ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 1, Winter 1980

REPRODUCING KERNEL SPACES AND ANALYTIC CONTINUATION

WILLIAM F. DONOGHUE, JR.*

We study here three forms of analytic continuation associated with certain reproducing kernel spaces of analytic functions of one or more variables. The first and simplest concerns the isometric mapping between two reproducing kernel spaces corresponding to the operation of restriction to a set of uniqueness, or, inversely, the analytic continuation from a set of uniqueness to a larger domain. The second concerns the sesqui-analytic continuation of Bergman kernels $K(z, w)$ from a domain $\mathscr{D} \times \mathscr{D}$ to a larger domain $\mathscr{D}^* \times \mathscr{D}^*$, and the easy extension of these results to the analytic continuation of almost positive matrices which are sesqui-analytic. Finally the third type of analytic continuation has to do with functions $V(z, w)$ initially defined on $M \times M$ but extendable to $\mathscr{D} \times \mathscr{D}$ where M is a set of uniqueness in $\mathscr{D}$. This third type of continuation is essentially due to Bergman-Schiffer [3] and has significant applications.

We have relied heavily here on a report written some years ago by FitzGerald [5]. Most of the results which we quote concerning reproducing kernels are really due to N. Aronszajn [1], but we prefer to make our references to the detailed exposition in Meschkowski [6].

In the following we are concerned with a domain $\mathscr{D}$ in $\mathbf{C}^n$ (where we do not exclude the simplest case $n = 1$) and functions defined in $\mathscr{D}$ or in $\mathscr{D} \times \mathscr{D}$. A function $S(z, w)$ defined in $\mathscr{D} \times \mathscr{D}$ is called *sesqui-analytic* if it is analytic in z and is conjugate analytic in w, i.e., the function $F(z, w) = S(z, \bar{w})$ is analytic in both variables. It is then clear what is to be understood by a sesqui-analytic continuation of a sesqui-analytic function. See, for example, [4].

A function $K(z, w)$ defined in $\mathscr{D} \times \mathscr{D}$ is called a *positive matrix* if and only if for every finite set of points $\{z_i\}$ in $\mathscr{D}$ and equally many complex numbers $\{a_i\}$, the number

$$\sum\sum K(z_i, z_j) a_i \bar{a}_j$$

is real and non-negative. See [1] and [6]. It is easy to show that if a sequence of functions $\{h_i(z)\}$ is defined on some set $\mathscr{D}$, then

*Research supported by NSF contact MCS 77–03536.
Received by the editors on April 13, 1978.

$$H(z, w) = \sum h_i(z)\overline{h_i(w)}$$

is a positive matrix on $\mathscr{D} \times \mathscr{D}$ provided that the sum makes sense, i.e., provided that

$$H(z, z) = \sum |h_i(z)|^2 < \infty.$$

It is also clear that if $K(z, w)$ is a positive matrix, so also is its conjugate $\overline{K(z, w)}$, and this function is easily shown to be equal to $K(w, z)$.

A function in a product domain which is simultaneously sesqui-analytic and a positive matrix is called a Bergman kernel function. In a known way, the kernel function gives rise to a Hilbert space of functions, all analytic in $\mathscr{D}$, and there is a canonical mapping of $\mathscr{D}$ into that space, given by

$$w \longrightarrow K_w$$

where $K_w = K_w(z)$ is the function $K(z, w)$.

