to show that $Q_0/\rho$ and $Q_0/\sigma$ are proper homotopy
equivalent by using an infinite dimensional proper Whitehead theorem
of E. Brown [30]; but the authors [72] have constructed a counter-example
to Brown's theorem. Using a proper Whitehead theorem of the authors
and some geometric constructions involving telescopes, the authors [70]
have been able to show that (nice) $\rho$ and $\sigma$ are equivalent if and only if
their ends, $E(Q_0/\rho)$ and $E(Q_0/\sigma)$, have the same shape. This still leaves
open the question of the existence of exotic actions. If $G$ is a finite group,
then West showed [181] that $E(Q_0/\rho)$ is a pro-system whose universal
covering has trivial shape. Call an inverse system $\{X_\alpha\}$ an Eilenberg-
MacLane pro-space if its inverse system of covering spaces, $\{\bar{X}_\alpha\}$, has
trivial shape, and let $K(G, 1)$ be the standard Eilenberg-MacLane space
(see [127]). If $\{X_\alpha\}$ is not shape equivalent to $K(G, 1)$, then we will call
it an exotic Eilenberg-MacLane pro-space. If $\rho$ is an exotic action of
$G$ on $Q_0$, then $E(Q_0/\rho)$ is an exotic Eilenberg-MacLane pro-space. The
authors [70] have constructed uncountably many exotic $K(Z_2, 1)$; but
we still do not know if there are any exotic $Z_2$ actions on $Q_0$ (on the other
hand, work of Tucker [172] implies the existence of uncountably many
different $Z$ actions on $Q_0$).

In [151] Siebenmann has introduced the concept of open $I$-regular
neighborhoods as a natural generalization of polyhedral regular neighbor-

hoods. Let $X$ be a compactum in a metric space $Y$. An $I$-nest for $X$ is a system $N_0 \subset N_1 \subset \cdots$ of neighborhoods of $X$ in $Y$ such that, given any neighborhood $W$ of $X$, there exists for each $m \geq 0$ an ambient isotopy of $Y$, fixing $Y \backslash N_{m+1}$ and a neighborhood of $X$, which pushes $N_m$ into $W$. Any neighborhood of $X$ in $Y$ expressible as $\bigcup_{m \geq 0} N_m$, where $\{N_m\}$ forms an $I$-nest, is called an $I$-regular neighborhood of $X$ in $Y$. $I$-regular neighborhoods are open and unique up to isotopy if they exist at all (more precisely, if $U$ and $U'$ are $I$-regular neighborhoods of $X$ in $Y$, then there exists an isotopy $g_t: U \to Y$, $0 \leq t \leq 1$, fixing a neighborhood of $X$, so that $g_0$ is the inclusion and $g_1(U) = U'$). Now let $Y$ be an $ANR$ and define the *extrinsic shape* of $X$ in $Y$ by $\mathrm{ExSh}(X \to Y) = \{U \backslash X | X \subset U \subset Y$ and $U$ is open$\}$. If $X$ has an $I$-regular neighborhood $U$, then it is easy to see that the natural inclusion $X \to U$ must be a shape equivalence. Hence, a necessary condition for $X$ to admit $I$-regular neighborhoods in an $ANR$ $Y$ is that $X$ have the shape of an $ANR$ (and hence of a $CW$-complex). Under certain conditions this necessary condition is also sufficient. In particular, Siebenmann has proven the following results.

THEOREM 4.8 ([151], [153]). *Let $Y$ be a topological manifold without boundary of dimension $n \geq 5$, and let $X$ be a compactum in $Y$. Then, $X$ admits open $I$-regular neighborhoods in $Y$ if either of the following conditions hold:*

i) $\mathrm{ExSh}\,(X \to Y)$ *has the shape of a finitely dominated complex;*
ii) $Y \backslash X$ *is $1$-$LC$ at $X$, $\mathrm{Dim}\, X \leq n - 3$, and $X$ has the shape of a complex.*

THEOREM 4.9 ([153]). *Let $Y$ be a $Q$-manifold and let $X$ be a compactum in $Y$. Then, $X$ admits open $I$-regular neighborhoods in $Y$ if and only if $\mathrm{ExSh}(X \to Y)$ has the shape of a finitely dominated complex. If $X$ is a $Z$-set in $Y$, then it is sufficient (since, for $Z$-sets, $\mathrm{ExSh}(X \to Y) = \mathrm{Sh}(X)$) to require that $X$ have the shape of a complex.*

THEOREM 4.10 ([153]). *Let $Y$ be an $S$-manifold and $X$ a compactum in $Y$. Then, $X$ admits open $I$-regular neighborhoods in $Y$ if and only if $X$ has the shape of a complex.*

REMARK 4.11. In the above theorems, the assumption that $\mathrm{ExSh}(X \to Y)$ has the shape of a finitely dominated complex is equivalent (see below) to assuming that $\mathrm{ExSh}(X \to Y)$ is shape dominated by a finite complex.

Wall [176] has shown that there exists a connected complex $Z$ which is homotopy dominated by a finite 2-dimensional complex $P$, but which is not homotopy equivalent to any finite complex. Let $Z \underset{d}{\overset{u}{\rightleftarrows}} P$ be the domination maps with $d \circ u$ homotopic to the identity on $Z$. Let $X$ be the inverse limit of the sequence

(4.12) $$P \xleftarrow{u \circ d} P \xleftarrow{u \circ d} P \longleftarrow \cdots$$

Then $X$ has the shape of $Z$ [66]. Hence, $X$ is a 2-dimensional compactum which has the shape of a complex but not the shape of any finite complex; we call such compacta *strange compacta* (by [184] $X$ cannot be an *ANR*, whether $X$ can have the homotopy type of a complex is still an open (and interesting) question). [Ferry has recently shown that any space homotopy dominated by a compactum has the homotopy type of a compactum.] By embedding $X$ nicely in the 5-sphere, $S^5$, and taking the complement $S^5 \backslash X$, one obtains [66] a very simple and heuristic construction of an open 5-manifold having a strange end (an end is strange if it is tame but does not admit a boundary). The proof that $S^5 \backslash X$ has a strange end is non-trivial and rests heavily on Siebenmann's theory of *I*-regular neighborhoods.

A natural and geometrically important question is: When does a pro-complex $\{X_\alpha\}$ have the shape of a complex $K$? This question includes as special cases the questions: 1. When does a topological space $X$ have the shape of a complex $K$? 2. When does the end, $E(M)$, of a locally compact *ANR* $M$ have the shape of a complex $K$? Given a pro-complex $\{X_\alpha\}$ we must first find a candidate for $K$, i.e., we desire to associate to $\{X_\alpha\}$ a complex $K$ and a (shape) map $q: K \to \{X_\alpha\}$ such that $K$ is the complex which best approximates $\{X_\alpha\}$ up to shape; in particular, if $\{X_\alpha\}$ has the shape of a complex, then $q$ should be a shape equivalence. Consider first the geometrically most important case of a tower $\{X_n\}$. Let $\mathrm{Ex}^\infty\{X_n\}$ denote the tower of fibrations obtained from $\{X_n\}$ by inductively converting the bonding maps of $\{X_n\}$ into fibrations. If we take the topological inverse limit of $\mathrm{Ex}^\infty\{X_n\}$, then we do *not*, in general, obtain a space having the homotopy type of a complex. So, instead, we first apply the singular functor $S: CW \to SS$ to $\mathrm{Ex}^\infty\{X_n\}$, then take the *simplicial* inverse limit, and then apply the geometric realization functor $R: SS \to CW$ to finally obtain

$$\mathrm{holim}\ \{X_n\} \equiv R \circ \lim \circ S \circ \mathrm{Ex}^\infty\{X_n\}$$

and a natural map (in pro-Ho($CW$)) $h: \mathrm{holim}\{X_n\} \to \{X_n\}$. A similar procedure can be followed for more general inverse systems (see [69] and [70]; Bousfield and Kan give a different (more mysterious) construction in [25]). Then the following result is proved by Edwards and Geoghegan in [65].

