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LYAPUNOV-TYPE FUNCTIONS FOR NTH ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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1. **Introduction.** The purpose of this paper is to introduce and utilize what we shall refer to as *Lyapunov-type functions* for the nth order real linear ordinary differential equation

$$\sum_{i=0}^{n} p_i(t)y^{(i)} = 0, \qquad t \in J \tag{1.1}$$

where J is the half-open interval $[\alpha, \infty)$, $p_i \in \mathscr{C}(J \to \mathbf{R})$ $(i = 0, \cdots, n)$ and $p_n(t) \neq 0$ for $t \in J$. We introduce the concept in § 1 and develop a *systematic procedure* for constructing Lyapunov-type functions in § 2. In § 3, we demonstrate how Lyapunov-type functions can be utilized to find coefficient criteria for (1.1) which guarantee that certain two-point boundary value problems are uniquely solvable. In a later paper, we apply Lyapunov-type functions to obtain oscillation results for higher order linear differential equations.

DEFINITIONS. Suppose the functions $a_{ij}(0 \leqq i, j \leqq n-1)$ and $b_i(0 \leqq i \leqq n-1)$ from J to $\mathbf{R}$ are such that

$$\frac{d}{dt}\left[\sum_{i,j=0}^{n-1} a_{ij}(t)y^{(i)}(t)y^{(j)}(t)\right] = \sum_{i=0}^{n-1} b_i(t)(y^{(i)}(t))^2 \tag{1.2}$$

for all $t \in J$ whenever y is a solution of (1.1). Then the function $\phi : \mathscr{C}^n(J \to \mathbf{R}) \to \mathscr{C}(J \to \mathbf{R})$ defined by

$$\phi(f) = \sum_{i,j=0}^{n-1} a_{ij}f^{(i)}f^{(j)} \tag{1.3}$$

is called a Lyapunov-type function for (1.1).

If $f \in C^n(J \to \mathbf{R})$ and $f_i \equiv f^{(i)}(i = 0, \cdots, n-1)$, then ϕ as defined in (1.3) can be regarded as a function from $J \times \mathbf{R}^n \to \mathbf{R}$ given by

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$$\phi(t, f_0, \cdots, f_{n-1}) = \sum_{i,j=0}^{n-1} a_{ij}(t) f_i f_j.$$

If the b_i's in (1.2) are either all nonnegative or all nonpositive on J and solutions of (1.1) are regarded as solutions of an appropriate first order system, then ϕ is *monotone along solution trajectories*. For this reason, we use the Lyapunov-terminology; of course, ϕ is not a Lyapunov function in the usual sense (cf. [1]) because $\phi(t, f_0, \cdots, f_{n-1})$ may change sign.

The results of § 3 are related to those obtained by Levin [3], Nehari [5], Hunt [2], and Ridenhour [6]; in particular, some special cases of those results are also special cases of the results obtained here.

2. **Construction of Lyapunov-type Functions.** We first illustrate a way in which a Lyapunov-type function can be found for the fourth order equation

$$y^{(4)} + p(t)y'' + q(t)y = 0. \tag{2.1}$$

We can multiply (2.1) by y and integrate by parts to obtain

$$yy''' - y'y'' + pyy' - \frac{1}{2}p'y^2$$
$$= \int \left[\left(-\frac{1}{2}p'' - q\right)y^2 + p(y')^2 - (y'')^2\right].$$

If we let

$$\phi(y) = yy''' - y'y'' + pyy' - \frac{1}{2}p'y^2, \tag{2.2}$$

then, for solutions y of (2.1), $(\phi(y))'$ is given by

$$(\phi(y))' = \left(-\frac{1}{2}p'' - q\right)y^2 + p(y')^2 - (y'')^2.$$

Hence, if $\phi(y)$ is as in (2.2), ϕ qualifies as a Lyapunov-type function for (2.1) (with the assumption that $p \in \mathscr{C}^1(J \to \mathbf{R})$).

