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ROTATING CHAIN FIXED AT TWO POINTS VERTICALLY ABOVE EACH OTHER

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0. **Introduction.** In 1955 Kolodner [1] treated the problem of a chain fixed at one point, the other end being free. Using methods characteristic of ordinary differential equations, he showed that there are infinitely many branches of solutions with one, two, three, $\cdots$ nodes bifurcating from the zero solution, at angular velocities ω_i, the latter sequence tending to infinity. By the zero or trivial solution we mean the solution for which the chain rotates at any angular velocity, but remains on the vertical line through the fixed point. We propose to study the analogous case where the second end of the chain is fixed to a point vertically below the first point at a distance smaller than the length of the chain. The mathematical problem is *not* analogous to that of Kolodner; the different boundary condition changes its character. An accessory equation is introduced which makes a treatment along the lines of Kolodner's attack rather difficult if not impossible.

So we turn to a different method, which has been used extensively in recent times. It involves establishing the existence of continua of solution via topological degree methods. Thus one obtains global extensions of local branches given by the implicit function theorem. A very practical and well known approach is that inaugurated by Rabinowitz in his treatment of global continua of solutions to certain bifurcation problems. Unfortunately the problem at hand resists that approach as well. The natural setup for this method would be a C^2 space, but the trivial solution of the problem does not belong to it. This solution is described by the following $x(s) = (x_1(s), x_2(s), x_3(s))$:

$$x_1(s) = x_2(s) = 0$$

$$x_3(s) = \begin{cases} s, & s \in \left[0, \dfrac{l+a}{2}\right], \\ l + a - s, & s \in \left[\dfrac{l+a}{2}, l\right]. \end{cases}$$

We have used the coordinate system shown in Figure 1.

The function describing the trivial solution is not differentiable, so bifurcation from it cannot be established by the usual analytical tools. But a heuristic argument shows that for infinite angular velocity there must be a definite solution, in shape roughly similar to a catenary.

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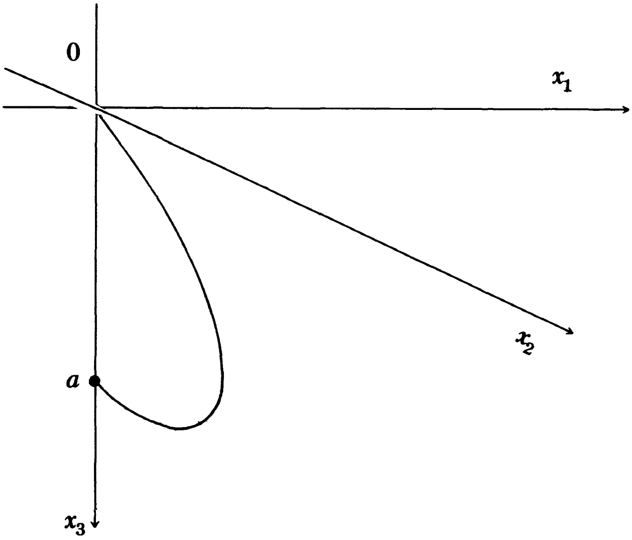

Figure 1.

There are actually infinitely many other solutions as well as we shall show. The idea then is to start from these solutions at infinity and to follow them back.

The plan of the paper is as follows. In §1, the equation for the chain configuration in question is derived and transformed into a different form more suitable for treatment. Some results on the nodes of x_3 and the behaviour of the tension τ are derived. In §2, the equation for the asymptotic configuration of the chain at infinite angular velocity is deduced and all its solutions are discussed. In §3, the general equation for finite angular velocity is studied and the existence of a "lowest" branch of solutions is established by global extension of local branches given by the implicit function theorem.

1. **Preliminaries.** The equations of motion of the chain are given by:

$$\ddot{x} = g + (\tau x')' \qquad g = (0, 0, g)$$

$$x' \cdot x' = 1 \qquad x(s, t) = (x_1(s, t), x_2(s, t), x_3(s, t))$$

$$x(0, t) = 0 \qquad x' \cdot x' = \sum_{i=1}^{3} (x_1')^2$$

$$x(l, t) = a = (0, 0, a),\ l > a > 0,\ \frac{dx}{dt} = \dot{x},\ \frac{dx}{ds} = x'$$

τ is the tension in the chain (it is a Lagrange multiplier associated with the condition $x' \cdot x' = 1$).

We are interested in solutions rotating uniformly around the vertical axis, and we have to express that condition in analytical terms. In order to do this in an expedient way, we define a complex function:

$$z(s, t) = x_1(s, t) + ix_2(s, t).$$

Uniform rotation at angular velocity ω is now expressed by

$$\begin{aligned} z(s, t) &= e^{i\omega t}z(s), \\ \dot{x}_3(s, t) &= 0. \end{aligned}$$

If we substitute this into the above equations of motion we get a new set of equations:

$$\begin{aligned} -\omega^2 z &= (\tau z')', \\ 0 &= g + (\tau x_3')', \\ (x_3')^2 + |z'|^2 &= 1, \\ z(0) &= z(l) = 0, \\ x_3(0) &= 0,\ x_3(l) = a. \end{aligned}$$

Our first aim is to eliminate τ and x_3 from the equations. The third equation leads to

$$x_3' = \pm(1 - |z'|^2)^{1/2}$$

(+ if the curve runs downwards, — otherwise). The second equation can be integrated to give

$$\tau x_3' = g(\sigma - s),$$

where the constant of integration σ is unknown (in Kolodner's case it can be determined from the boundary condition for τ at the free end).

The other equations are

$$\begin{aligned} -\omega^2 x_1 &= (\tau x_1')', \\ -\omega^2 x_2 &= (\tau x_1')'. \end{aligned}$$

If we multiply by x_1', add and integrate, we get

$$\begin{aligned} &-\omega^2 \int_0^s (x_1x_1' + x_2x_2')\, dr \\ &= \int_0^s ((\tau x_1')'x_1' + \tau x_2')'x_2')\, dr. \end{aligned}$$

Multiplying the equation for x_3 by x_3', we get

$$0 = \int_0^s g x_3'\, dr + \int_0^s (\tau x_3')' x_3'\, dr.$$

Adding the two and integrating by parts leads to

$$-\frac{\omega^2}{2}\int_0^s \frac{d}{dr}\left(x_1^2 + x_2^2\right) dr - \int_0^s g\, x_3'\, dr = \int_0^s \tau' dr.$$

We have used the equation $\Sigma_1^3 (x_i')^2 = 1$ and its derivative $\Sigma_1^3 x_i' x_i'' = 0$.