THEOREM 4.13. *Let* $\{X_\alpha\} \in$ *pro-$CW_0$ and let* $h: \mathrm{holim}\ \{X_\alpha\} \to \{X_\alpha\}$ *be the natural map in* pro-Ho($CW_0$). *Then $h$ is an isomorphism in* pro-Ho($CW_0$) *if either of the following conditions hold*:

    A. $\{X_\alpha\}$ *is dominated in* pro-Ho($CW_0$) *by a complex $P$;*

*B.* $\sup_\alpha\{\text{Dim } (X_\alpha)\} < \infty$ *and for all* $i \geq 1$, $\{\pi_i(X_\alpha)\} \simeq \lim_\alpha\{\pi_i(X_\alpha)\}$ *in pro-groups.*

REMARKS 4.14. The first thing that one must show is that $h$ induces isomoprhisms on homotopy pro-groups. This is proved by first showing that $\lim^s$ vanishes on stable pro-groups and then applying a spectral sequence argument (first given by Porter in [138]; for towers the argument is easier and was first given in [68]). In case *A.* one then applies Brown's Representability Theorem [34] to show that $\{X_\alpha\}$ has the shape of a complex $K$. The ordinary Whitehead Theorem now shows that $h$ is a shape equivalence. In case *B.* one then uses the Artin-Mazur ♮-Whitehead Theorem [13; p. 37] (see §3) and [176] to show that holim $\{X_\alpha\}$ has the homotopy type of a finite dimensional complex. The extra "coherence" data in $h$ (see [25; p. 297]) allows one to use the *proof* of the Whitehead Theorem in [64] to conclude that $h$ is a shape equivalence.

REMARKS 4.15. Using [31], Geoghegan and Lacher [81] were able to give a geometric proof that a finite dimensional 1-*UV* (i.e., pro-$\pi_1(X) \simeq 0$) compactum $X$ whose integral Čech cohomology $\check{H}^*(X; Z)$ is finitely generated has the shape of a finite complex. Siebenmann has a geometric proof (unpublished)—using his theory of *I*-regular neighborhoods—that finite dimensional compacta and ends with stable pro-homotopy groups are themselves stable, i.e., have the shape of a (not necessarily finite) complex (he also has cohomological criteria using cohomology with coefficients in the integral group ring $Z[\check{\pi}_1(X)]$). The stability proof given in [65] has the advantage, over the geometric proofs just mentioned, that it is more general, easier, and only uses algebraic topology to prove a theorem in algebraic topology. See [80] for the most recent and simplest proofs of the stability and Whitehead theorems.

REMARKS 1.16. A strong shape (see §5) version of the domination theorem (case *A.*) was given by Porter in [138]. Dydak [59] has shown that a pointed connected movable compactum $X$ such that pro-$\pi_\omega(X) \simeq \check{\pi}_\omega(X)$ is stable ($\pi_\omega(P) \equiv [\bigvee_{i>0}S^i, P]$).

The philosophy of Čech Theory can also be used in the study of compact topological groups. A *Lie Series* is an inverse system of compact Lie groups $\{G_\alpha\}$. $\{G_\alpha\}$ is said to be *associated* with a compact topological group $G$ if $G$ and $\lim_\alpha\{G_\alpha\}$ are isomorphic as topological groups. Lie Series are the topological group analogue of Alexandroff's projection spectra (see §2), and just as Alexandroff was able to show that any compact metric space has projection spectra associated to it, Pontrjagin was able to show that any compact metric group has Lie Series associated to it. (Note: The metric hypothesis is unnecessary, and, of course, the group case is much deeper than the space case.) Using [[75]; Theorem

11.9 p. 287] one can show that any two projection spectra for a compactum have the same shape, i.e., are isomorphic in pro-Ho($CW$). Keesling (private communication) has recently shown that any two Lie Series for a compact group are isomorphic in pro-Lie Groups. More generally, since the category of Lie Groups under a topological group $G$, $(G \downarrow$ Lie Group), is a filtering category, one has a natural functor $L$: Top Groups $\rightarrow$ pro-Lie Groups. Keesling has begun the study of the shape theory of compact topological groups using Lie series (see for example [107] and [108]; [108] contains a very interesting example which was used by the authors in [70] to show that $\lim^1$ does *not* necessarily vanish on uncountable pro-groups which satisfy the Mittag-Leffler condition). The authors have shown [73] that the inverse system of classifying spaces $\{BG_\alpha\}$ associated to a Lie Series $\{G_\alpha\}$ for $G$ classifies (in an appropriate sense) the isomorphism classes of principal $G$ fibrations (in the sense of [51]) over compacta (recall that the classifying space $BG$ only classifies principal $G$ fibre bundles in general, though, by a result of Gleason [84], if $G$ is a compact *Lie* group, then every principle $G$-fibration over a compactum is a principal $G$-fibre bundle).

Some analysts have recently shown interest in Čech Theory. J. Taylor [165, 166, 167] has considered the general problem of determining the relationship between the structure of a commutative Banach algebra $A$ with identity and the various Čech invariants of its maximal ideal space $\Delta(A)$ ($\Delta(A)$ is always a compact Hausdorff space—usually *horrible*). The following are some typical results of this theory (stated by Taylor in [165], though not necessarily due to him).

THEOREM 4.17. *The Gelfand transform induces isomorphisms*: $\check{H}^0$. $Q(A) \simeq \check{H}^0(\Delta(A); Z)$, *where $Q(A)$ denotes the additive subgroup of $A$ generated by the idempotents*;

$\check{H}^1$. $A^{-1}/\exp(A) \simeq \check{H}^1(\Delta(A); Z)$, *where $A^{-1}$ is the invertible group of $A$ and* $\exp(A)$ *is the subgroup consisting of elements with logarithms in $A$*;

$\check{H}^2$. $\mathrm{Pic}(A) \simeq \check{H}^2(\Delta(A); Z)$, *where* $\mathrm{Pic}(A)$ *is the Picard group of $A$ (i.e., the invertible group of the semi-group of isomorphism classes of finitely generated projective, A-modules under tensor product)*:

$\check{K}^0$. $K_0(A) \simeq \check{K}^0(\Delta(A))$, *where $K_0(A)$ is the Grothendieck group of the semi-group of isomorphism classes of finitely generated projective A-modules under direct sum and $\check{K}^0$ is the Čech extension of the Atiyah-Hirzebruck K-theory (which happens to equal $K^0(\Delta(A))$, the Grothendieck group of isomorphism classes of finite dimensional complex vector bundles over $\Delta(A)$)*.

THEOREM (4.17) turns out to be a corollary of the following.