If we had multiplied by another derivative of y, say $y^{(i)}$ where $1 \leqq i \leqq 4$, we could have integrated by parts in the same manner to obtain a different Lyapunov-type function. In general, we will denote by ϕ_i the Lyapunov-type funtion obtained by multiplying by $y^{(i)}$ and integrating by parts in the above way.

As pointed out on pp. 152–153 of [7], the technique of multiplying an equation by y and integrating by parts in the above way is an old idea which goes back at least as far as the paper of Mammana [4] in 1931.

Before carrying out the general derivation of ϕ_i, we establish some notation. As usual, $\binom{h}{l}$ denotes a binomial coefficient and $[|x|]$ denotes the greatest integer less than or equal to x. If i and j are integers with $0 \leqq i \leqq j-1$ and p and y are functions in $\mathscr{C}^j(J \to \mathbf{R})$, define $f_{j,i}(y)$, $g_j(p, y)$, $F_j(p, y)$, and $I_j(p, y)$ by

$$f_{j,i}(y) \equiv \sum_{k=0}^{[|(j-i-1)/2|]} \frac{(-1)^{i+k}}{1+[|(k+1)/(j-i-k)|]} \binom{i+k}{i} y^{(k)} y^{(j-i-k-1)}, \tag{2.3}$$

$$\begin{aligned} g_j(p, y) \equiv \frac{1}{2}\Bigg[(-1)^j p^{(j)} y^2 + \sum_{k=1}^{[|j/2|]} (-1)^{j+k} \Bigg\{ \binom{j-k}{k} \\ + \binom{j-k-1}{k-1} \Bigg\} p^{(j-2k)} (y^{(k)})^2 \Bigg], \end{aligned} \tag{2.4}$$

$$F_j(p, y) \equiv \sum_{k=0}^{j-1} p^{(k)} f_{j,k}(y), \tag{2.5}$$

and

$$I_j(p, y) \equiv \int p y y^{(j)}. \tag{2.6}$$

We also define $g_0(p, y)$, $F_0(p, y)$ and $I_0(p, y)$ by

$$g_0(p, y) \equiv p y^2, \; F_0(p, y) \equiv 0, \text{ and } I_0(p, y) \equiv \int p y^2. \tag{2.7}$$

Although $I_j(p, y)$, being an antiderivative, is unique only up to a constant, this will create no difficulty.

LEMMA 2.1. *If $p, y \in \mathscr{C}^j(J \to \mathbf{R})$, then*

$$I_j(p, y) = F_j(p, y) + \int g_j(p, y), \; j = 0, 1, 2 \cdots. \tag{2.8}$$

PROOF. The proof is by induction. From (2.7), we see that (2.8) is true when $j = 0$. Integration by parts gives

$$I_1(p, y) = \int p y y' = \frac{1}{2} p y^2 - \int \frac{1}{2} p' y^2$$

which equals $F_1(p, y) + \int g_1(p, y)$.

Assuming that (2.8) is true for $j = 0, \cdots, m$, we need to prove (2.8) is true when $j = m + 1$. On integrating by parts and using the inductive assumption, we get

$$\begin{aligned} I_{m+1}(p, y) &= pyy^{(m)} - I_{m-1}(p, y') - I_m(p', y) \\ &= pyy^{(m)} - F_{m-1}(p, y') - \int g_{m-1}(p, y') \\ &\quad - F_m(p', y) - \int g_m(p', y). \end{aligned}$$

The proof is completed by showing that

$$-g_{m-1}(p, y') - g_m(p', y) = g_{m+1}(p, y) \tag{2.9}$$

and

$$pyy^{(m)} - F_{m-1}(p, y') - F_m(p', y) = F_{m+1}(p, y). \tag{2.10}$$

In proving (2.9), it is best to consider cases in which m is even or odd. Also, in proving (2.10), one uses that $[|k/(m - i - k)|] = [|(k + 1)/(m - i - k + 1)|]$ when $1 \leqq k \leqq [|(m - i)/2|]$. We leave the details of these proofs to the reader.