Let us remark that the mapping just described should be thought of as conjugate analytic rather than analytic. More exactly, given a continuous linear functional F on the reproducing kernel space $\mathscr{H}(\mathscr{D})$ associated with $K(z, w)$, the function

$$F(K_z) = F(z)$$

is the conjugate of an analytic function. To see this, let F be represented in the space by the element f; we then have

$$\begin{aligned} F(z) &= F(K_z) = (K_z, f) \\ &= \overline{(f, K_z)} = \overline{f(z)} \end{aligned}$$

where $f(z)$ is a function in the space $\mathscr{H}(\mathscr{D})$ and is therefore analytic. This elementary remark is also easily verified for the example of the Bergman kernel associated with the unit disk in one complex variable:

$$K(z, w) = K_w(z) = \frac{1}{\pi(1 - z\bar{w})^2}\,.$$

A subset M of $\mathscr{D}$ is called a *set of uniqueness* for $\mathscr{D}$ if it has the property that every function analytic in $\mathscr{D}$ and vanishing on M vanishes identically on $\mathscr{D}$. Clearly any set M containing an open set is a set of uniqueness, and any set M' containing a set of uniqueness is also a set of uniqueness. We see that if a set of uniqueness is decomposed into a finite collection of subsets, at least one of the subsets in the decomposition is itself a set of uniqueness. Moreover, a countable dense subset of a set of uniqueness is again a set of uniqueness.

LEMMA 1. *Let M' be a set of uniqueness for $\mathscr{D}'$, an open set in the space of n' complex variables; similarly, let M'' be a set of uniqueness for $\mathscr{D}''$,* *open in the space of n'' complex variables. Then the product $M' \times M''$ is a set of uniqueness for $\mathscr{D}' \times \mathscr{D}''$ in the space of $n' + n''$ complex variables.*

PROOF. Let $F(z, w)$ be analytic in $\mathscr{D}' \times \mathscr{D}''$ where z is in $\mathscr{D}'$ and w in $\mathscr{D}''$ and suppose that $F(z, w) = 0$ on $M' \times M''$. Fix w in M''; then for z in M' we have $F(z, w) = 0$, and since M' is a set of uniqueness fo $\mathscr{D}'$, $F(z, w) = 0$ for all z in $\mathscr{D}'$ provided, of course, that w is in M''. Now for any z in $\mathscr{D}'$ we have $F(z, w) = 0$ for w in M'', hence for all w in $\mathscr{D}''$ since M'' is a set of uniqueness for $\mathscr{D}''$. Thus $F(z, w)$ vanishes throughout $\mathscr{D}' \times \mathscr{D}''$. This shows that $M' \times M''$ is a set of uniqueness for $\mathscr{D}' \times \mathscr{D}''$.

We next consider a slightly simpler case.

LEMMA 2. *Let M be a set of uniqueness for $\mathscr{D}$, and let $S(z, w)$ be sesquianalytic in $\mathscr{D} \times \mathscr{D}$ with the property that $S(z, w) = 0$ on $M \times M$; then $S(z, w)$ vanishes throughout $\mathscr{D} \times \mathscr{D}$.*

PROOF. As before, we fix w in M and note that $S(z, w) = 0$ for z in M and therefore for all z in $\mathscr{D}$. For any fixed z in $\mathscr{D}$, then, the analytic function $h(w) = \bar{S}(z, w)$ is equal to 0 for w in M and therefore vanishes for all w in $\mathscr{D}$. Thus $S(z, w) = 0$ for all z in $\mathscr{D}$ and w in $\mathscr{D}$, as desired.

We now study the first form of analytic continuation mentioned above. Let $\mathscr{D}$ be an open connected set in $\mathbf{C}^n$ and let M be a set of uniqueness in $\mathscr{D}$. Let $\mathscr{H}(\mathscr{D})$ be the reproducing kernel space over $\mathscr{D}$ associated with the Bergman kernel function $K(z, w)$ defined on $\mathscr{D} \times \mathscr{D}$. Let $\mathscr{H}(M)$ be the reproducing kernel space of functions defined on M associated with the kernel $K(z, w)$ here considered only as a function on $M \times M$. Since M is a set of uniqueness, the collection of vectors K_w for w in M spans a linear subspace of $\mathscr{H}(\mathscr{D})$ which is dense in that space. The easy proof follows: if an element f in $\mathscr{H}(\mathscr{D})$ were orthogonal to all such K_w we would have $f(w) = (f, K_w) = 0$ for all w in M, whence, since M is a set of uniqueness $f(z) = 0$ identically in $\mathscr{D}$. Thus f is the zero element of $\mathscr{H}(\mathscr{D})$.