THEOREM 4.18 ([167]). *Let $F$ be a closed complex submanifold of a domain*

$U \subset \mathbf{C}^n$. *Then one can define a subset* $A_F$ *of* $A^n$ *(if* $A$ *is semi-simple, then* $A_F = \{\alpha \in A^n |$ *the joint spectrum* $\alpha$ *is contained in* $F\}$*) such that if* $F$ *is a discrete union of complex homogeneous spaces, then* $A_F$ *is locally path connected and the Gelfand transform induces a bijection*

$$\pi_0(A_F) \simeq [\Delta(A), F].$$

REMARKS 4.19. Taylor makes the following observations in [165; p. 5]. Theorem (1.18) can be used to relate the homotopy invariants of the form $[\Delta(A), F]$ to the structure of $A$ whenever $F$ is a space which has the homotopy type of a complex homogeneous space or of a direct limit of complex homogeneous spaces. The invariants that arise in $K$-theory are all of this form. On the other hand, while the functors $\check{H}^p(-; Z)$ of Čech cohomology have the form $\Delta \mapsto [\Delta, K(Z, p)]$ for spaces $K(Z, p)$ (the Eilenberg-MacLane spaces), only for $p = 0, 1, 2$ can these spaces be approximated by complex homogeneous spaces. Taylor has recently informed us that he can now identify $\check{H}^3(\Delta(A); Z)$ with a Brauer group of $A$. Is it possible that Theorem (4.18) can be extended to more general $F$ by replacing $\pi_0(A_F)$ with the set ${}^s\pi_0(A_F)$ of "Steenrod" path components of $A_F$ (${}^s\pi_0(A_F)$ is equal to the set of *strong* net homotopy classes of maps from a point to $A_F$ in the sense of Christie [50] (see §2, §5 and below)]. A similar situation has arisen in some recent work of I. Craw. He is developing a "Galois theory" for commutative Banach Algebras $A$ and hoped to relate a Galois group of $A$ to the fundamental group of $\Delta(A)$. Unfortunately, this only worked when $\Delta(A)$ was locally nice. What is probably needed is the Steenrod fundamental group of $\Delta(A)$, ${}^s\pi_1(\Delta(A))$ and an associated theory of covering spaces. (A similar situation probably also holds in algebraic geometry: where they presently use the fundamental pro-group, they might be able to use a strong étale fundamental group; see [76], [117], [1], [13]; also see [131]; p. 114 for a possible application of Steenrod homology (see below)).

Taylor is interested in the Banach Algebra $M(G)$ of all finite regular Borel measures on a locally compact abelian group $G$ under convolution multiplication. $M(G)$ and $\Delta(M(G))$ are very complicated. However, $M(G)$ contains a relatively simple subalgebra $M_1(G) = \oplus_\tau L(G_\tau)$, where $\tau$ ranges over topologies on $G$ which dominate the original topology and for which $G$ is still a locally compact topological group, and $L(G_\tau)$ is the algebra of measures absolutely continuous with respect to Haar measure on $G_\tau$ (Note: the $L(G_\tau)$ depend upon the topology on $G$, but $M(G)$ depends only upon the Borel structure on $G$; it is for this reason that $M_1(G)$ is much simpler than $M(G)$). Taylor has shown that the natural map of maximal ideal spaces, $i^*: \Delta(M(G)) \to \Delta(M_1(G))$, induced by the inclusion map $i: M_1(G) \to M(G)$, is a $CE$-map. Hence, by a Vietoris theorem implicit in [112], $i^*$ induces an isomorphism on Čech cohomology.

This isomorphism, Theorem (4.17), and computations of the Čech cohomology of $M_1(G)$, yield some information about the structure of $M(G)$. Taylor [165] then asked whether $i^*$ also induces an isomorphism of Čech $K$-theory, and, more generally, is it in fact a shape equivalence. One can show that $i^*$ induces a $\natural$-isomorphism (unpointed) in shape theory, and hence, if $\Delta(M(G))$ and $\Delta(M_1(G))$ were finite dimensional (or if $i^*$ were movable (see [74] and [67], then $i^*$ would be a shape equivalence (a geometric proof that every $CE$-map between finite dimensional compacta is a shape equivalence is given in [148]). Unfortunately, these conditions do not appear to apply here—in particular, the maximal ideal spaces are infinite dimensional—and it is known that many finite dimensional results fail for infinite dimensional spaces. The infinite dimensional counter-examples almost all derive from the work of Adams [2, 3, 4, 5] or alternatively, Toda [170].

THEOREM 4.20. ([3, p. 22]). *Let $p$ be an odd prime, $g \colon S^{2q-1} \to S^{2q-1}$ a map of degree $p^f$, and $Y$ the Moore space $S^{2q-1} \cup_g e^{2q}$ (thus $\tilde{K}(Y) = Z_{p^f}$). $S^{2r}Y$ will denote the $2r$-fold suspension of $Y$ with $r = (p-1)p^{f-1}$. Then, for suitable $q$, there is a map*

$$A \colon S^{2r}Y \to Y$$

*which induces an isomorphism*

$$A^* \colon \tilde{K}(Y) \to \tilde{K}(S^{2r}Y).$$

*Therefore the composite*

$$A_s = A \circ S^{2r}A \circ S^{4r}A \circ \cdots \circ S^{2r(s-1)}A \colon S^{2rs}Y \to Y$$

*induces an isomorphism of $\tilde{K}$, and hence is essential for every $s$.*

The inverse system

$$\{ Y \xleftarrow{A_1} S^{2r}Y \xleftarrow{A_2} S^{4r}Y \longleftarrow \cdots \}$$

and its inverse limit, $X_A$, can be used to obtain counter-examples to the extension of many finite dimensional results, e.g., the Hopf Classification Theorem and the Shape Whitehead Theorem (see [104], [98], [56], [72]). Taylor [164] showed that one could define a $CE$-map $f \colon X_A \to Q$. Since $X_A$ does not have trivial shape but $Q$ does, this shows that not every $CE$-map is a shape equivalence. (Keesling [109] turned this around to obtain a $CE$-map $g \colon Q \to X_A$, thus obtaining a trivial shape decomposition of $Q$ whose decomposition space is not an absolute retract; compare [111].) Thus the question of whether $i^*$ is a shape equivalence cannot be settled by general results concerning mappings. See also recent work of S. Mardešić and T. B. Rushing [125, 126] D. Coram and P. Duvall,

Jr. [49], J. Bryant and C. Lacher [38], L. Husch [100, 101], R. Goad [85], and T. C. McMillan [129] on shape fibrations and $CE$ maps.