The relation we have derived is valid only for solutions in $C^2([0, l])$. Another admissible choice would be $L_2^2([0, l])$. Performing the integrations brings us to the following expression:

$$\tau(s) = \tau(0) - \frac{\omega^2}{2}\left(x_1^2(s) + x_2^2(s)\right) - g\, x_3(s).$$

This relation proves that τ is finite on the whole interval $[0, l]$ if $\tau(0)$ is finite. This we assume in all that follows.

This then implies that x_3' has at most one zero in $[0, l]$ located at σ; otherwise τ would become infinite at that point. If σ is not in $[0, l]$, then x_3' has no zero.

We shall show that σ *is positive*. Assume, on the contrary, that $\sigma \leqq 0$. Then $g(\sigma - s) \leqq 0$. This implies that τ and x_3' have different signs. x_3' must have positive values because the boundary conditions $x_3(l) = a$ and $x_3(0) = 0$ imply that $\int_0^l x_3'\, dr = a > 0$. As σ is negative or zero, x_3' has at most one zero at zero (if $\sigma = 0$; otherwise none). So x_3 is positive on $[0, l]$. Therefore τ is negative and this implies $z(s) \equiv 0$ because $-\omega^2 z = (\tau z')'$, $z(0) = z(l) = 0$ has only the trivial solution if $\tau < 0$. But then $x_3' \equiv 1$ and $\int_0^l x_3' dz = l + a$. This contradiction proves that $\sigma > 0$.

We now prove that $\tau > 0$ (consequently sgn $x_3' =$ sgn $(\sigma - s)$). In fact, the equation $\tau x_3' = g(\sigma - s)$ shows that τ has at most one zero at σ. If σ is actually a zero of τ then x_3' cannot have one at σ and so must be positive on $[0, l]$. But then τ changes sign at σ and is negative on $(\sigma, l]$ with $\tau(\sigma) = 0$. On the interval $[\sigma, l]$, we have again an equation

$$-\omega^2 z = (\tau z')'$$
$$z(l) = 0,\ \tau(\sigma) = 0,\ \tau \leqq 0,$$

which has only the trivial solution. So we certainly have $z(\sigma) = 0$. But this is impossible, because on $[0, \sigma]$ we have now

$$\begin{aligned}
-\omega^2 z &= (\tau z')' \\
z(0) &= z(\sigma) = 0 \\
\tau(\sigma) &= 0.
\end{aligned}$$

This again implies $z \equiv 0$ on $[0, \sigma]$, because this together with the equation for x_3 is just Kolodner's case (a free end at σ because $\tau(\sigma) = 0$), and the only solution with $z(\sigma) = 0$ is the trivial one. This contradiction proves the contention $\tau > 0$.

We can also show that any solution lies in a plane going through the x_3-axis. For that we recall that a Sturm-Liouville problem of the form $-(\tau z')' - \omega^2 z = 0$, $z(0) = z(l) = 0$ has at most one linearly independent real solution. This implies that x_1 and x_2 are linearly dependent. By suitable rotation of the coordinate system around the x_3-axis, we can achieve that $x_2 \equiv 0$. That shows that we can assume z to be real, and denoting this real quantity by x, we have the following nonlinear system of equations for it:

$$\begin{aligned}
&-\left(\frac{g(\sigma - s)}{(1 - (x')^2)^{1/2}}\, x'\right)' - \omega^2 x = 0, \\
(1) \qquad & x(0) = x(l) = 0, \\
&\int_0^l (1 - (x')^2)^{1/2}\, ds = a.
\end{aligned}$$

The square roots are understood to change their sign from $+$ to $-$ at $s = \sigma$.

This equation is not very nice because of the singularity in the expression $g(\sigma - s)/(1 - (x')^2)^{1/2}$. It is expedient to use the following transformation (Kolodner).

$$u(s) := \frac{g(\sigma - s)}{\omega^2(1 - (x')^2(s))^{1/2}}\, x'(s), \quad \omega \neq 0.$$

This can be solved for x' :

$$(x')^2(s) = \frac{u^2(s)}{(g^2/\omega^4)(\sigma - s)^2 + u^2(s)}$$

or

$$(x')(s) = \frac{u(s)}{+ ((g^2/\omega^4)(\sigma - s)^2 + u^2(s)^{1/2}}.$$

From

$$1 - (x')^2 = \frac{(g^2/\omega^4)(\sigma - s)^2}{(g^2/\omega^4)(\sigma - s)^2 + u^2},$$

we infer that

$$\frac{(g/\omega^2)(\sigma - s)}{+ (g^2/\omega^4(\sigma - s)^2 + u^2)^{1/2}} = (1 - (x')^2)^{1/2}.$$

These substitutions transform (1) into $- u' - x = 0$. This immediately implies the boundary conditions $u'(0) = u'(l) = 0$ for u. The last equation of (1) becomes

$$\int_0^l \frac{(g/\omega^2)(\sigma - s)}{((g^2/\omega^4)(\sigma - s)^2 + u^2(s))^{1/2}} ds = a.$$

Differentiation of $- u' - x = 0$ finally leads to the following system where we have set $\mu = g/\omega^2$ $(\omega \neq 0)$:

$$- u'' - \frac{u}{(\mu^2(\sigma - s)^2 + u^2(s))^{1/2}} = 0,$$

(2) $$u'(0) = u'(l) = 0,$$

$$\int_0^l \frac{\mu(\sigma - s)}{(\mu^2(\sigma - s)^2 + u^2(s))^{1/2}} ds = a.$$

(From now on square roots are always understood as nonnegative unless indicated otherwise.)

Let $\Gamma_\sigma = L^2_{2,0}(0, l) \times \mathbf{R} \times \mathbf{R}$ where $L^2_{2,0}(0, l) = \{u \in L_2{}^2(0, l) \mid u'(0) = u'(l) = 0\}$. $\Sigma_\sigma = \{(u, \sigma, \mu) \in \Gamma_\sigma \mid \mu > 0, \sigma \in [0, l], u(\sigma) = 0\}$. If $(u, \sigma, \mu) \in \Gamma_\sigma$ and satisfies (2), we call it a solution.