Now consider the subspace of $\mathscr{H}(\mathscr{D})$ consisting of all finite sums of the form

$$u(z) = \sum a_j K(z, w_j) \text{ for } w_j \text{ in } M$$

and consider the subspace determined by exactly the same expressions for the space $\mathscr{H}(M)$. The norm of u in either space is the same, since we have

$$\|u\|^2 = \|\sum a_j K_{w_j}\|^2 = \sum\sum a_j \bar{a}_k (K_{w_j}, K_{w_k})$$

and this depends only on the values of $K(z, w)$ on $M \times M$. The subspace in question is dense in $\mathscr{H}(\mathscr{D})$ as well as in $\mathscr{H}(M)$ and so there is established a natural isometry between these two spaces. Accordingly, $u(z)$ in $\mathscr{H}(\mathscr{D})$ corresponds isometrically to $u(z)$ in $\mathscr{H}(M)$, and the latter function is merely the restriction of the former to M. Any function $v(z)$ in $\mathscr{H}(M)$ now admits a continuation to a uniquely determined function $v(z)$ in $\mathscr{H}(\mathscr{D})$ having the same norm, and $v(z)$ is clearly an analytic continuation of v, initially given on M, to all of $\mathscr{D}$.

We infer: Let M be a set of uniqueness in $\mathscr{D}$ and $\mathscr{H}(\mathscr{D})$ the reproducing kernel space of functions analytic in $\mathscr{D}$ associated with the kernel function $K(z, w)$. Let $\mathscr{H}(M)$ be the space of restrictions of functions in $\mathscr{H}(\mathscr{D})$ to M. Then the mapping between these spaces given by the restriction map is an isometry onto, and its inverse, the analytic continuation of functions in $\mathscr{H}(M)$ to $\mathscr{H}(\mathscr{D})$, is also an isometry onto.

The process which we have just sketched is one of the most important types of analytic continuation which we shall study.

Consider a space $\mathscr{H}(\mathscr{D})$. When is a function $f(z)$ an element of the space? Of course, here $f(z)$ is analytic in $\mathscr{D}$.

THEOREM 1. *$f(z)$ is an element of $\mathscr{H}(\mathscr{D})$ if and only if the sesqui-analytic function*

$$CK(z, w) - f(z)\overline{f(w)}$$

is itself a Bergman kernel on $\mathscr{D} \times \mathscr{D}$ for an appropriate positive C.

PROOF. (We should remark that the theorem is well-known, but rarely explicitly stated in the literature.) If $f(z)$ is an element of the space $\mathscr{H}(\mathscr{D})$, we may choose a positive number c so that $cf(z)$ is of norm 1. Now the kernel admits an expansion in terms of an arbitrary orthonormal set of the form

$$K(z, w) = \sum_{k=1}^{\infty} g_k(z)\overline{g_k(w)}$$

and we may suppose that the orthonormal set $\{g_k\}$ is so chosen that the first element is $cf(z)$. So now

$$K(z, w) = c^2 f(z)\overline{f(w)} + \sum_{k=2}^{\infty} g_k(z)\overline{g_k(w)}$$

and the second term on the right is manifestly a positive matrix on $\mathscr{D} \times \mathscr{D}$. Conversely, if $K(z, w)$ admits such a decomposition:

$$K(z, w) = Cf(z)\overline{f(w)} + K^*(z, w)$$

where $K^*(z, w)$ is a Bergman kernel in $\mathscr{D} \times \mathscr{D}$, then, as shown, for example, in **[6]**, the reproducing kernel space associated with $K(z, w)$ contains the functions in the spaces associated with the summands. In particular, then, the space contains the functions generated by the kernel $Cf(z)\overline{f(w)}$, and this space contains $f(z)$, which is therefore in $\mathscr{H}(\mathscr{D})$.