Simple iterative devices may be used to calculate the magnitude of the numerical coefficients in (2.3) and (2.4). To be specific, the magnitude of the coefficient of $y^{(k)}y^{(j-i-k-1)}$ in the expression for $f_{j,i}(y)$ is the jth number down the ith diagonal of Pascal's triangle unless $k = j - i - k - 1$ in which case that number is multiplied by 1/2. For example, using this rule, we get

$$\begin{aligned} F_6(p, y) = {} & p[yy^{(5)} - y'y^{(4)} + y''y'''] \\ & + p' \left[-yy^{(4)} + 2y'y''' - \frac{3}{2}(y'')^2 \right] \\ & + p''[yy''' - 3y'y''] + p'''[-yy'' + 2(y')^2] \\ & + p^{(4)}[yy'] + p^{(5)} \left[-\frac{1}{2} y^2 \right]. \end{aligned}$$

To find $g_j(p, y)$, we construct an array of numbers as follows (see (2.11) below): (1) The number 1/2 is put in the first column of each row. (2) An element may be found by adding the element two rows up and one to the left to the element directly above (when such elements exist). (3) The number 1 is added at the end of even numbered rows after all possible elements have been inserted using rules (1) and (2).

To obtain the coefficients for $g_j(p, y)$, we use the jth row of (2.11). For example,

$$
\begin{array}{llll}
\frac{1}{2} & & & \\
\frac{1}{2} & 1 & & \\
\frac{1}{2} & \frac{3}{2} & & \\
\frac{1}{2} & 2 & 1 & \\
\frac{1}{2} & \frac{5}{2} & \frac{5}{2} & \\
\frac{1}{2} & 3 & \frac{9}{2} & 1 \\
\vdots & \vdots & \vdots & \vdots
\end{array}
\tag{2.11}
$$

$$g_5(p, y) = \frac{1}{2} p^{(5)}y^2 + \frac{5}{2} p'''(y')^2 - \frac{5}{2}p'(y'')^2$$

while

$$g_6(p, y) = \frac{1}{2} p^{(6)}y^2 - 3p^{(4)}(y')^2 + \frac{9}{2}p''(y'')^2 - p(y''')^2.$$

Although we will develop general formulas for Lyapunov-type functions, it is easier when given a specific equation to find a Lyapunov-type function using the above devices than it is to consult the general formula.

Now suppose i is an integer with $0 \leqq i \leqq n$. If the coefficients $p_0, \cdots, p_n$ in (1.1) are such that $p_j \in \mathscr{C}^{|i-j|}(J \to \mathbf{R})$, define $\phi_i : \mathscr{C}^n(J \to \mathbf{R}) \to \mathscr{C}(J \to \mathbf{R})$ and $P_{i,k} : J \to \mathbf{R}$, $k = 0, \cdots, [|(i + n)/2|]$ by

$$\phi_i(y) \equiv \sum_{j=0}^{i-1} F_{i-j}(p_j, y^{(j)}) + \sum_{j=i+1}^{n} F_{j-i}(p_j, y^{(i)}), \tag{2.12}$$

$$(2.13)\quad P_{i,k} = \begin{cases} (-1)^{i-k} p_k^{(i-k)} + \sum_{j=0}^{k-1} (-1)^{i+k} \left\{ \binom{i-k}{k-j} \right. \\ \left. + \binom{i-k-1}{k-j-1} \right\} p_j^{(i+j-2k)} & \text{if } 0 \leqq k \leqq [|i/2|] < i, \\ (-1)^{i-k} p_k^{(i-k)} + \sum_{j=2k-i}^{k-1} (-1)^{i+k} \left\{ \binom{i-k}{k-j} \right. \\ \left. + \binom{i-k-1}{k-j-1} \right\} p_j^{(i+j-2k)} & \text{if } [|i/2|] < k < i, \\ 2p_i + \sum_{j=i+1}^{n} (-1)^{j-1} p_j^{(j-i)} & \text{if } k = i, \\ \sum_{j=2k-i}^{n} (-1)^{j+k} \left\{ \binom{j-k}{k-i} \right. \\ \left. + \binom{j-k-1}{k-i-1} \right\} p_j^{(i+j-2k)} & \text{if } i < k \leqq [|(i+n)/2|]. \end{cases}$$