We can show that any (x, σ, μ) with $x \in L_2{}^2(0, l)$ satisfying (1) leads to a solution (u, σ, μ) of (2) by Kolodner's substitution. This is obvious from the given calculations. But we also show by contradiction that in this case $(u, \sigma, \mu) \notin \Sigma_\sigma$. Namely, assume $\sigma \in [0, l]$ and $u(\sigma) = 0$. We have shown that x_3' has its zero at σ, so $x'(\sigma) = \pm 1$ because $(x_3')^2(\sigma) + (x')^2(\sigma) = 1$. We also know that $g(\sigma - s)/(1 - (x')^2(s))^{1/2} = \tau(s) > 0$ on $[0, l]$. It follows that $u(s) = g(\sigma - s)/(1 - x')^2(s))^{1/2}x'(s)$ is unequal to zero at σ, that is $u(\sigma) = \pm \tau(\sigma) \neq 0$.

If we are given a solution (u, σ, μ) of (2), then we define

$$x(s) = \int_0^s \frac{u(t)}{(\mu^2(\sigma - t)^2 + u^2(t))^{1/2}} dt.$$

This will lead to a solution of (1) with $x \in C^2(0,l)$ if $(u, \sigma, \mu) \notin \Sigma_\sigma$; otherwise x'' will have a discontinuity at σ. For instance $(0, (l + a)/2,$

μ) is, for any μ, a solution of (2), but it corresponds to the trivial solution already mentioned, which has a discontinuous derivative at σ. This is proven by the fact that for $(u, \sigma, \mu) \notin \Sigma_\sigma$, $u(s)/(\mu^2(\sigma - s) + u^2(s))^{1/2}$ is as many times differentiable with square intergrable derivatives as u is. For $(u, \sigma, \mu) \notin \Sigma_\sigma$, a Taylor development shows that $x'(\sigma + 0) = - x'(\sigma - 0) = \pm u'(\sigma)/(\mu^2 + (u')^2(\sigma))^{1/2}$.

Let us now look for solutions of (2) in $\Sigma_\sigma \cdot u$ is a solution of the following integral equation:

$$u(s) = \alpha(s - \sigma) - \int_\sigma^s dt \int_\sigma^t \frac{u(r)}{(\mu^2(\sigma - r)^2 + u^2(r))^{1/2}} dr,$$

where $u'(\sigma) = \alpha$. On the space $C^*(\sigma, \tau) = \{u \in C(\sigma, \tau) \, |\sup| \, u(s)/(s - \sigma) \, | = |u|_* < \infty\}$ with norm $| \ |_*$, the above operator with given α is a contraction for $\tau > \sigma$ sufficiently close to σ, on a ball $|u|_* \leqq 2\alpha$. It then follows that the solution $(u, \sigma, \mu) \in \Sigma_\sigma$ of the first equation of (2) is uniquely determined by $u'(\sigma) = \alpha$ and we also must have $u(t - \sigma) = - u(2\sigma - t)$. If $\alpha = 0$, then $u \equiv 0$.

Consequently if u has exactly one zero, then $\sigma = l/2$ because by concavity or convexity there is exactly one point s where $u'(s) = 0$ for $s < \sigma$ and then $u'(2\sigma - s) = 0$. But from $u'(0) = 0$, we have $s = 0$ and from $u'(l)$ we get $2\sigma = l$. The solution u is therefore antisymmetric around $\sigma = l/2$.

Obviously the integral in the second line of (2) is then zero. So, no solution of (2) with exactly one zero lies in Σ_σ. Those which do are chains with $a = 0$, and a little thought shows that they are folded upon themselves forming a double chain with free end at $s = l/2$. This is in accordance with the result on stationary kinks in [3].

2. **The asymptotic equation.** We are now interested in the behaviour of (2) when $\omega \to \infty$, that is when $\mu \to 0$. We set $u(s) = v(s) + c$ with $\int_0^l v(s)\, ds = 0$. (2) then becomes

$$- v''(s) - \frac{v(s) + c}{(\mu^2(\sigma - s)^2 + (v(s) + c)^2)^{1/2}} = 0,$$

(2a) $\quad v'(0) = v'(l) = 0,$

$$\int_0^l \frac{\mu(\sigma - s)}{(\mu^2(\sigma - s)^2 + (v(s) + c)^2)^{1/2}} ds = a.$$

Integration of the first equation leads to

$$\int_0^l \frac{v(s) + c}{(\mu^2(\sigma - s)^2 + (v(s) + c)^2)^{1/2}} ds = 0.$$

The operator $-d^2/ds^2$ with boundary conditions $v'(0) = v'(l) = 0$ is boundedly invertible on the space $L_\perp{}^2(0, l)$, the space of square integrable functions orthogonal to the constant function. This implies that v is bounded in $L_2{}^2(0, l)$, and therefore in $C^1(0, l)$, independently of μ and σ, because

$$\left| \frac{v(s) + c}{(\mu^2(\sigma - s)^2 + (v(s) + c)^2)^{1/2}} \right| \leqq 1 \text{ for any } \mu, \sigma \text{ and } v.$$

Any nontrivial solution u of (2) must have a zero. If not we have either $u > 0$ or $u < 0$ and (2) then implies that $u'' < 0$ or $u'' > 0$. Since $u'(0) = u'(l) = 0$ it follows that $u \equiv 0$ because a concave or convex function with a continuous first derivative and horizontal tangents at the endpoints of an interval has to be constant. The only constant function satisfying (2) is 0. For all μ and σ, v stays uniformly bounded in sup-norm. Therefore c must stay uniformly bounded, otherwise $v(s) + c = u(s)$ would have no zero. This proves that $u(s)$ stays uniformly bounded in sup-norm for all μ and σ. Since u is even bounded in C^1-sup-norm, the set of solutions u is precompact in C^0. From

$$u(s) = \frac{\mu(\sigma - s)}{(1 - (x')^2(s))^{1/2}} x'(s),$$

or if we set $\mu\sigma = \kappa$ from

$$u(s)(1 - (x')^2(s))^{1/2} + \mu s x'(s) = \kappa x'(s),$$

we conclude that κ is bounded as $\mu \to 0$. This is because for any μ, there must be an $s(\mu) \in (0, l)$ such that $|x'(s(\mu))| \geqq \epsilon > 0$ for a suitable ϵ; otherwise $\int_0^l (1 - (x')^2(s))^{1/2} ds > a$.