COROLLARY. *Let $f(z)$ be defined on M, a set of uniqueness for $\mathscr{D}$ and suppose that for some real constant c the function*

$$K(z, w) - c^2 f(z)\overline{f(w)}$$

is a positive matrix on $M \times M$; then $f(z)$ is the restriction to M of a uniquely determined function defined on $\mathscr{D}$ and belonging to the reproducing kernel space over $\mathscr{D}$ associated with the kernel $K(z, w)$.

We now turn to a slightly different situation.

THEOREM 2. *Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be domains in $\mathbf{C}^n$ overlapping in the open set $M = \mathscr{D}_1 \cap \mathscr{D}_2$, a set which is a set of uniqueness for either domain. Let $K(z, w)$ be a function defined in $(\mathscr{D}_1 \times \mathscr{D}_1) \cup (\mathscr{D}_2 \times \mathscr{D}_2)$ and sesqui-analytic there. We shall also suppose that $K(z, w)$ is a positive matrix in both $\mathscr{D}_1 \times \mathscr{D}_1$ as well as $\mathscr{D}_2 \times \mathscr{D}_2$. Then $K(z, w)$ admits a uniquely determined sesqui-analytic continuation to the product*

$$(\mathscr{D}_1 \cup \mathscr{D}_2) \times (\mathscr{D}_1 \cup \mathscr{D}_2)$$

and is a positive matrix there.

PROOF. Consider the mapping from $\mathscr{H}(\mathscr{D}_1)$ to $\mathscr{H}(M)$ given by restricting functions in the first space to M. We know this mapping to be an isometry onto. The extension mapping, carrying $\mathscr{H}(M)$ onto $\mathscr{H}(\mathscr{D}_1)$, is also an isometry. Similarly, the map from $\mathscr{H}(M)$ to $\mathscr{H}(\mathscr{D}_2)$ is an isometry. Accordingly, the functions in $\mathscr{H}(M)$ extend in a unique way to functions defined on $\mathscr{D}_1 \cup \mathscr{D}_2$ and we obtain in this way a space of functions analytic in that set and normed by the norm of $\mathscr{H}(M)$. The valuation functionals are clearly continuous relative to that norm and so there exists an associated reproducing kernel which we also write as $K(z, w)$. It is manifest that this kernel is sesqui-analytic, i.e., a Bergman kernel function, and it coincides with the initial $K(z, w)$ on $M \times M$. Because M is a set of uniqueness for $\mathscr{D}_1$, $K(z, w)$ coincides with our initial $K(z, w)$ in $\mathscr{D}_1 \times \mathscr{D}_1$ and also in $\mathscr{D}_2 \times \mathscr{D}_2$. This proves the theorem.

It is important to notice that no hypothesis of simple connectedness occurs in the previous theorem.

There follows from this a result essentially due to FitzGerald.

THEOREM 3. *Let $\mathscr{D}$ be a domain in $\mathbf{C}^n$ and let G be an open connected subset of $\mathscr{D} \times \mathscr{D}$ containing the diagonal—i.e., containing all points of the form (z_0, z_0) for z_0 in $\mathscr{D}$. Suppose that $K(z, w)$ is a sesqui-analytic function defined in G and that for every z_0 in $\mathscr{D}$ there exists a neighborhood $\mathscr{D}_0$ so that $K(z, w)$ is a positive matrix in $\mathscr{D}_0 \times \mathscr{D}_0$. Then $K(z, w)$ admits a uniquely determined sesqui-analytic continuation to the product $\mathscr{D} \times \mathscr{D}$ where it is a positive matrix, i.e., a Bergman kernel function.*

PROOF. The proof is an immediate consequence of the previous theorem; the sets $\mathscr{D}_0$ being all sets of uniqueness in $\mathscr{D}$.

We turn next to the second type of analytic continuation which we need to understand.