THEOREM 2.1. *If i is an integer with $0 \leqq i \leqq n$ and $p_j \in \mathscr{C}^{|i-j|}(J \to \mathbf{R})$ for $j = 0, \cdots, n$, then ϕ_i is a Lyapunov-type function for* (1.1) *and, when y is a solution of* (1.1),

$$(2.14)\qquad (\phi_i(y))' = -\frac{1}{2} \sum_{k=0}^{[|(i+n)/2|]} P_{i,k}(y^{(k)})^2.$$

PROOF. The idea of the proof has already been mentioned; i.e., the Lyapunov-type function ϕ_i is obtained by multiplying (1.1) by $y^{(i)}$ and systematically integrating by parts through use of Lemma 2.1. We mention a few of the details.

When we multiply (1.1) by $y^{(i)}$ and integrate, we obtain

$$\phi_i(y) = -\sum_{j=0}^{i-1} \int g_{i-j}(p_j, y^{(j)}) - \int g_0(p_i, y^{(i)}) - \sum_{j=i+1}^{n} \int g_{j-i}(p_j, y^{(i)})$$

when y is a solution of (1.1). Then

$$
\begin{aligned}
(\phi_i(y))' = {} & \frac{1}{2}\sum_{j=0}^{i-1} (-1)^{i-j} p_j^{(i-j)} (y^{(j)})^2 \\
& - \frac{1}{2}\sum_{j=0}^{i-1} \sum_{k=1}^{[|(i-j)/2|]} (-1)^{i-j+k} \left\{ \binom{i-j-k}{k} \right. \\
& \left. + \binom{i-j-k-1}{k-1} \right\} p_j^{(i-j-2k)} (y^{(j+k)})^2 \\
& - p_i (y^{(i)})^2 - \frac{1}{2} \sum_{j=i+1}^{n} (-1)^{j-i} p_j^{(j-i)} (y^{(i)})^2 \\
& - \frac{1}{2} \sum_{j=i+1}^{n} \sum_{k=1}^{[|(j-i)/2|]} (-1)^{j-i+k} \left\{ \binom{j-i-k}{k} \right. \\
& \left. + \binom{j-i-k-1}{k-1} \right\} p_j^{(j-i-2k)} (y^{(i+k)})^2.
\end{aligned} \tag{2.15}
$$

With the aid of the facts that, for arbitrary h defined on lattice points,

$$
\begin{aligned}
\sum_{j=0}^{i-1} \sum_{k=1}^{[|(i-j)/2|]} h(j, k) &= \sum_{j=0}^{i-1} \sum_{k=j+1}^{[|(i+j)/2|]} h(j, k-j) \\
&= \sum_{k=1}^{[|i/2|]} \sum_{j=0}^{k-1} h(j, k-j) \\
&\quad + \sum_{k=[|i/2|]+1}^{i-1} \sum_{j=2k-1}^{k-1} h(j, k-j)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=i+1}^{n} \sum_{k=1}^{[|(j-i)/2|]} h(j, k) &= \sum_{j=i+1}^{n} \sum_{k=i+1}^{[|(i+j)/2|]} h(j, k-i) \\
&= \sum_{k=i+1}^{[|(i+n)/2|]} \sum_{j=2k-i}^{n} h(j, k-i),
\end{aligned}
$$

one can rearrange the right side of (2.15) to obtain (2.14).

For convenient reference later, we mention that $\phi_0(y)$ is given by

$$\phi_0(y) = \sum_{j=1}^{n} \sum_{i=0}^{j-1} \sum_{k=0}^{[|(j-i-1)/2|]} \frac{(-1)^{i+k}}{1 + [|(k+1)/(j-i-k)|]} \binom{i+k}{i} p_j^{(i)} y^{(k)} y^{(j-i-k-1)}. \tag{2.16}$$

From this point on in the paper, we will as a matter of convenience not explicitly mention the smoothness conditions on the coefficients when stating results for a differential equation. However, any coefficient appearing in the statement of a result is implicitly assumed to have continuous derivatives up through the highest order mentioned. It always suffices to have $p_j \in \mathscr{C}^{|i-j|}(J \to \mathbf{R})$ when applying the Lyapunov-type function ϕ_i.