The conclusion is that we can choose a sequence $\mu_i \to 0$ such that $\sigma_i \to \infty$, $\kappa_i \to \kappa_0 < \infty$ and $u_i \to u_0$ in C^0. If $\kappa_0 > 0$ then (2) leads to

$$\begin{aligned} & -u_0'' - \frac{u_0}{(\kappa_0{}^2 + u_0{}^2)^{1/2}} = 0, \\ (3) \qquad & u_0'(0) = u'(l) = 0, \\ & \int_0^l \frac{\kappa_0}{(\kappa_0{}^2 + u_0{}^2)^{1/2}} ds = a. \end{aligned}$$

This is the asymptotic equation of a chain. The chain itself is given by

$$x(s) = \int_0^s \frac{u_0(t)}{(\kappa_0{}^2 + u_0{}^2(t))^{1/2}} dt$$

and

$$x_3(s) = \int_0^s \frac{\kappa_0}{(\kappa_0{}^2 + u_0{}^2(t))^{1/2}}\, dt.$$

In (3) κ_0 must be unequal to zero. This does not imply that 0 is not an accumulation point of possible κ_0, as we shall see later.

The next problem to deal with is to find all solutions of (3). On the road to this goal we derive a general relation valid for (2) as well as for (3). From

$$\frac{uu' - \mu^2(\sigma - s)}{(\mu^2(\sigma - s)^2 + u^2)^{1/2}} = \frac{d}{ds}(\mu^2(\sigma - s)^2 + u^2)^{1/2}$$

and $u'u'' = (1/2)(d/ds)u^{1/2}$, as well as from $- u'u'' - uu'/(\mu^2(\sigma - s)^2 + u^2)^{1/2} = 0$, which comes from (2) by multiplication with u', we conclude

$$\frac{d}{ds}\left[- \frac{1}{2} (u')^2 - (\mu^2(\sigma - s)^2 + u^2)^{1/2} \right]$$
$$- \frac{\mu^2(\sigma - s)}{(\mu^2(\sigma - s)^2 + u^2)^{1/2}} = 0.$$

Integration leads to

$$-\frac{1}{2} (u')^2 \Big|_0^l - (\mu^2(\sigma - s)^2 + u^2)^{1/2} \Big|_0^l - \mu\, a = 0,$$

or equivalently

$$- (\mu^2(\sigma - l)^2 + u^2(l))^{1/2} + (\mu^2\sigma^2 + u^2(0))^{1/2} = \mu a.$$

For (3) an analogous computation gives $u^2(0) = u^2(l)$. (3) is an ordinary differential equation, the dependence with respect to u_0 being Lipschitz because $\kappa_0 > 0$. Therefore u_0 has only simple zeroes. u_0 must have at least one zero because it is an eigenvector for the eigenvalue zero of the Sturm-Liouville operator

$$-\frac{d^2}{ds^2} - \frac{1}{(\kappa_0{}^2 + u_0{}^2)^{1/2}},\ h'(0) = h'(l) = 0.$$

But zero is certainly not the lowest eigenvlaue of this operator because obviously

$$-\frac{d^2}{ds^2} - \frac{1}{(\kappa_0{}^2 + u_0{}^2)^{1/2}} < -\frac{d^2}{ds^2}\,.$$

This implies that u_0 changes sign at least once. The equation is autonomous and of second order so u_0 must be symmetric or antisymmetric around the point $s = l/2$, that is $u_0(s) = \pm\, u_0(l - s)$. This is because

$u_0^2(0) = u_0^2(l)$ or $u_0(0) = \pm u_0(l)$; but the solution u_0 with initial conditions $u_0(0)$, $u_0'(0) = 0$ solved in the forward direction must then be identical with the solution $\pm u_0$ with end conditions $u_0(l) = \pm u_0(0)$, $u_0'(l) = 0$ solved in the backward direction.

This fact, together with the existence of at least one zero, allows us to describe all solutions of (3) in the following way: Let $s_i^{(n)} = il/2n$, $i = 0, 1, 2, \cdots, 2n$. Any nonnegative solution of

$$- v_0'' - \frac{v_0}{(\kappa_0{}^2 + v_0{}^2)^{1/2}} = 0,\ s \in [0, s_1^{(n)}],$$
$$v_0'(0) = v_0(s_1^{(n)}) = 0,$$
$$\int_0^{s_1^{(n)}} \frac{\kappa_0}{(\kappa_0{}^2 + v_0{}^2)^{1/2}}\, ds = \frac{a}{2n}$$

gives rise to a solution of (3) with n zeroes at the points $s_{1+2i}^{(n)}$, $i = 0, \cdots, n-1$ by the definition

$$u_0(s_i^{(n)} + s) = (-1)^{[1/2]} \cdot \begin{cases} v_0(s)\ , & i \text{ even} \\ -v_0(l/2n - s) & i \text{ odd} \end{cases},\quad s \in [0, l/2n].$$

Any solution u_0 of (3) with n zeroes has these zeroes at the points $s_{1+2i}^{(n)}$, and u_0 restricted to $[0, s_1^{(n)}]$ is a solution of the above-mentioned equations.

Thus we have reduced the problem of solving (3) to that of finding all nonnegative solutions of

$$- v_0'' - \frac{v_0}{(\kappa_0{}^2 + v_0{}^2)^{1/2}} = 0,$$
$$v_0'(0) = v_0(l/2n) = 0,$$
$$\int_0^{l/2n} \frac{\kappa_0}{(\kappa_0{}^2 + v_0{}^2)^{1/2}}\, ds = \frac{a}{2n},$$

with $\kappa_0 > 0$.

By the scaling $u_0(s) = \kappa_0^{-1} v_0(ls/2n)$, $s \in [0, 1]$, the system of equations changes into

$$- u_0'' - \left(\frac{l}{2(\kappa_0)^{1/2} n} \right)^2 \frac{u_0}{(1 + u_0{}^2)^{1/2}} = 0,$$
$$(4) \qquad u_0'(0) = u_0(1) = 0,$$
$$\int_0^1 \frac{1}{(1 + u_0{}^2)^{1/2}} = \frac{a}{l}.$$

Our first aim is to establish the uniqueness of the solutions of (4). We start by studying the equation:

$$- u_0'' - \gamma \frac{u_0}{(1 + u_0^2)^{1/2}} = 0, \qquad \gamma > 0,$$

$$u_0'(0) = u_0(1) = 0.$$

$u_0 \equiv 0$ is a solution for any $\gamma > 0$. So, this is a typical bifurcation problem.