THEOREM 4. *Let $\mathscr{D}$ be a bounded domain in $\mathbf{C}^n$ and $K(z, w)$ a sesqui-analytic function defined on $\mathscr{D} \times \mathscr{D}$. Let M be a set of uniqueness in $\mathscr{D}$ and suppose that $K(z, w)$ is a positive matrix on $M \times M$ and that it is a bounded function on $\mathscr{D} \times \mathscr{D}$. Then $K(z, w)$ is a positive matrix throughout $\mathscr{D} \times \mathscr{D}$.*

PROOF. The proof is long and basically depends on the consideration of an integral operator on $L^2(\mathscr{D})$ with the kernel $K(z, w)$. We have to show first that the operator in question is bounded and self-adjoint; then we show that it is positive, i.e., that $K(u, u) \geqq 0$ for all u in $L^2(\mathscr{D})$. From the positiveness of that operator we will then deduce that $K(z, w)$ is a positive matrix in $\mathscr{D} \times \mathscr{D}$.

Let us begin with the definition of our integral operator: we will have

$$(Kg)(z) = \int_{\mathscr{D}} K(z, w)g(w)dA_w = (g, K_z)$$

where dA_w is the element of volume in $\mathbf{C}^n$, and where

$$K_z(w) = \overline{K(z, w)},$$

a bounded function which is surely an element of $L^2(\mathscr{D})$. It is easy to verify that $\|Kg\|^2 \leqq M^2 A \|g\|^2$ where M is a bound for the kernel in $\mathscr{D} \times \mathscr{D}$ and A is the volume of $\mathscr{D}$. Thus K is clearly a bounded integral operator.

In order to show that K is self-adjoint, we must verify the relation

$$K(z, w) = \overline{K(w, z)}$$

for all points in $\mathscr{D} \times \mathscr{D}$. So we consider

$$f(z, w) = K(z, w) - \overline{K(w, z)},$$

a sesqui-analytic function in $\mathscr{D} \times \mathscr{D}$ which vanishes on $M \times M$. From Lemma 2 it follows that $f(z, w)$ vanishes identically and therefore that the integral operator K is self-adjoint.

Let us remark next that the elements in the range of K may be taken to be analytic functions in $\mathscr{D}$ since the measure $g(w)dA_w$ is a measure of finite total mass. This circumstance leads us to consider another Hilbert space, the space of all functions $f(z)$ analytic in $\mathscr{D}$ such that the quantity

$$\|f\|^2 = \int_{\mathscr{D}} |f(z)|^2 dA_z$$

is finite. (Here dA_z refers to the ordinary Lebesgue measure in $\mathscr{D}$.) This is evidently a reproducing kernel space $A^2(\mathscr{D})$ associated with some Bergman kernel $B(z, w)$ and may be identified with a subspace of $L^2(\mathscr{D})$, the subspace consisting of all equivalence classes in $L^2(\mathscr{D})$ which contain an analytic function. Note also that because K is a self-adjoint operator, it is enough for us to show the positivity of (Ku, u) for u in the range of K, i.e., for u in the space of analytic functions associated with the kernel $B(z, w)$.

Next we define a measure μ on M as follows. Choose a countable dense subset $\{z_k\}$ of M and put a positive mass m_k at z_k so chosen that

$$\sum m_k B(z_k, z_k) = 1.$$

Given a function $f(z)$ in $A^2(\mathscr{D})$ we consider its restriction $Tf(z)$ to M and think of Tf as an element of $L^2(\mu)$. We will find that

$$\begin{aligned}\|Tf\|^2 &= \sum |f(z_k)|^2 m_k = \sum |(f, B_{z_k})|^2 m_k \leqq \|f\|^2 \sum B(z_k, z_k) m_k \\ &= \|f\|^2\end{aligned}$$

and so T is a bounded linear operator of norm at most 1. It is also clear that T has no null space, for if f in $A^2(\mathscr{D})$ is such that $Tf = 0$ in $L^2(\mu)$, then $f(z_k) = 0$ for all k, and since the sequence $\{z_k\}$ is dense in M and f is continuous, $f(z) = 0$ for all z in M and therefore for all z in $\mathscr{D}$, M being a set of uniqueness. A consequence of this fact is that the range of the adjoint map T^* from $L^2(\mu)$ to $A^2(\mathscr{D})$ is dense in the second space.