3. **Two-point Boundary-value Problems.** In this section, we use Lyapunov-type functions to find conditions under which certain two-point boundary-value problems have a unique solution. The basic idea itself is not new and seems to have originated with Mammana [4] (refer again to pp. 152–153 of Swanson's book [7]). The following is well known and establishes the basic connection between boundary-value problems and zeros of solutions.

LEMMA 3.1. *Suppose that* $i_1, \cdots, i_p$ *are distinct integers in the interval* $[0, n]$, *that* $j_1, \cdots, j_q$ *are distinct integers in the interval* $[0, n]$, *that* $p + q = n$, *and that* $\alpha, \beta \in J$ *with* $\alpha < \beta$. *For each choice of numbers* $A_1, \cdots, A_p, B_1, \cdots, B_q \in \mathbf{R}$, *there is a unique solution of* (1.1) *satisfying the boundary conditions*

$$y^{(i_1)}(\alpha) = A_1, \cdots, y^{(i_p)}(\alpha) = A_p,\ y^{(j_1)}(\beta) = B_1, \cdots, y^{(j_q)}(\beta) = B_q \tag{3.1}$$

if and only if no nontrivial solution y *of* (1.1) *satisfies*

$$y^{(i_1)}(\alpha) = \cdots = y^{(i_p)}(\alpha) = 0 = y^{(j_1)}(\beta) = \cdots = y^{(j_q)}(\beta).$$

PROOF. The proof is by Cramer's rule.

As terminology, we say that $\{(i_1, \cdots, i_p)$ *at* α; $(j_1, \cdots, j_q)$ *at* $\beta\}$*-problems for* (1.1) *are uniquely solvable* if for each choice of $A_1, \cdots, A_p$, $B_1, \cdots, B_q \in \mathbf{R}$ there is a unique solution y of (1.1) satisfying (3.1). In a similar way, we say that $\{(i_1$ *or* $i_1', \cdots, i_p$ *or* $i_p')$ *at* α; $(j_1$ *or* $j_1', \cdots, j_q$ *or* $j_q')$ *at* $\beta\}$*-problems are uniquely solvable* provided that all $\{(i_1'', \cdots, i_p'')$ at α; $(j_1'', \cdots, j_q'')$ at $\beta\}$-problems are uniquely solvable whenever i_k'' is a number selected from i_k or i_k' $(k = 1, \cdots, p)$ and j_k'' is a number selected from j_k or j_k' $(k = 1, \cdots, q)$. With the interpretation being obvious, we will mix these notations and talk, for example, of $\{(i_1''$

or $i_1', i_2, \cdots, i_p$) at α; $(j_1, \cdots, j_p)$ at $\beta\}$-problems. We also adopt the convention of using a circumflex to delete a particular number from a list of numbers; for example, $1, \cdots, \hat{4}, \cdots, 7$ denotes 1, 2, 3, 5, 6, 7, and $0, \cdots, \hat{4}$ denotes 0, 1, 2, 3. As another convention, we will suppress the arguments in inequalities that are meant to hold on J; for example, $p > 0$ means $p(t) > 0$ for all $t \in J$.

Lyapunov-type functions can be used to draw numerous conclusions about two-point boundary-value problems for (1.1). We intend to illustrate this with some theorems; however, stating all possible implications of the technique would represent needless repetition which the reader could easily produce himself.

For each $i = 0, 1, 2$, let (H_i) denote the following hypotheses on the coefficients of (1.1) (here $P_{i,k}$ is as in (2.13)):

(H_i) $\quad P_{i,i} < 0$ and $P_{i,k} \leqq 0$ for $0 \leqq k \leqq [|(i + n)/2|]$ and $k \neq i$.