Besides the Čech $K$-theory studied by Taylor, other generalized Čech cohomology theories and generalized Steenrod homology theories are being studied by analysts. In [6] Alexander and Yorke have proven a global version of the Hopf Bifurcation Theorem for autonomous differential systems. One of their main tools is the use of the generalized homology theory called framed bordism (see [159]), its associated Čech cohomology theory, and the relative Alexander duality between them. Recall that Alexander duality states that if $X$ is a compactum in $S^n$, and $p$ and $q$ are non-negative integers for which $p + q = n - 1$, then there is a natural isomorphism

$$H_p^c(S^n \backslash X; Z) = \check{H}^q(X; Z),$$

where $H_*^c$ denotes homology with compact supports. A similar duality theorem holds for generalized theories. In [156] Steenrod sought a similar duality theorem relating $H^p(S^n \backslash X; Z)$ to some homology theory of $X$. The resulting homology theory is now called the Steenrod homology theory and will be denoted by $^sH_*$. Because the inverse limit functor is not exact, Čech homology fails, in general, to satisfy the Eilenberg-Steenrod [75] exactness axiom; but it compensates for this failure by instead satisfying the continuity axiom, i.e., if $\{X_n\}$ is a projection spectrum for $X$, then the natural map $\check{H}_*(X) \to \lim_n\{H_*(X_n)\}$ is an isomorphism. Milnor [130] has shown (see also [154]) that Steenrod homology theory satisfies all the Eilenberg-Steenrod axioms plus the following substitute for the continuity axiom: There is a natural short exact sequence

$$(4.21) \qquad 0 \to \lim_n^1\{H_{q+1}(X_n)\} \to {}^sH_q(X) \to \lim_n\{H_q(X_n)\} \to 0.$$

(If $\{G_n\}$ is an inverse sequence of groups and $S: \Pi_n G_n \to \Pi_n G_n$ is the shift map given by

$$S(g_1, g_2, \cdots) = (g_1 - p_1(g_2), g_2 - p_2(g_3), \cdots),$$

where $p_n: G_{n+1} \to G_n$ is the bonding homomorphism, then $\lim_n\{G_n\}$ is the kernel of $S$ and $\lim_n^1\{G_n\}$ is the cokernel of $S$. The fact that $^sH_*$ satisfies all the Eilenberg-Steenrod axioms plus (4.21) is actually due to Steenrod (unpublished).) Milnor also shows that $^sH_*$ is uniquely characterized up to natural equivalence by the Eilenberg-Steenrod axioms, (4.21), plus invariance under relative homeomorphism. Kaminker and Schochet [105] have made the following definition.

DEFINITION 4.22. A *generalized Steenrod homology theory* consists of a sequence $h_* = \{h_n | n \in Z\}$ of covariant, homotopy invariant functors from the category of compact metric spaces to abelian groups satisfying the following axioms:

(E) If $(X, A)$ is a compact metric pair, then the natural sequence

$$h_n(A) \to h_n(X) \to h_n(X/A)$$

is exact for all $n$.

(S) There is a sequence of natural equivalences

$$\sigma_n: h_n \to h_{n+1} \circ S$$

called suspension, where $S$ is unreduced suspension.

(W) If $X = \lim_n\{\bigvee_{j=1}^n X_j\}$ is the *strong* wedge of a sequence of pointed compact metric spaces, then the natural projections $X \to X_j$ induce an isomorphism

$$h_*(X) \to \prod_j h_*(X_j).$$

REMARKS 4.23. An interesting problem is to give a construction for extending a generalized homology theory $h_*$ defined for compact *CW*-complexes to a generalized Steenrod homology theory $^s h_*$ defined for all compact metric spaces (or beyond?). In §5 we outline a general solution to this problem. Brown, Douglas and Fillmore [36, 37] (see below) have constructed a Steenrod *K*-theory. In light of the work of Alexander and Yorke, the development of a Steenrod bordism theory might yield dividends in the study of differential equations. Steenrod homology theories do *not*, in general, factor through the shape category; but they factor through the *strong shape category* (see §5).

DEFINITION 4.24 ([36]). Let $H$ be a separable complex Hilbert space, $L(H)$ the algebra of bounded linear operators on $H$, $K(H)$ the ideal of compact operators on $H$, $A(H)$ the Calkin algebra $L(H)/K(H)$, $T: L(H) \to A(H)$ the quotient map, $X$ a compactum and $C(X)$ the $C^*$-algebra of continuous complex valued functions on $X$. An *extension* of $K$ by $C(X)$ is a pair $(E, \phi)$, where $E$ is a $C^*$-subalgebra of $L(H)$ that contains $K(H)$ and the identity operator, and $\phi$ is a *-homomorphism of $E$ onto $C(X)$ with kernel $K$. Extensions $(E_1, \phi_1)$ on $H_1$ and $(E_2, \phi_2)$ on $H_2$ are *equivalent* if there exists a *-isomorphism $\psi: E_1 \to E_2$ such that $\phi_1 = \phi_2\psi$. The set of equivalence classes of extensions of $K$ by $C(X)$ is denoted by $\text{Ext}(X)$. Let

$$E_n(X) = \begin{cases} \text{Ext}(X) & \text{if } n \text{ is odd} \\ \text{Ext}(SX) & \text{if } n \text{ is even.} \end{cases}$$

Brown's Bott Periodicity Theorem for Ext [35] (i.e., there is a natural isomorphism $Bp: \text{Ext}(S^2X) \to \text{Ext}(X)$) together with results from [36] yields the following result (see [105]).

THEOREM 4.25. *$E_*$ is a generalized Steenrod homology theory on the category of compact metric spaces. The coefficient groups are given by*

$$E_n(S^0) = \begin{cases} Z & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Brown, Douglas and Fillmore have used Ext to prove the following striking generalization of a classical result of H. Weyl and J. von Neumann.

THEOREM 4.26 ([36]). *Every normal element* $S \in A(H)$ *determines a class* $[S] \in$ Ext $(\sigma(S))$, *where* $\sigma(S)$ *is the spectrum of* $S$. *Two such normal elements* $S$, $T \in A(H)$ *are unitarily equivalent if and only if* $\sigma(S) = \sigma(T) = X$ *and* $[S] = [T]$ *in* Ext $(X)$. *A normal element* $S \in A(H)$ *is the image of a normal operator* $N \in L(H)$ *if and only if* $[S] = 0$ *in* Ext $(\sigma(S))$.

REMARKS 4.27. The Brown-Douglas-Fillmore theory of $E_*$ can probably be extended to yield a theory $E_*^{\text{II}}$ which would realize Steenrod homology with coefficients in $R$. This would be done in the following manner. A von Neumann algebra $B$ is a *-subalgebra of $L(H)$ which is closed in the weak topology. Many of the standard results of functional analysis, such as the spectral theorem, hold in von Neumann algebras (see [146]). If the center of $B$ consists of scalar multiples of the identity operator, then $B$ is called a factor. Murray and von Neumann [134] showed that one could define a "dimension" function on the lattice of projections of a factor with values in one of the following sets:

$$I_n = \{0, 1, 2, \cdots, n\},$$
$$I_\infty = \{0, 1, 2, \cdots, \infty\},$$
$$II_n = [0, n],$$
$$II_\infty = [0, \infty],$$
$$III_\infty = \{0, \infty\}.$$

Factors of type $I_n$ are isomorphic to $M(n, \mathbf{C})$, and factors of type $I_\infty$ are isomorphic to $L(H)$. Factors of type $II_n$, $II_\infty$, and $III_\infty$ do occur, but are not unique (see [103] for a survey of the history of factors). Now let $B$ be a factor of type $II_1$. Then $B \otimes L(H)$ is a factor of type $II_\infty$ and we have a natural inclusion $L(H) \to B \otimes L(H)$. Let $K_{\text{II}}$ denote the closure of the ideal of operators of "finite" rank in $B \otimes L(H)$, $A_{\text{II}}$ the "Calkin" algebra $B \otimes L(H)/K_{\text{II}}$, and $F_{\text{II}}$ the "Fredholm" operators $\pi^{-1}(G(B \otimes L(H)/K_{\text{II}}))$. $F_{\text{I}}$ is a classifying space for $K$-theory [15] and $F_{\text{II}}$ is a classifying space for a $K$-theory based upon type II-bundles [28] or, equivalently, for $H^{ev}(-; R)$ $= \oplus H^{2k}(-; R)$. The natural inclusion $F_{\text{I}} \to F_{\text{II}}$ induces the Chern character. If one now defines $\text{Ext}^{\text{II}}(X)$ by using extensions of $K_{\text{II}}$ by $C(X)$, one should be able to obtain the appropriate analogs of theorems (4.25) and (4.26).