As we have already remarked, to show that K is a positive operator it is enough to show that $(Ku, u) \geqq 0$ for all u in $A^2(\mathscr{D})$ when that space is identified with a subspace of $L^2(\mathscr{D})$. Because of the continuity of the operator, it is even enough to show this for elements u in the range of T^*, a set which is dense in $A^2(\mathscr{D})$. So all that remains is to show the positivity of (KT^*g, T^*g) for all g in $L^2(\mu)$, and from the definition of the adjoint, this is the same thing as (TKT^*g, g).

Now $(KT^*g)(z) = \int K(z, w)T^*g(w)dA_w = (T^*g, K_z) = (g, TK_z) = \sum g(z_k)K(z_k, z)m_k$ whence (TKT^*g, g) is given by the sum

$$\sum\sum g(z_k)K(z_k, z_j)\overline{g(z_j)}m_k m_j,$$

a quantity which is positive because $K(z, w)$ is a positive matrix on $M \times M$.

From the positivity of the operator K we easily infer that $K(z, w)$ is a positive matrix over $\mathscr{D} \times \mathscr{D}$. Let us select an arbitrary finite set of points $\{z_i\}$ in $\mathscr{D}$ and equally many complex numbers $\{a_i\}$. For each j, let $g_j(z)$ be the characteristic function of a sphere centered at z_j suitably normalized so that $\int g_j(z)dA_z = 1$. The function $g_j(z)$ clearly belongs to $L^2(\mathscr{D})$. Now form the function

$$u(w) = \sum a_j g_j(w),$$

whence

$$Ku(z) = \sum a_j \int K(z, w) g_j(w) dA_w = \sum a_j K(z, z_j)$$

because the sesqui-analytic $K(z, w)$ is harmonic in w for fixed z. Now $(Ku, u) = \sum\sum a_j \bar{a}_k \int K(z, z_j)\overline{g_k(z)dA_z} = \sum\sum a_j \bar{a}_k K(z_k, z_j)$ and this quantity is positive because K is a positive operator. It follows that $K(z, w)$ is a positive matrix over $\mathscr{D} \times \mathscr{D}$, completing the proof of the theorem.

It is not necessary to give a proof of the following more general theorem.

THEOREM 5. *Let $\mathscr{D}$ be a domain in $\mathbf{C}^n$, not necessarily bounded, and let $K(z, w)$ be sesqui-analytic in G, a connected neighborhood of the diagonal in $\mathscr{D} \times \mathscr{D}$. Let M be a set of uniqueness in some open set $\mathscr{D}_0$ with compact closure in $\mathscr{D}$ and suppose that $K(z, w)$ is a positive matrix over $M \times M$. Then $K(z, w)$ admits a unique sesqui-analytic continuation to $\mathscr{D} \times \mathscr{D}$ where it is a Bergman kernel function.*

As FitzGerald has observed, we obtain an interesting application of the previous theorem in the proof of the following somewhat simplified version of a result of Widder [7].