Using arguments similar to those in [2] we can show that there is a unique branch of nonnegative solutions u_0 parametrized by γ bifurcating from zero and defined on the interval $[\pi^2/4, \infty)$. All nonnegative solutions belong to that branch.

To prove these contentions we define on $L^2_{2,0}(0, 1) \times \mathbf{R}$ a function $F \cdot L^2_{2,0}(0, 1)$ is the space of functions which together with their generalized derivatives up to second order, are square integrable and satisfy the prescribed boundary conditions. $F : L^2_{2,0}(0, 1) \times \mathbf{R} \to L^2(0, 1)$ is given by

$$F(u, \gamma) = - u'' - \gamma \frac{u}{(1 + u^2)^{1/2}}.$$

Suppose (u_0, γ_0) is a solution of $F(u, \gamma) = 0$ with $u_0 \geqq 0$, $u_0 \neq 0$ and $\gamma_0 > 0$. The Frechet derivative of F with respect to u is given by

$$F_u'(u_0, \gamma_0)h = - h'' - \gamma_0 \frac{h}{(1 + u_0^2)^{1/2}} + \gamma_0 \frac{u_0^2 h}{(1 + u_0^2)^{3/2}}.$$

The operator

$$L(u_0, \gamma_0) = - \frac{d^2}{ds^2} - \frac{\gamma_0}{(1 + u_0^2)^{1/2}} + \frac{\gamma_0 u_0^2}{(1 + u_0^2)^{3/2}}$$

is boundedly invertible as an operator from $L^2_{2,0}(0, 1)$ to $L^2(0, 1)$. This we show by proving that the spectrum of this selfadjoint operator does not contain zero. This in turn is demonstrated by a perturbation argument. The operator $- (d^2/ds^2) - (\gamma_0/(1 + u_0^2)^{1/2})$ has zero as an eigenvalue with the nontrivial eigenfunction $u_0 \geqq 0$. The operator is of Sturm-Liouville type. Therefore $u_0 \geqq 0$ implies that 0 is the first eigenvalue, otherwise u_0 would change sign. $L(u_0, \gamma_0)$ comes from that operator. Thus, $L(u_0, \gamma_0)$ is also selfadjoint and has a discrete spectrum which is shifted to the right by a positive amount.

The implicit function theorem establishes the existence of a local branch of solutions around (u_0, γ_0), because F_u' is invertible at that point. Let K be inverse of the operator $- d^2/ds^2$ with the prescribed boundary conditions. The equation $F(u, \gamma) = 0$ can then be written as

$$u = K\frac{\gamma u}{(1 + u^2)^{1/2}}.$$

This implies that there exists a constant C such that $\|u\| \, L_2{}^2 \leqq C \cdot \gamma$ for all solutions (u, γ). But K is a compact operator. It follows that any bounded infinite set of solutions has a solution as accumulation point. Differentiation of $F(u(\gamma), \gamma) = 0$ leads to

$$-\frac{du}{d\gamma} - \frac{u}{(1 + u^2)^{1/2}} - \frac{\gamma \dfrac{du}{d\gamma}}{(1 + u^2)^{1/2}} + \frac{\gamma u_2 \dfrac{du}{d\gamma}}{(1 + u^2)^{3/2}} = 0.$$

This implies that

$$\frac{du}{d\gamma} = L^{-1}(u, \gamma) \frac{u}{(1 + u^2)^{1/2}}.$$

$L^{-1}(u, \gamma)$ maps nontrivial nonnegative functions into positive functions. It follows that $(d/d\gamma) \, \|u\|^2_{L^2} = 2(u, du/d\gamma) = 2(u, L^{-1}(u, \gamma)u/(1 + u^2)^{1/2}$ is positive. This, together with the compactness argument given before, shows that the branch around (u_0, γ_0) can be followed back until it reaches $u = 0$. But that is only possible at the lowest eigenvalue of the linearized problem which is $\pi^2/4$. Since bifurcation is unique, all positive solutions must lie on this branch of solutions, which can be continued indefinitely for $\gamma \to \infty$. Otherwise there would be a solution (u_0, γ_0) with a maximal γ_0, which is impossible because the extension argument above would work again.

We go on to prove that (4) has at most one positive solution. The idea is to put the unique branch $u(\gamma)$ into the integral and to see if it takes the prescribed value for some γ in $[\pi^2/4, \infty)$. We differentiate the integral with respect to γ and get

$$- \int_0^1 \frac{u\dfrac{du}{d\gamma}}{(1 + u^2)^{3/2}} \, ds =$$

$$- \int_0^1 \frac{u}{(1 + u^2)^{3/2}} \, L^{-1}(u, \gamma) \, \frac{u}{(1 + u^2)^{1/2}} \, ds < 0.$$

This proves that there is at most one γ for which the integral takes the value a/l.

What is left open is the question as to whether there is a solution at all. We resolve it by Schauder's fixed point theorem. We transform (4) into an equivalent form. For this purpose we introduce K, the inverse of $-\, d^2/ds^2$ with the prescribed boundary conditions. Then (4) can be written in the form

$$
(5) \qquad \begin{aligned}
u_0 &= K\frac{\gamma u_0}{(1+u_0{}^2)^{1/2}},\\
\kappa_0 &= \kappa_0 + \left(\int_0^1 \frac{1}{(1+u_0{}^2)^{1/2}}\,ds - \frac{a}{l}\right)^2.
\end{aligned}
$$

Let us write $K^+ = \{u \in C^0 \mid u \geqq 0\}$, and $\mathbf{R}^+ = \{a \mid a \geqq 0\}$. The closed convex cone $K^+ \times \mathbf{R}^+$ in $C^0 \times \mathbf{R}$ is mapped into itself by the mappings defined by the right hand side of (5). This mapping is compact. By the Schauder fixed point theorem it has a fixed point in $K^+ \times \mathbf{R}^+$.

We conclude this paragraph with a summary of what we have proved for the asymptotic equation. The system of equations

$$
\begin{aligned}
&-u_0'' - \frac{u_0}{(\kappa_0{}^2 + u_0{}^2)^{1/2}} = 0,\\
&u_0'(0) = u_0'(l) = 0,\\
&\int_0^l \frac{\kappa_0}{(\kappa_0 + u_0{}^2)^{1/2}}\,ds = a,
\end{aligned}
$$

has, for any integer $n \geqq 1$, exactly one solution u_0 with n simple zeroes located at $s^{(n)}_{1+2i} = ((1+2i)/2n)\cdot l\ (i = 0, 1, \cdots, n-1)$ and $\kappa_0 < (l/n\pi)^2$.