The importance of (H_i) is that $\phi_i(y)$ is nondecreasing (see (2.14)) when (H_i) holds and y is a solution of (1.1).

We distinguish between the cases where n is even or odd in (1.1) by considering separately the equations

$$\sum_{j=0}^{2n} p_j(t)y^{(j)} = 0 \tag{3.2}$$

and

$$\sum_{j=0}^{2n+1} p_j(t)y^{(j)} = 0. \tag{3.3}$$

THEOREM 3.1. *Consider the following various assumptions on the coefficients of* (3.2) *and* (3.3).

(P_1) $$(-1)^n[np_{2n}' - p_{2n-1}] \geqq 0$$

(P_2) $$(-1)^n p_{2n+1} < 0$$

(P_3) $$(-1)^{n+1}[\tfrac{1}{2}n(n+1)p''_{2n+1} - np_{2n}' + p_{2n-1}] \geqq 0, (-1)^n p_{2n+1} < 0$$

(P_4) $$p_0 \leqq 0, (-1)^{n-1}p_{2n} > 0$$

(P_5) $$p_0 \leqq 0$$

(P_6) $$p_1 \geqq 0, p_0' \geqq 0$$

(P_7) $$p_0' \leqq 0, (-1)^{n+1}p_{2n+1} < 0, p_1 \leqq 0.$$

The following conclusions are valid.

(0) (H_0) *implies* $\{(0, \cdots, n-1)$ *at* α; $(0, \cdots, n-1)$ *at* $\beta\}$*-problems for* (3.2) *are uniquely solvable.*

(1) (H_0) *and* (P_1) *imply* $\{(0, \cdots, \widehat{n-1}, n)$ *at* α; $(0, \cdots, n-1)$ *at* $\beta\}$*-problems for* (3.2) *are uniquely solvable.*

(2) (H_0) *and* (P_2) *imply* $\{(0, \cdots, n)$ *at* α; $(0, \cdots, n-1)$ *at* $\beta\}$*-problems for* (3.3) *are uniquely solvable.*

(3) (H_0) *and* (P_3) *imply* $\{(0, \cdots, \widehat{n-1}, n, n+1)$ *at* α; $(0, \cdots, n-1)$ *at* $\beta\}$*-problems for* (3.3) *are uniquely solvable.*

(4) (H_1) *and* (P_4) *imply* $\{(0, \cdots, n-1)$ *at* α; $(1, \cdots, n)$ *at* $\beta\}$*-problems for* (3.2) *are uniquely solvable.*

(5) (H_1) *and* (P_5) *imply* $\{(0, \cdots, n)$ *at* α; $(1, \cdots, n)$ *at* $\beta\}$*-problems for* (3.3) *are uniquely solvable.*

(6) (H_2) *and* (P_6) *imply* $\{(0, \hat{1}, \cdots, n)$ *at* α; $(1, \cdots, n)$ *at* $\beta\}$*-problems for* (3.2) *are uniquely solvable.*

(7) (H_2) *and* (P_7) *imply* $\{(1, \cdots, n+1)$ *at* α; $(0, \hat{1}, \cdots, n)$ *at* $\beta\}$*-problems for* (3.3) *are uniquely solvable.*

PROOF. Suppose (H_0) holds and y is a solution of (3.2) such that

$$y^{(i)}(\alpha) = 0 = y^{(i)}(\beta) \quad (i = 0, \cdots, n-1). \tag{3.4}$$

From (2.16) and (3.4) one sees that $(\phi_0(y))(\alpha) = (\phi_0(y))(\beta) = 0$. Since (H_0) holds, $\phi_0(y)$ is monotone nondecreasing and one sees in succession that $\phi_0(y)$, $(\phi_0(y))'$, and y are all identically zero on (α, β). Conclusion (0) then follows from Lemma 3.1.