THEOREM 6. *Let U be a connected neighborhood in the plane of the closed interval $[-1, 1]$ and let M be the closed interval $[-1/2, 1/2]$; let $g(z)$ be a function analytic in U such that the function $K(x, y) = g(x + y)$ is a positive matrix over $M \times M$; then $g(z)$ admits an analytic continuation over the strip $-1 < \text{Re } z < 1$.*

PROOF. Select the positive ε so small that the interval $[-1 - \varepsilon, 1 + \varepsilon]$ is contained in U, and for $\mathscr{D}$ take the strip $|\text{Re } z| < 1/2 + \varepsilon$. Now let

$$K(z, w) = g(z + \bar{w})$$

and let G be the neighborhood of the diagonal consisting of all (z, w) such that $z + \bar{w}$ is in U. We infer from the previous theorem that $K(z, w)$ admits a uniquely determined sesqui-analytic continuation to $\mathscr{D} \times \mathscr{D}$. Now we get an analytic continuation of $g(z)$ from U to $\mathscr{D}$ by putting

$$g(z) = K(z, 0)$$

and similarly, if we consider $K(z, 1/2)$, we obtain a function analytic in $\mathscr{D}$ and which, for certain z in U, is equal to $g(z + 1/2)$. In the same way $K(z, -1/2) = g(z - 1/2)$ and these equations make it clear that $g(z)$ can be extended from $\mathscr{D}$ to its two translates by $1/2$ and $-1/2$, i.e., $g(z)$ becomes an analytic function in the strip $-1 < \text{Re } z < 1$ as desired.

As an immediate consequence of the previous theorems we obtain results concerning the sesqui-analytic continuation of almost positive matrices. Let us recall (Donoghue **[4]**) the definition of an almost positive matrix $A(x, y)$ defined in some product set $S \times S$. We must have, for every finite subset $\{x_i\}$ of S and equally many complex numbers $\{a_i\}$,

$$\sum \sum A(x_i, x_j)a_i\bar{a}_j \geqq 0$$

for all choices of the a_i subject to the condition $\sum a_i = 0$. It is well-known that the function

$$K(x, y) = A(x, y) - A(z, y) - A(x, z) + A(z, z)$$

is then a positive matrix over $S \times S$ for any choice of z in S and this is a necessary and sufficient condition for $A(x, y)$ to be almost positive. Indeed, if for some function $F(x)$ defined on S we have

$$A(x, y) + F(x) + \overline{F(y)}$$

as a positive matrix, then $A(x, y)$ is almost positive.

Let us suppose that $A(z, w)$ is defined and sesqui-analytic on the set $(\mathscr{D} \times \mathscr{D}) \cup (\mathscr{D}^* \times \mathscr{D}^*)$ where the intersection $\mathscr{D} \cap \mathscr{D}^*$ is open. We can select z_0 in $\mathscr{D} \cap \mathscr{D}^*$ and form the function

$$F(z) = -A(z, z_0) + A(z_0, z_0)/2$$

which is defined in the union $\mathscr{D} \cup \mathscr{D}^*$. Now if $A(z, w)$ is almost positive in $\mathscr{D} \times \mathscr{D}$ and in $\mathscr{D}^* \times \mathscr{D}^*$, the function

$$K(z, w) = A(z, w) + F(z) + \overline{F(w)}$$

is a positive matrix on those two sets and hence admits an analytic continuation to a Bergman kernel over $(\mathscr{D} \cup \mathscr{D}^*) \times (\mathscr{D} \cup \mathscr{D}^*)$. Thus $A(z, w)$ is similarly extendable. We therefore have the following result.

THEOREM 7. *Let $\mathscr{D}$ be a domain in $\mathbf{C}^n$, not necessarily bounded, and let $A(z, w)$ be sesqui-analytic in G, a connected neighborhood of the diagonal in $\mathscr{D} \times \mathscr{D}$. Let M be a set of uniqueness in some open subset $\mathscr{D}_0$ of $\mathscr{D}$ having a compact closure in $\mathscr{D}$ and suppose that $A(z, w)$ is an almost positive matrix over $M \times M$. Then $A(z, w)$ admits a uniquely determimed sesqui-analytic continuation to $\mathscr{D} \times \mathscr{D}$ where it is an almost positive matrix.*

The applications of this theorem are important and well-known and we shall not discuss them here.

There is a third form of analytic continuation associated with reproducing kernel spaces to which we now turn. The next theorem, due essentially to Carl FitzGerald, is tedious to state but it is of great importance.