3. **The general equation.** If we replace $\mu\sigma$ by κ in the system of equations (2) then we get

$$
\begin{aligned}
&-u'' - \frac{u}{((\kappa - \mu s)^2 + u^2)^{1/2}} = 0,\\
&u'(0) = u'(l) = 0,\\
&-\int_0^l \frac{(\kappa - \mu s)}{((\kappa - \mu s)^2 + u^2)^{1/2}}\,ds + a = 0.
\end{aligned}
$$

We solve the first equation for u depending upon κ and μ. Define

$$
\Gamma_\kappa = \{(u, \kappa, \mu) \mid u \in L^2_{2,0}(0, l), \mu, \kappa \in \mathbf{R}, (\mu, \kappa) \neq 0\}
$$

and

$$
G : \Gamma_\kappa \to L^2(0, l) \text{ by}
$$

$$
G(u, \kappa, \mu) = \frac{u}{((\kappa - \mu s)^2 + u^2)^{1/2}}.
$$

Let

$$\Sigma_\kappa = \left\{ (u, \kappa, \mu) \in \Gamma_\kappa \mid \gamma \neq 0, \frac{\kappa}{\mu} \in [0, l], u\left(\frac{\kappa}{\mu}\right) = 0 \right\}.$$

Then Σ_κ is a closed set in Γ_κ.

For points in Σ_κ, G is not well behaved. But on $\Gamma_\kappa \setminus \Sigma_\kappa$, G is Frechet-differentiable. We have proved that no solution of (6) lies in Σ_κ. Let P be the orthogonal projection on the constant functions in $L^2(0, l)$ and $Q = 1 - P$. The operator $-d^2/ds^2$ with boundary conditions $u'(0) = u'(l) = 0$ has an inverse on $QL^2(0, l)$ which we denote by K. Then the equation

$$G(u, \kappa, \mu) = -u''$$

is equivalent to $KQG(u, \kappa, \mu) + Pu = u$. Let us denote this map by H. We have

$$H : \Gamma_\kappa \rightarrow L^2_{2,0}(0, l).$$

A little thought shows that given any $\epsilon > 0$, H is a compact map for all κ, γ with $|\kappa| \geqq \epsilon > 0$. On sets M which have a positive distance from Σ_κ, this is obvious because in $QL^2(0, l)$ we have

$$Hu = KQ \frac{u}{((\kappa - \mu s)^2 + u^2)^{1/2}}.$$

Since $(\kappa - \mu s)^2 + u^2(s) \neq 0$ for all $s \in [0, l]$ and $u \in L^2_{2,0}(0, l)$ we have $u \in C^{1+(1/2)}(0, l)$, and there must be a $\delta > 0$ such that $(\kappa - \mu s)^2 + u^2(s) \geqq \delta$ for all $s \in [0, l]$ and all $u \in M$. It is then straightforward to prove that

$$(u, \kappa, \mu) \rightarrow \frac{u}{((\kappa - \mu s)^2 + u^2)^{1/2}}$$

is a bounded map from $M \subset \Gamma_\kappa$ to $L^2_{2,0}(0, l)$. This proves compactness of H on M.

If $M \cap \Sigma_\kappa \neq \phi$, things are more delicate. If $u \in \Sigma_\kappa$ then $\kappa/\mu = \sigma$ is in $[0, l]$ and $u(\sigma) = 0$. But $u \in C^{1+(1/2)}(0, l)$, so there is a finite Taylor-development $u(s) = u'(\sigma)(s - \sigma) + v(s - \sigma)$ with $v(s - \sigma) = o(|s - \sigma|)$ and $v \in C^{1+(1/2)}(0, l)$. This implies that $v(s - \sigma)/(s - \sigma)$ is in $C^{1/2}(0, l)$. Therefore we get

$$\frac{u}{((\kappa - \mu s)^2 + u^2(s))^{1/2}} = \frac{u'(\sigma)(s - \sigma) + v(s - \sigma)}{(\mu^2(\sigma - s)^2 + (u'(\sigma)(s - \sigma) + v(s - \sigma))^2)^{1/2}}$$

$$= \frac{u'(\sigma) + \dfrac{v(s - \sigma)}{(s - \sigma)}}{\left(\mu^2 + \left(u'(\sigma) + \dfrac{v(s - \sigma)}{s - \sigma} \right)^{2'} \right)^{1/2}}$$

If $u \notin \Sigma_\kappa$ then the same representation is valid, but the function $v(s - \sigma)/(s - \sigma)$ is better than $C^{1/2}(0, l)$.

Since $\mu \neq 0$ this function is of the same class as $v(s - \sigma)/(s - \sigma)$. Furthermore $(u, \kappa, \mu) \to u/((\kappa - \mu s)^2 + u^2)^{1/2}$ is bounded from Γ_κ to $C^{1/2}(0, l)$. But $C^{1/2}(0, l)$ is compactly imbedded in $C^0(0, l)$ and that is continuously imbedded in $L^2(0, l)$. Therefore $KQ\ u/((\kappa - \mu s)^2 + u^2)^{1/2}$ defines a compact map. In both cases the second part $P\ u/((\kappa - \mu s)^2 + u^2)^{1/2}$ being an integral over a function absolutely bounded by 1, is certainly compact.

We have shown earlier that the set of fixed points of H is bounded independently of μ and κ. Therefore this set is compact for $\kappa \geqq \epsilon > 0$. H is also a continuous map on Γ_κ. For points not in Σ this is obvious. So let $u(\sigma) = 0$. If u_1 is close to u in $L_2{}^2$-norm, it is so in C^1-norm. We have to show that

$$KQ \left[\frac{u}{(\kappa - \mu s)^2 + u^2)^{1/2}} - \frac{u_1}{(\kappa_1 - \mu_1 s)^2 + u_1{}^2)^{1/2}} \right]$$

becomes arbitrarily small in $L_2{}^2$ when $u_1 \to u$ in $L_2{}^2$ and $(\mu_1, \kappa_1) \to (\mu, \kappa)$. K is an integral operator with bounded kernel. If we split the interval of integration into a small interval around σ and the rest, and if μ_1 and κ_1 are sufficently near to μ and κ, the term involving the nonlinear terms will be small in C^0-norm outside the small interval, so this term is all right. On the small interval that term is absolutely bounded by an expression involving the norms of u and u_1. Therefore the integration leads to an $L_2{}^2$ function of arbitarily small $L_2{}^2$-norm as the size of the interval is decreased. This proves continuity.