Now suppose (H_0) holds and y is a solution of (3.2) satisfying

$$\begin{aligned} &y^{(i)}(\alpha) = 0 \ (i = 0, \cdots, \widehat{n-1}, n) \text{ and} \\ &y^{(i)}(\beta) = 0 \ (i = 0, \cdots, n-1). \end{aligned} \tag{3.5}$$

It follows from (2.16) and (3.5) that

$$(\phi_0(y))(\alpha) = \frac{1}{2}(-1)^n[np'_{2n}(\alpha) - p_{2n-1}(\alpha)](y^{(n-1)}(\alpha))^2$$

and $(\phi_0(y))(\beta) = 0$. If (H_0) and (P_1) hold, then y is again identically zero on (α, β). Hence, (1) also follows from Lemma 3.1.

Conclusions (2)–(7) follow similarly. The Lyapunov-type function ϕ_0 is used to establish (2) and (3), ϕ_1 is used to prove (4) and (5), and ϕ_2 is utilized in the proofs of (6) and (7).

Conclusion (0) of Theorem 3.1 is not a new result—it follows from Theorem 1 of [6]. We could have drawn several other implications about two-point boundary value functions but have chosen not to do so;

to be specific, ϕ_1 and ϕ_2 can be used to draw implications similar to (2) and (3) in Theorem 3.1. Also, if (H_0) holds and the direction of the inequality in (P_1) is reversed, then $\{(0, \cdots, n-1)$ at α; $(0, \cdots, \widehat{n-1}, n)$ at $\beta\}$-problems for (3.2) are uniquely solvable; and we have omitted conclusions such as this in the statement of Theorem 3.1. In total, the author has been able to draw twenty-seven distinct conclusions which follow by applying ϕ_0, ϕ_1 and ϕ_2 to equations (3.2) and (3.3) in the above way.

Only the Lyapunov-type functions ϕ_0, ϕ_1 and ϕ_2 were used in Theorem 3.1. For the general equations (3.2) and (3.3), the same technique does not yield any results using ϕ_i with $i \geqq 3$ because one needs to assign too many zeros (more than the order of the equation) to y at α and β in order to determine the sign of $\phi_i(y)$ at α and β. However, such Lyapunov-type functions can be used to obtain results for equations where some of the coefficients are identically zero. We will illustrate this fact later.

In general, if some coefficients are identically zero, one can obtain more information with fewer hypotheses. At one extreme are equations (3.2) and (3.3) where none of the coefficients is assumed to be identically zero. At the other extreme, are the two-term equations

$$y^{(2n)} + p(t)y = 0 \tag{3.6}$$

and

$$y^{(2n+1)} + p(t)y = 0 \tag{3.7}$$

for which we give the following theorem.

THEOREM 3.2. *The following are valid:*

(1) *If* $(-1)^n p > 0$, *then* $\{(0$ *or* $2n-1, \cdots, n-1$ *or* $n)$ *at* α; $(0$ *or* $2n-1, \cdots, n-1$ *or* $n)$ *at* $\beta\}$*-problems for* (3.6) *are uniquely solvable.*

(2) *If* $(-1)^n p > 0$, *then* $\{(0$ *or* $2n, \cdots, n-1$ *or* $n+1, n)$ *at* α; $(0$ *or* $2n, \cdots, n-1$ *or* $n+1)$ *at* $\beta\}$*-problems for* (3.7) *are uniquely solvable.*

(3) *If* $(-1)^n p < 0$, *then* $\{(0$ *or* $2n, \cdots, n-1$ *or* $n+1)$ *at* α; $(0$ *or* $2n, \cdots, n-1$ *or* $n+1, n)$ *at* $\beta\}$*-problems for* (3.7) *are uniquely solvable.*

PROOF. If y is a solution of (3.6), then $\phi_0(y)$ and $(\phi_0(y))'$ reduce to

$$\phi_0(y) = yy^{(2n-1)} - y'y^{(2n-2)} + \cdots + (-1)^{n-1}y^{(n-1)}y^{(n)}$$

and

$$(\phi_0(y))' = -py^2 + (-1)^{n+1}(y^{(n)})^2.$$

If y is a solution of (3.6) such that

$$y^{(i)}(\alpha)y^{(2n-1-i)}(\alpha) = 0 = y^{(i)}(\beta)y^{(2n-1-i)}(\beta) \quad (i = 0, \cdots, n-1),$$

then $(\phi_0(y))(\alpha) = (\phi_0(y))(\beta) = 0$. Conclusion (1) follows from Lemma 3.1; the other conclusions follow similarly.