We conclude this section with a fascinating conjecture of J. Wagoner. If $R$ is a ring with unit, then Quillen has defined a classifying space $BR^+$

whose homotopy groups, $\pi_i(BR^+)$, are defined to be the algebraic $K$-groups of $R$, $K_i(R)$ (see [83]). If $R$ is a topological ring, then $BR^+$ inherits a natural topology from that on $R$. Wagoner conjectures that for appropriate $R$ (such as $Q_l$) the compactly generated shape homotopy groups of $BR^+$ (see [158], [144]) are isomorphic to the *continuous* algebraic $K$-groups of $R$. By using a delooping $B_R$ of $BR^+$ [175], he can show that the compactly generated shape homotopy groups of $B_R$ are isomorphic to the continuous algebraic $K$-groups of $R$; it is the failure of Hurewicz fibrations to yield long exact sequences of shape homotopy groups ([74]) which makes Wagoner's conjecture a conjecture and not a theorem.

5. **Steenrod Homotopy Theory.** In [139] Quillen gave the following axiomatization of homotopy theory.

DEFINITIONS 5.1 [142, p. 233–235]). A *closed model category* is a category $C$ endowed with three distinguished families of maps called cofibrations, fibrations, and weak equivalences satisfying the axioms $CM1$–$CM5$ below.

$CM1$. $C$ is closed under finite projective and inductive limits.

$CM2$. If $f$ and $g$ are maps such that $gf$ is defined, then if two of $f$, $g$, and $gf$ are weak equivalences, so is the third.

Recall that the maps in $C$ form a category $AC$ having commutative squares for morphisms. We say that a map $f$ in $C$ is a retract of $g$ if there are morphisms $\phi: f \to g$ and $\psi: g \to f$ in $AC$ such that $\psi\phi = 1_f$.

$CM3$. If $f$ is a retract of $g$ and $g$ is a fibration, cofibration, or weak equivalence, so is $f$.

A map which is both a fibration (resp. cofibration) and weak equivalence will be called a trivial fibration (resp. trivial cofibration).

$CM4$. (Lifting). Given a solid arrow diagram

(5.2)

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\scriptstyle i \downarrow & \nearrow & \downarrow \scriptstyle p \\
B & \longrightarrow & Y
\end{array}
$$

the dotted arrow exists in either of the following situations:
   (i) $i$ is a cofibration and $p$ is a trivial fibration;
   (ii) $i$ is a trivial cofibration and $p$ is a fibration.

$CM5$. (Factorization). Any map $f$ may be factored in two ways:
   (i) $f = pi$, where $i$ is a cofibration and $p$ is a trivial fibration;
   (ii) $f = pi$, where $i$ is a trivial cofibration and $p$ is a fibration.

We say that a map $i: A \to B$ in a category has the *left lifting property* (*LLP*) with respect to another map $p: X \to Y$ and $p$ is said to have the right lifting property (*RLP*) with respect to $i$ if the dotted arrow exists in any diagram of the form (5.2). An object $X$ of $C$ is called *cofibrant* if the map

$\phi \to X$ ($\phi$ is the initial object of $C$ which exists by $CM1$) is a cofibration and *fibrant* if $X \to e$ ($e$ is the final object) is a fibration. If $A \vee A$, $\text{in}_i\colon A \to A \vee A$, $i = 1, 2$ is the direct sum of two copies of $A$, we define a cylinder object for $A$ to be an object $A_\mathrm{I}$ together with maps $\partial_i\colon A \to A_\mathrm{I}$, $i = 0, 1$, and $\sigma\colon A_\mathrm{I} \to A$ such that $\partial_0 + \partial_1\colon A \vee A \to A_\mathrm{I}$ is a cofibration, $\sigma$ is a weak equivalence and $\sigma\partial_i = I_A$, $i = 0, 1$. Here $\partial_0 + \partial_1$ denotes the unique map with $(\partial_0 + \partial_1)\text{in}_i = \partial_{i-1}$. If $f, g \in C(A, B)$, a *left homotopy* from $f$ to $g$ is defined to be a map $h\colon A_\mathrm{I} \to B$, where $A_\mathrm{I}$ is a cylinder object for $A$, such that $h\partial_0 = f$ and $h\partial_1 = g$. $f$ is said to be left homotopic to $g$ if such a left homotopy exists. When $A$ is cofibrant, the relation "is left homotopic to" is an equivalence relation [139; Lemma 4, §1] on $C(A, B)$. The notions of path object and right homotopy are defined in a dual manner. If $A$ is cofibrant and $B$ is fibrant, then the left and right homotopy relations on $C(A, B)$ coincide and we denote the set of equivalence classes by $[A, B]$. We let $\pi C_{cf}$ denote the category whose objects are the objects of $C$ which are both fibrant and cofibrant with $\pi C_{cf}(A, B) = [A, B]$, and with composition induced from that of $C$. The *homotopy category* $\mathrm{Ho}(C)$ of a closed model category is defined to be the localization (see [79]) of $C$ with respect to the class of weak equivalences. The canonical functor $\pi\colon C \to \mathrm{Ho}(C)$ induces $\bar{\pi}\colon \pi C_{cf} \to \mathrm{Ho}(C)$ which is an equivalence of categories [139]; furthermore, $\pi(f)$ is an isomorphism in $\mathrm{Ho}(C)$ if and only if $f$ is a weak equivalence. If $C$ is pointed, i.e., the initial object equals the final object, then loop and suspension functors and families of fibration and cofibration sequences exist in $\mathrm{Ho}(C)$.

EXAMPLES 5.3. (A). (*Simplicial*). The category $SS$ of simplicial sets is the prototype for Quillen's theory of model categories. Fibrations in $SS$ are Kan fibration (see [127]), cofibrations are (dimensionwise) injective maps, and weak equivalences are those maps which become ordinary homotopy equivalences after the geometric realization functor $R\colon SS \to CW$ is applied. If $A$ is a category, then $sA$, the category of simplicial objects over $A$, is often a model category (see [139; Chapter 2, §4]); in particular, simplicial groups, simplicial abelian groups, simplicial rings, etc. are closed model categories. Simplicial *finite* sets do *not* form a model category. K. S. Brown [34] and the second author [91] have shown that Kan's category of simplicial spectra (see [185]) is a closed model category.

(B). (*Topological*). Let Top denote the category of topological spaces with cofibrations and fibrations defined by the homotopy-extension and covering homotopy properties, respectively, and weak equivalences defined to be ordinary homotopy equivalences. Then, Strøm [160] showed that Top is a closed model category. The similar structure on $CG$, the category of compactly generated spaces [158], has also been shown to be a closed model category [92]. Let Sing denote the category of topological spaces

with the following *singular* structure: cofibrations are pushouts of inclusions of subcomplexes of $CW$-complexes, fibrations are Serre fibrations, weak equivalences are weak homotopy equivalences (i.e., maps inducing isomorphisms for the functions $[K, -]$, where $K$ is a finite complex). Then Sing is a closed model category [139; Chapter 2, §3]. Furthermore, the adjoint functors Sing $_R\rightleftarrows^S SS$ induce an equivalence of homotopy theories Ho(Sing) $_R\rightleftarrows^S$ Ho($SS$).