These properties of H show that a sequence (u_i, κ_i, μ_i) of solutions of $H(u, \kappa, \mu) = 0$ with $\kappa_i \to \kappa_0 > 0$ and $\mu_i \to \mu_0$ has a solution (u_0, κ_0, μ_0) as accumulation point.

We now return to the equivalent equation $G(u, \kappa, \mu) = 0$. We have constructed a solution $(u_0, \kappa_0, 0)$ of (6) where u_0 has exactly one zero. Let $(\bar{u}, \bar{\kappa}, \bar{\mu})$ be any solution of (6) such that $\bar{u}$ has exactly one zero, and let $\bar{\kappa} > \bar{\mu}l/2$. Since $(\bar{u}, \bar{\kappa}, \bar{\mu}) \notin \Sigma_\kappa$, G is differentiable at that solution and we get

$$G_u'(\bar{u}, \bar{\kappa}, \bar{\mu})h = -h'' - \frac{h}{((\bar{\kappa} - \bar{\mu}s)^2 + \bar{u}^2)^{1/2}}$$
$$+ \frac{\bar{u}^2 h}{((\bar{\kappa} - \bar{\mu}s)^2 - \bar{u}^2)^{3/2}}$$
$$= -h'' - \frac{(\bar{\kappa} - \bar{\mu}s)^2}{((\bar{\kappa} - \bar{\mu}s)^2 - \bar{u}^2)^{3/2}} h$$
$$= L(\bar{u}, \bar{\kappa}, \bar{\mu}) \cdot h + \frac{\bar{u}^2 h}{((\bar{\kappa} - \bar{\mu}s)^2 + \bar{u}^2)^{3/2}} h.$$

The linear operator L has $\bar{u}$ as an eigenvector for the eigenvalue 0 because $L(\bar{u}, \bar{\kappa}, \bar{\mu}) \cdot \bar{u} = G(\bar{u}, \bar{\kappa}, \bar{\mu}) = 0 \cdot \bar{u}$. As $\bar{u}$ has exactly one zero it must belong to the second eigenvalue. L is a selfadjointed operator with discrete spectrum and the multiplication operator $\bar{u}^2/((\bar{\kappa} - \bar{\mu}s)^2 + \bar{u}^2)^{3/2}$ is a relatively compact symmetric perturbation of it, so the sum of both is selfadjoint with discrete spectrum. Since the multiplication operator is positive, the spectrum of L gets shifted to the right. The worst to happen is that 0 becomes the first eigenvalue of the new operator. But that is impossible because the eigenfunction would be of constant sign, so that

$$-h'' = \frac{(\bar{\kappa} - \bar{\mu}s)^2}{((\bar{\kappa} - \bar{\mu}s)^2 + \bar{u}^2)^{3/2}}$$

would imply that h is concave or convex with $h'(0) = h'(l) = 0$. This forces h to be identically zero.

But if 0 is not an eigenvalue for that operator, then $G_u'(\bar{u}, \bar{\kappa}, \bar{\mu})$ is invertible and by the implicit function theorem there is a neighbourhood U of $(\bar{\kappa}, \bar{\mu})$ and a function $g : U \longrightarrow L^2_{2,0}(0, l)$ such that $G(g(\kappa, \mu), \kappa, \mu)$ for all $(\kappa, \mu) \in U$.

The only problem left is to show that for fixed μ, we can find a κ such that the last equation of (6) is satisfied. We attack that question by showing that the function

$$J(u, \kappa, \mu) := \int_0^l \frac{(\kappa - \mu s)}{((\kappa - \mu s)^2 + u^2)^{1/2}} ds$$

takes all values between 0 and l as κ runs through its maximal interval of definition. The first step is, of course, to prove continuity of J, which is done in the same way as for H. The next step is to show that we can extend the local branch $g(\kappa, u)$ for fixed μ and variable κ. Let us first ask how far κ can be decreased. $(\bar{u}, \bar{\kappa}, \bar{\mu})$ is the solution we started with. From what we have shown before, we know that $(\bar{u}, \bar{\kappa}, \bar{\mu}) \notin \Sigma_\kappa$. Since

$J(u, \kappa, \mu) = a$ we can, for a given $\alpha > 0$, choose a neighborhood V of $\bar{\mu}$ so that for all $\mu \in V$ and $(\mu, \kappa) \in U$ we shall have $J(g(\kappa, \mu), \kappa, \mu) \geqq \alpha$. It follows that $(g(\kappa, \mu), \kappa, \mu) \notin \Sigma_\kappa$ for these (κ, μ). We now fix a $\mu \in V$, with $(\bar{\kappa}, \mu) \in U$ and extend $g(\kappa, \mu)$ for fixed μ and $\kappa < \bar{\kappa}$ as far as possible. The extension continues as long as for the extended function $\tilde{g}(\kappa, \mu)$ an inequality $J(\tilde{g}(\kappa, \mu), \kappa, \mu) > 0$ is valid. The continuation ends when J becomes zero and Σ_κ is reached. For the limiting value $\tilde{\kappa}$ of κ, we have $\tilde{\kappa} = l\mu/2$. If J becomes zero but Σ_κ were not reached, then the continuation would continue and we could decrease κ to zero. This is impossible because there are no solutions for $\kappa = 0$, i.e., $\sigma = 0$.

We now try to extend $g(\kappa, \mu)$ for $\kappa > \tilde{\kappa}$. If we could extend it to $\kappa = \infty$ we would be done. This is because $J(\tilde{g}(\kappa, \mu), \kappa, \mu) \to l$ when $\kappa \to \infty$ irrespective of the behaviour of $\tilde{g}(\kappa, \mu)$, since we know already that the latter function is uniformly bounded in L_2^2. So we assume that the continuation business breaks down before we reach $\kappa = \infty$. Thus there exists a finite $\hat{\kappa} > \bar{\kappa}$ such that $\tilde{g}(\kappa, \mu)$ cannot be extended beyond $\hat{\kappa}$. $(\tilde{g}(\kappa, \mu), \kappa, \mu)$ has an accumulation point, $(\hat{u}, \hat{\kappa}, \mu)$ which is a solution of $G(u, \kappa, \mu) = 0$. In order that continuation does not work at this point, it is necessary that $(\hat{u}, \hat{\kappa}, \mu) \in \Sigma_\kappa$, which implies that $J(\hat{u}, \hat{\kappa}, \mu) = 0$. But we have $\bar{\kappa} > l\bar{\mu}/2$, and supposing that V was chosen sufficiently small, we have $\bar{\kappa} > l\mu/2$ for all $\mu \in V$.
Therefore along the whole extended branch $\tilde{g}(\kappa, \mu)$, we have $\hat{\kappa} \geqq \kappa \geqq \bar{\kappa} > l\mu/2$, which implies that $J(\hat{u}, \hat{\kappa}, \mu) \neq 0$. This contradiction shows that the extension cannot break down at $\kappa = \hat{\kappa} < \infty$.