Note that Theorem 3.2 specifies 2^{2n} different kinds of boundary-value problems which are uniquely solvable for (3.6) when $(-1)^n p > 0$. Lyapunov-type functions other than ϕ_0 could be used to study (3.6) and (3.7) and equations intermediate between the two-term equations and (1.1) could be systematically investigated but we leave this to the interested reader.

We now make use of ϕ_4 to study the equation

$$y^{(10)} + p_4(t)y^{(4)} + p_0(t)y = 0 \tag{3.8}$$

in order to illustrate the use of a Lyapunov-type function other than ϕ_0, ϕ_1, or ϕ_2 and also to illustrate a restriction on coefficients which arises when non-identically zero coefficients, p_0 and p_4 in this case, are "widely separated" (as opposed to adjacent coefficients p_i and p_{i+1}).

If y is a solution of (3.8), $\phi_4(y)$ and $(\phi_4(y))'$ are given by

$$\begin{aligned}\phi_4(y) = y^{(4)}y^{(9)} - y^{(5)}y^{(8)} + y^{(6)}y^{(7)} + p_0[yy''' - y'y''] \\ + p_0'[-yy'' + (y')^2] + p_0''yy' - 1/2\, p_0'''y^2\end{aligned}$$

and

$$\begin{aligned}(\phi_4(y))' = -\frac{1}{2}p_0^{(4)}y^2 + 2p_0''(y')^2 \\ - p_0(y'')^2 - p_4(y^{(4)})^2 + (y^{(7)})^2.\end{aligned}$$

For monotonicity purposes, we assume the inequalities $p_0^{(4)} \leqq 0$, $p_0'' \geqq 0$, $p_0 \leqq 0$ and $p_4 \leqq 0$ hold. However, $p_0^{(4)} \leqq 0$, $p_0'' \geqq 0$ and $p_0 \leqq 0$ restrict p_0 to being a linear function for sufficiently large t. We assume therefore that p_0 is a linear function which is negative on J. Arguing as before, we obtain the following theorem.

THEOREM 3.3. *If $p_4 \leqq 0$ and p_0 is a negative linear function on J, then $\{(0, 1, 4$ or $9, 5$ or $8, 6$ or $7)$ at α; $(0, 1, 4$ or $9, 5$ or $8, 6$ or $7)$ at $\beta\}$-problems, $\{(0, 1, 4$ or $9, 5$ or $8, 6$ or $7)$ at α; $(0, 2, 4$ or $9, 5$ or $8, 6$ or $7)$ at $\beta\}$-problems, and $\{(0, 1, 4$ or $9, 5$ or $8, 6$ or $7)$ at α; $(2, 3, 4$ or $9, 5$ or $8, 6$ or $7)$ at $\beta\}$-problems for (3.8) are uniquely solvable.*

Some of the strict inequalities in hypotheses of theorems in this section can be weakened. For example, in Theorem 3.1(0), it is assumed that $P_{0,0} < 0$ and $P_{0,k} \leqq 0$ for $k = 1, \cdots, n$; the same conclusion follows if it is only assumed that $P_{0,k} \leqq 0$ for $k = 0, \cdots, n$. Proving the stronger result usually involves ruling out the existence of certain polynomial solutions. Even so, simple examples show that certain inequalities need to be strict. For instance, in $n = 2$ and $p > 0$, Theorem 3.2(1) implies that $\{(2, 3)$ at α; $(2, 3)$ at $\beta\}$-problems for (3.6) are uniquely solvable. The same is not true if $p \equiv 0$ on J (linear functions are counter examples). For the same equation, however, $\{(0, 1)$ at α; $(0, 1)$ at $\beta\}$-problems are uniquely solvable assuming only $p \geqq 0$. When we assume a strict inequality, it always suffices to assume it is strict at some point γ between α and β.

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