*Question.* Are there closed model structures on Top and on Schemes corresponding to Čech theory and étale theory?

(C). (*Algebraic*). Let $A$ be an abelian category with sufficiently many projectives and let $C = C_+(A)$ be the category of differential complexes $K = \{K_q, d: K_q \to K_{q-1}\}$ of objects of $A$ which are bounded below, i.e., $K_q = 0$ for $q \ll 0$. Then [139] $C$ is a model category where weak equivalences are maps inducing isomorphisms on homology, where fibrations are the epimorphisms in $C$, and where cofibrations are the monomorphisms $i$ such that Coker $i$ is a complex having a projective object of $A$ in each dimension. Every object in $C$ is fibrant and the cofibrant objects are the projective complexes. $C_{cf}$ is what is denoted $K^-(P)$ in [89], where $P$ is the additive sub-category of projectives in $A$, while Ho($C$) is the derived category $D^-(A)$ or $D_+(A)$.

APPLICATIONS 5.4. Quillen [142] defines rational homotopy theory to be the study of the rational homotopy category, Ho$_Q$(Sing$_2$), obtained from the category of 1-connected pointed spaces by localizing with respect to the family of those maps such that $\pi_* f \otimes Q$ is an isomorphism. He then proves that rational homotopy theory is equivalent to the homotopy theory of reduced differential graded Lie Algebras over $Q$, Ho($DGL_1$), and also to the homotopy theory of 2-reduced differential graded cocommutative coalgebras over $Q$, Ho($DGC_2$). In particular, he obtains pairs of adjoint functors

$$\text{Sing}_2 \leftrightarrows DGL_1 \leftrightarrows DGC_2$$

which induce natural equivalences

$$\text{Ho}_Q(\text{Sing}_2) \leftrightarrows \text{Ho}(DGL_1) \leftrightarrows \text{Ho}(DGC_2).$$

Furthermore, these equivalences have the property that the graded Lie algebra $\pi_{*-1}(X) \otimes Q$ under Whitehead product and the homology coalgebra $H_*(X; Q)$ of a space $X$ are canonically isomorphic to the homology of the corresponding differential graded Lie algebra and coalgebra, respectively. He thus obtained simple algebraic models for rational homotopy theory and also simultaneously solved certain problems posed by Thom and Hopf (see [142]).

Qullen observed in [139; Chapter II, p. 9.3] that the Artin-Mazur theory of pro-Ho(SS) did *not* fit his framework, i.e., pro-Ho(SS) is not the homotopy category of a model structure on pro-SS. Porter [138] seems to have been the first person to define and use the "correct" strong pro-homotopy category, Ho(pro-SS). Let $\Sigma$ be the class of morphisms in pro-SS which may be represented by level (weak) equivalences. Then Porter defines Ho(pro-SS) $\equiv$ (pro-SS) $[\Sigma^{-1}]$, i.e., localize pro-SS at $\Sigma$. In [90] Hastings showed that pro-SS admits a natural closed model structure with homotopy theory Ho(pro-SS). More generally, the following results are proved in [70] for a fairly wide class of closed model categories which includes all those we actually care about.

THEOREM 5.5. *The categories $C^I$ (where $I$ is a cofinite directed set) and pro-C admit natural closed model structures. In $C^I$, cofibrations and weak equivalences are defined degreewise, and fibrations are defined by the right-lifting-property. These classes of maps are extended to pro-C by forming the appropriate retracts and composites.*

Our structures are natural in the following sense.

THEOREM 5.6. *The inclusions $C \to C^I$ and $C \to$ pro-C preserve model structures. The inverse limit functors* lim: $C^I \to C$ *and* lim: pro-C $\to C$ *preserve fibrations and trivial fibrations.*

THEOREM 5.7. *The inclusions* Ho(C) $\to$ Ho($C^I$) *and* Ho(C) $\to$ Ho(pro-C) *admit adjoints* holim: Ho($C^I$) $\to$ Ho(C) *and* holim: Ho(pro-C) $\to$ Ho(C).

REMARKS 5.8. Edwards and Geoghegan (unpublished, see [70]) gave a geometric description of Ho(Tow-Top) (objects are inverse systems indexed by the natural numbers) using infinite mapping telescopes, i.e., fundamental complexes.

Bousfield and Kan [25] gave another closed model structure on $C^I$ with the same homotopy category; however, Theorem (5.6) fails for their structure. Grossman [86] gave a coarser closed model structure on Tow-SS.

The natural functor $\pi$: Ho(pro-C) $\to$ pro-Ho(C) is *not* an embedding. Let $C_*$ be a "nice" pointed closed model category.

THEOREM 5.9. *Let $\{X_i\}$ and $\{Y_j\}$ be objects of Tow-$C_*$. Then, there is a natural short exact sequence of pointed sets*

$$0 \longrightarrow \lim_j^1 \mathrm{colim}_i \{\mathrm{Ho}(C_*)(X_i, Y_j)\}$$
$$\longrightarrow \mathrm{Ho}(\mathrm{Tow}\text{-}C_*)(\{X_i\}, \{Y_j\})$$
$$\xrightarrow{\pi} \mathrm{Tow}\text{-}\mathrm{Ho}(C_*)(\{X_i\}, \{Y_j\}) \longrightarrow 0.$$

REMARK 5.10. Grossman [87] proved Theorem (5.9) in his coarser setting.

THEOREM 5.11. *Let $X$, $Y \in$ Tow-$C$, and let $f: X \to Y$ be an isomorphism in* Tow-Ho($C$). *Then there is an isomorphism $g: X \to Y$ in* Ho(Tow-$C$) *with $\pi(g) = f$ in* Tow-Ho($C$). *Hence, the isomorphism classification of towers is the same in* Tow-Ho($C$) *and in* Ho(Tow-$C$).

REMARK 5.12. Dydak [62] has constructed an example of a morphism $f$ in Ho(Tow-$SS$) which is invertible in Tow-Ho($SS$) but *not* invertible in Ho(Tow-$SS$).

Following Quillen [139; Chapter II, §5], let $C$ be a model category and let $C_{ab}$ be the category of abelian group objects and homomorphisms in $C$. Assume the abelianization $X_{ab}$ of any object $X$ of $C$ exists so that there are adjoint functors

$$(5.13) \qquad\qquad C \underset{i}{\overset{ab}{\rightleftarrows}} C_{ab},$$

were $i$ is the faithful inclusion functor. Assume also that (5.13) induces functors

$$(5.14) \qquad\qquad \mathrm{Ho}(C) \underset{\mathrm{Ri}}{\overset{\mathrm{Lab}}{\rightleftarrows}} \mathrm{Ho}(C_{ab}),$$

where Lab (Ri) is called the left (right) derived functor of ab ($i$) (see [139; Chapter $I$, §4]). Finally assume that Ho($C_{ab}$) satisfies the following two conditions:

(5.15$A$) The adjunction map $\theta : A \to \Omega\Sigma A$ is an isomorphism for all $A \in H_0(C_{ab})$.

(5.15$B$) If

$$A' \xrightarrow{\ i\ } A \xrightarrow{\ j\ } A'' \xrightarrow{\ \delta\ } \Sigma A'$$

is a cofibration sequence, then

$$\Omega\Sigma A' \xrightarrow{-i \circ \theta^{-1}} A \xrightarrow{\ j\ } A'' \xrightarrow{\ \delta\ } \Sigma A'$$

is a fibration sequence.