This proves that we can extend $g(\kappa, \mu)$ in a unique fashion to a region $A = \{(\kappa, \mu) \mid \mu \in [0, \mu^*\}, \kappa \geqq l\mu/2\}$, and that there is a connected $A^* \subset A$ whose projection onto the μ-axis is given by $[0, \mu^{**}]$, such that $J(\tilde{g}(\kappa, \mu), \kappa, \mu) = a$ for $(\kappa, \mu) \in A^*$. We can even prove that any solution with exactly one zero belongs to that sheet $\tilde{g}(\kappa, \mu)$. This follows from the fact that on a neighbourhood of the line $\mu = 0$ the sheet is unique. For $\mu = 0$ we know all the solutions of $G(u, \kappa, 0) = 0$; this is just part of the asymptotic equation of the chain. Given any $\kappa > 0$, the solutions of $G(u, \kappa, \mu) = 0$ for $\mu \geqq 0$ are unique in some neighbourhood $[0, \epsilon(\kappa)]$ by the implicit function theorem. But since any sheet $\tilde{g}(\kappa, \mu)$ extends to $\mu = 0$, they all coincide there. But if they coincide there, they coincide everywhere by the unique continuation process.

One can even prove that the solution of (6) is unique in a one-sided neighbourhood of $\mu = 0$. This is done by showing that the system of equations

$$G(u, \kappa, \mu) = 0,$$

$$J(u, \kappa, \mu) = a,$$

with solution $(u_0, \kappa_0, 0)$ has an invertible Frechet derivative with respect to (u, κ) at $(u_0, \kappa_0, 0)$, so that a unique solution $(u(\mu), \kappa(\mu), \mu)$ exists for a neighbourhood of $\mu = 0$.

To prove uniqueness globally seems to be an extremely difficult job which we have not been able to accomplish. As for the asymptotic equation one would have to prove that $J(\tilde{g}(\kappa, \mu), \kappa, \mu)$ is strictly monotone in κ. Differentiation of J with respect to κ leads to expressions which are not manageable because for $\mu \neq 0$ the differential equation is no longer independent of s; this destroys all those arguments working so nicely for the asymptotic equation.

Finally, we ask whether $\mu^{**} = \infty$. From the physics involved we expect that μ^{**} is finite because at low ω (that is big μ) the centrifugal force is not strong enough to pull the chain out of the vertical.

Let (u_i, κ_i, μ_i) be a sequence of solutions with $u_i \to 0$, $\mu_i \to \mu^{**}$. Then $\sigma_i = \kappa_i/\mu_i$ tends to $(l + a)/2$ as we have shown earlier. Let x_i be the function corresponding to u_i. Then we get

$$\int_0^l x_i' \frac{(\sigma_i - s)}{\sqrt{1 - (x_i')^2}} x_i' ds = \frac{1}{\mu} \int_0^l x_i^2 \, ds.$$

From $\tau/g = (\sigma - s)/(1 - (x')^2)^{1/2} = (1/\mu)\, (\mu^2(\sigma - s)^2 + u^2)^{1/2} \geqq |\sigma - s|$ we conclude that

$$\frac{1}{\mu_i} = \frac{\int_0^l x_i' \frac{\tau}{g} x_i' \, ds}{\int_0^l x_i^2 \, ds} \geqq \frac{\int_0^l x_i' \, |\sigma_i - s| \, x_i' \, ds}{\int_0^l x_i^2 \, ds}.$$

This leads to

$$\frac{1}{\mu_i} \geqq \frac{\int_0^l x_i' \, |\sigma_i - s| \, x_i' \, ds}{\int_0^l x_i^2 \, ds} = \alpha_i.$$

But

$$\frac{\int_0^l x_i' \left| \frac{l + a}{2} - s \right| x_i' \, ds}{\int_0^l x_i^2 \, ds} \geqq \epsilon > 0,$$

so that for i sufficiently big $\alpha_i \geqq \epsilon/2$. It follows that $\mu^{**} \leqq 2/\epsilon$.

The following proposition gives an account of what we have proved:

PROPOSITION *Equation* (1) *of the chain admits at least one solution* x *with no node in the interior of the parameter interval* $(0, l)$ *for any* $\omega \in [\omega^{**}, \infty]$, $\omega^{**} > 0$. *As* $\omega \to \infty$ *the corresponding solutions tend to the uniquely determined solution* x_0 *corresponding to the solution* (u_0, κ_0) *of the asymptotic equation* (3), u_0 *having exactly one zero.*

Uniqueness remains an open problem. It is also to be conjectured that there are branches of solutions with one, two, etc. nodes starting at values ω_i^{**} which form an increasing sequence. For these solutions the continuation process via an application of the implicit function theorem breaks down. It could be replaced by topological degree but the problem is to determine the index of the higher solutions of (3).

That there do exist solutions with nodes is obvious from Kolodner's result. Just take one of Kolodner's solutions with many nodes and fix the chain at the lowest node to the axis, cutting off the remainder of the chain with the free end. This is then a solution of (1) with nodes. Nevertheless this does not furnish the existence proof we are looking for, because the length l and the distance a are fixed from the outset, while the argument above just gives some length l and some distance a.

REFERENCES

1. I. I. Kolodner, *Heavy rotating chain-A nonlinear eigenvalue problem*, Commun. Pure Appl. Math 8 (1955), 395–408.

2. M. Reeken, *General theorem on bifurcation and its application to the Hartree equation of the helium atom*, Journal of Math. Physics 11 (1970), 8.

3. ———, *The equation of motion of a chain*, Math. Z. 155 (1977), 219–237.

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