DEFINITION 5.16. The *cohomology groups* of an object $X \in$ Ho($C$) with coefficients from an object $A$ of Ho($C_{ab}$) are defined to be

$$H^q_M(X;\ A) = \mathrm{colim}_N[\mathrm{Lab}\ (X),\ \Omega^{q+N}\Sigma^N A]$$
$$= \mathrm{colim}_N[X,\ \mathrm{Ri}\ \Omega^{q+N}\Sigma^N A].$$

If $C$ is pointed, then

$$(5.17) \qquad\qquad H^q_M(\Sigma X;\ A) = H^{q+1}_M(X;\ A).$$

If $X \to Y \to C \xrightarrow{\delta} \cdots$ is a cofibration sequence, then there is a long exact sequence

(5.18)     $\rightarrow H^q_M(C; A) \rightarrow H^q_M(Y; A) \rightarrow H^q(X; A) \xrightarrow{\delta} H^{q+1}_M(C; A) \rightarrow$ .

If $A'' \rightarrow A \rightarrow A'' \rightarrow \Sigma A'$ is a cofibration sequence in $\text{Ho}(S_{ab})$, then there is a long exact sequence

(5.19)    $\rightarrow H^q_M(X; A') \rightarrow H^q_M(X; A) \rightarrow H^q_M(X; A'') \xrightarrow{\delta} H^{q+1}_M(X; A') \rightarrow$ .

DEFINITION 5.20 ([139]). An object of $\text{Ho}(C)$ of the form $\text{Ri}(A)$ is called a *generalized Eilenberg-MacLane object* and the object $\text{Lab}(X) \in \text{Ho}(C_{ab})$ is called the *homology* of $X$.

The formula

(5.21)                $H^0_M(X; A) = [\text{Lab}(X), A]$

is a universal coefficient theorem, and

(5.22)                $H^0_M(X; A) = [X; \text{Ri}(A)]$

is a representability theorem for cohomology.

REMARKS 5.23. Quillen proves [139; Chapter II, §5, Theorem 5] that his model category cohomology groups, $H^q_M$, are canonically isomorphic with the Grothendieck sheaf cohomology groups $H^q_{GT}$ (see §3).

Consider the case $C = SS$. Then $SS_{ab} = s(Ab)$, the category of simplicial abelian groups, and $\text{Lab}(X) = X_{ab} = ZX$, the simplicial free abelian group generated by $X$. Also, $\pi_i(X_{ab}) = [S^i, ZX] = H_i(X; Z)$, justifying calling $X_{ab}$ the homology of $X$. Furthermore, if $A$ is a discrete abelian group, then $\text{Ri}(A) = K(A, 0) = A$, and hence, $H^q_M(X; A) = H^q(X; A)$, thus justifying calling $\text{Ri}(A)$ a generalized Eilenberg-MacLane object. For $X \in SS_*$, $ZX$ is the simplicial free abelian group generated by $X$ modulo the subgroup generated by the base point. Then, the natural map $X \rightarrow ZX$ induces the Hurewicz homomorphism

$$\pi_i(X) \rightarrow \pi_i(ZX) = H_i(X; Z).$$

We have cohomology with coefficients; what about homology with coefficients? Let $R$ be a commutative ring with identity, and $RX$ the associated simplicial free $R$-module generated by $X$. Then $\pi_i(RX) \simeq H_i(X; R)$. Hence, $RX$ may be called the $R$-homology of $X$.

More generally, we may obtain generalized homology and cohomology theories as follows. Consider the diagram of categories and functors

$$SS_* \underset{i}{\overset{ab}{\rightleftarrows}} SS_{ab} = s(AB)$$

$$\text{Stab} \Big\downarrow \qquad\qquad \Big\downarrow \qquad\qquad \Big\downarrow \text{Stab}$$

$$Sp \underset{i}{\overset{ab}{\rightleftarrows}} Sp_{ab} = Sp(Ab);$$

here Sp denotes Kan's category of simplicial spectra (see Examples (5.3) (A)), Sp(Ab) the category of simplicial abelian group spectra (see [185]), and Stab is the stabilization functor. There is an induced diagram of homotopy categories

$$
(5.24) \quad
\begin{array}{ccc}
\mathrm{Ho}(SS_*) \xrightleftharpoons[i]{ab} & \mathrm{Ho}(SS_{ab}) = \mathrm{Ho}(s(Ab)) \\[2mm]
\Big\downarrow{ab} \qquad \simeq \Big\downarrow{u} \qquad \simeq \Big\downarrow{u} \\[2mm]
\mathrm{Ho}(Sp) \xrightleftharpoons[i]{ab} & \mathrm{Ho}(Sp_{ab}) = \mathrm{Ho}(Sp(Ab)).
\end{array}
$$

(The functor $u$ associates to a simplicial spectrum $X$ the 0th stage $PX(0)$ of the associated $\Omega$-prespectrum (see [185]).)

We have thus factored Quillen's homology functor

$$ ab: \mathrm{Ho}(SS_*) \rightarrow \mathrm{Ho}(sAb) $$

through the stable category Ho(Sp). But Ho(Sp) is already an abelian category which satisfies conditions (5.15A) and (5.15B). In fact, Alex Heller [93] showed that the functor

$$ (5.25) \qquad\qquad \mathrm{Stab}: \mathrm{Ho}(SS_*) \rightarrow \mathrm{Ho}(Sp) $$

is the universal homology theory on $SS_*$ (actually Heller worked in the categories of pointed $CW$-complexes and $CW$ spectra, but the associated homotopy categories are equivalent to $\mathrm{Ho}(SS_*)$ and $\mathrm{Ho}(Sp)$, respectively).

We therefore consider the functor Stab: $\mathrm{Ho}(SS_*) \rightarrow \mathrm{Ho}(Sp)$ as a *strengthened* Quillen homology theory. Of course the associated homology groups $\mathrm{Ho}(Sp)(\mathrm{Stab}(S^i), \mathrm{Stab}(X))$ are just the stable homotopy groups of $X$, $\pi_i^s(X)$. G. W. Whitehead's theory of generalized cohomology and homology theories (see [185]) is now available. If $E$ denotes a simplicial spectrum, the $E$-homology and $E$-cohomology groups of a simplicial set $X$ are given by

$$ E_i(X) = [S^i, X \wedge E], \text{ and} $$
$$ E^i(X) = [X \wedge S^{-i}, E] $$

(we suppress the stabilization and write [ , $E$] for Ho(Sp) ( , $E$)). We call the simplicial spectrum $X \wedge E$ the $E$-homology of $X$.

If $E$ is the Eilenberg-MacLane spectrum $K(Z) = Z(S^0)$ in Sp(Ab),

$$ X \wedge K(Z) = X \wedge Z(S^0) \sim ZX $$

where $\sim$ denotes equivalence in Ho(Sp). This connects the Quillen and generalized homology theories.

We now consider the case $C = \text{pro-}SS_*$. There is a commutative diagram

$$\text{pro-}s(Ab)$$

$$\text{pro-}SS_* \xrightarrow[i]{ab} (\text{pro-}SS_*)_{ab},$$

where $ab$: pro-$SS^*$ → pro-$s(Ab)$ is defined levelwise. Also Lab $(\{X_\alpha\})$ = $\{ZX_\alpha\}$. Define the Steenrod homotopy, homology, and cohomology groups of $\{X_\alpha\}$ by

$$\begin{aligned}
&{}^s\pi_q(\{X_\alpha\}) &&= [S^q, \{X_\alpha\}] = \text{Ho}(\text{pro-}SS_*)\,(S^q, \{X_\alpha\}),\\
(5.26)\quad &{}^sH_q(\{X_\alpha\}; Z) &&= [S^q, \{ZX_\alpha\}],\\
&{}^sH^q(\{X_\alpha\}; \{A_\beta\}) &&= \text{colim}_N[\{X_\alpha\}; \Omega^{q+N}\Sigma^N\{A_\beta\}].
\end{aligned}$$

More generally, for a simplicial spectrum (or even a pro-simplicial spectrum) $E$

$$\begin{aligned}
(5.27)\quad &{}^sE_i(\{X_\alpha\}) = [S^i, \{X_\alpha \wedge E\}] = \text{Ho}(\text{pro-Sp})(S^i, \{X_\alpha \wedge E\}),\\
&{}^sE^i(\{X_\alpha\}) = [\{X_\alpha \wedge S^{-i}\}, E].
\end{aligned}$$

One applies the above theory to topology and algebraic geometry by rigidifying the Čech and étale constructions by using pointed coverings (see [99]) to obtain functors

$$\begin{array}{c}
\text{Top}\\
\searrow{\scriptstyle C}\\
\phantom{xxxxxx}\longrightarrow\ \text{pro-}SS\\
\nearrow{\scriptstyle E}\\
\text{Schemes}
\end{array}$$

In particular, for a pointed topological space $X$, one defines

$${}^s\pi_i(X) \equiv \pi_i(C(X)) = [S^i, C(X)] = \text{Ho}(\text{pro-}SS_*)\,(S^i, C(X))$$

and

$${}^sE_i(X) = \pi_i(C(X) \wedge E) = \text{Ho}(\text{pro-Sp}_*)\,(S^i, C(X) \wedge E).$$

These definitions yield good Steenrod homotopy and generalized homology theories for compact topological spaces [70], and possibly for schemes using a rigid étale functor. D. S. Kahn, Kaminker, and Schochet [105] give another construction of Steenrod homology for spaces.

Another geometric application is to proper homotopy theory. Let $P$ denote the proper category of locally compact spaces and $HP$ the usual proper homotopy category. $P$ is *not* a model category; e.g., there are not enough fibrations in $P$. On the other hand, there are natural full embeddings of $P$ into closed model categories.

The associations

$$X \mapsto \{(X, X \backslash C)\} \mapsto (X \leftarrow \{X \backslash C\}),$$

where $X \in P$ and $C$ varies over compact subsets of $X$, determines functors

$$P \xrightarrow{i} \text{pro-Maps(Top)} \xrightarrow{j} \text{Maps (pro-Top).}$$

Both pro-Maps (Top) and Maps (pro-Top) are closed model categories [70] and $i$ and $j \circ i$ are easily seen to be full embeddings. One gets induced functors

$$HP \xrightarrow{i_*} \text{Ho(pro-Maps (Top))} \xrightarrow{j_*} \text{Ho(Maps (pro-Top)).}$$

The functors $i_*$ and $j_* \circ i_*$ are full embeddings when restricted to HPANR. One can now study *proper* shape theory in either of the following ways (see [19] and [21] for a Borsuk style approach to proper shape theory).

1. First make $P$ into a model category and then get rid of local pathology, i.e., study the functors

$$P \longrightarrow \text{pro-Maps (Top)} \longrightarrow \text{Maps (pro-Top))}$$
$$\downarrow c \qquad\qquad\qquad\qquad \downarrow c$$
$$\text{pro-Maps}(SS) \longrightarrow \text{Maps (pro-}SS))$$

2. First get rid of local pathology (by using the Čech functor based upon canonical coverings [154] and then go into a model category, i.e., study the functors

$$P \xrightarrow{C} \text{pro-PANR} \longrightarrow \text{pro-Maps(Top)} \longrightarrow \text{Maps(pro-Top).}$$

It is clear that there is much work left to be done in developing Steenrod homotopy theory and its applications.

6. **Open Problems.** We conclude this survey with the statements of some open problems that we find particularly interesting.

PROBLEM 1. Does there exist a tower $X = \{X_n\}$ of pointed connected complexes such that:

i. $\{\pi_1(X_n)\}$ is pro-isomorphic to a finite group and $\{\bar{X}_n\}$ is pro-homotopy equivalent to a point;

ii. $\{X_n\}$ is pro-homotopy equivalent to the end of a locally finite simplicial complex?

REMARKS. We can show that *no* tower of *finite* complexes can satisfy $i$; but we *do* know how to construct infinite examples which satisfy condition $i$, but not condition ii. The reason we are interested in Problem 1 is that we can reduce West's problem of classifying free finite groups actions on $Q_o$ to it.

PROBLEM 2. Does there exist a compactum which is shape dominated by a complex, but does not have the shape of any complex?

REMARKS. Dydak has constructed a tower of connected complexes which is Čech homotopy dominated by a complex, but *not* Čech homotopy equivalent to any complex. (Such phenomena can never occur in Steenrod homotopy theory.) Edwards and Geoghegan [66] have shown that such phenomena cannot occur in pointed Čech homotopy theory. Dydak's example also shows that there is no unpointed Brown's theorem. Dydak's example warns us that we have to be very careful about basepoints in Čech homotopy theory. Similar questions have been raised by McMillan concerning the relationship between pointed and unpointed movability (see Ball's survey article [20] for a discussion of this and other related geometric questions.)

PROBLEM 3. When does a shape fibration admit $\varepsilon$-cross sections which are embeddings?

REMARKS. In [49] Coram and Duvall introduced the notion of an approximate fibration between compact ANR's, and showed, for instance, that the fibers of an approximate fibration had constant shape and that any map which could be approximated by fibrations was, in fact, an approximate fibration. Later work investigated when an approximate fibration could in fact be approximated by fibrations [85], [100], [101] and when they couldn't [100, 101]. More recently, Mardešić and Rushing [125, 126] have extended these ideas to compacta by introducing the notion of a shape fibration. The above problem is a generalization of the question of when a $CE$-map between compact manifolds can be approximated by homeomorphisms. The problem was stated to us by Rushing, and he and his students have some preliminary results. Along the same lines, one wonders if one could develop a simple shape theory analogous to simple homotopy theory by using hereditary shape equivalences? [There has been recent work on this problem by Ferry and Hastings and Hollingsworth.] Also, Lacher and others have been using approximate fibrations to "resolve the singularities" of homology manifolds.

PROBLEM 4. Does there exist a geometric model of real homotopy theory?

REMARKS. In [53], Sullivan and company define algebraic models of rational and real homotopy theory. They show that their algebraic model of the rational homotopy theory of $X$ is equivalent to studying its rational Postnikoff tower $\{Q_s X\}$ in Ho (Tow-$SS_*$). Problem 4 asks if there is some analogous construction for real homotopy theory; possibly to be obtained by "completing" the rational Postnikoff tower to obtain a "real"

Postnikoff tower which might be some sort of tower of simplicial spaces. Similarly, one might also hope to obtain geometric models of "p-adic" homotopy theory.

As a final remark, we observe that pathological compacta have been recently arising more frequently in applied mathematics; for example, as the strange attractors of dynamical systems [128] [M-O], and as "fractals" in Mandelbrot's [121] theory of random sets. Where there occur compacta, can shape theory be far behind?

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