THEOREM 8. *Let $\mathscr{D}'$ be a domain in the space of n' complex variables and $K'(z, w)$ a sesqui-analytic function in $\mathscr{D}' \times \mathscr{D}'$; similarly, let $\mathscr{D}''$ be a domain in n'' variables with $K''(z, w)$ sesqui-analytic in $\mathscr{D}'' \times \mathscr{D}''$. Let M'*

be a set of uniqueness for $\mathscr{D}'$ *and* M'' *a set of uniqueness for* $\mathscr{D}''$, *and suppose that* $f(z, w)$ *is a function defined on* $M' \times M''$ *with the property that for every pair of finite sets* $\{z_i\}$ *in* M' *and* $\{w_j\}$ *in* M'' *the inequality*

$$\left|\sum\sum a_i b_j f(z_i, w_j)\right|^2 \leqq \left(\sum\sum a_i \bar{a}_j K'(z_i, z_j)\right)\left(\sum\sum b_i \bar{b}_j K''(w_i, w_j)\right)$$

is valid, whatever be the choices of the coefficients $\{a_j\}$ *and* $\{b_j\}$. *Then there exists a uniquely determined function* $F(z, w)$, *analytic in* $\mathscr{D}' \times \mathscr{D}''$, *which coincides with* $f(z, w)$ *on* $M' \times M''$.

PROOF. It is easy to see from the inequality above that since the choices of the z's and w's are arbitrary, all expressions of the form

$$\sum\sum a_i \bar{a}_j K'(z_i, z_j)$$

have the same sign. It follows that the function $K'(z, w)$ is a positive matrix on $M' \times M'$ or is the negative of a positive matrix. We change its sign, if necessary, to make it a positive matrix on $M' \times M'$, and then, in view of Theorem 5, it is a positive matrix on all of $\mathscr{D}' \times \mathscr{D}'$. Of course we argue similarly for $K''(z, w)$, obtaining a positive matrix on $\mathscr{D}'' \times \mathscr{D}''$.

Let $\mathscr{H}(\mathscr{D}')$ be the reproducing kernel space over $\mathscr{D}'$ associated with the kernel $K'(z, w)$; similarly we have $\mathscr{H}(\mathscr{D}'')$ associated with $K''(z, w)$. Consider an element u in $\mathscr{H}(\mathscr{D}')$ which is a finite sum of the form

$$u = \sum \bar{a}_i K'_{z_i}$$

and compute its norm in the space $\mathscr{H}(\mathscr{D}')$. We find

$$\|u\|^2 = \sum\sum a_i \bar{a}_j K'(z_i, z_j).$$

In a similar way we look at an element v in $\mathscr{H}(\mathscr{D}'')$ of the form

$$v = \sum \bar{b}_j K''_{w_j}$$

to determine $\|v\|^2 = \sum\sum b_i \bar{b}_j K''(w_i, w_j)$.

Let $\mathscr{M}'$ be the space of all u in $\mathscr{H}(\mathscr{D}')$ of the form

$$u = \sum \bar{a}_i K'_{z_i}$$

where the numbers $\{z_i\}$ are in M'. This is a dense linear subspace of $\mathscr{H}(\mathscr{D}')$. In a similar way we define $\mathscr{M}''$, a dense subspace of $\mathscr{H}(\mathscr{D}'')$ associated with the set M''.

Next we define a bilinear form on the space $\mathscr{M}' \times \mathscr{M}''$ by setting, for u and v as above,

$$B(u, v) = \sum\sum \bar{a}_i \bar{b}_j \overline{f(z_i, w_j)}.$$

Our hypothesis then reads

$$|B(u, v)|^2 \leqq \|u\|^2 \, \|v\|^2$$

and it is plain that the bilinear form, first defined only for $\mathscr{M}' \times \mathscr{M}''$ admits a unique extension by continuity to all of $\mathscr{H}(\mathscr{D}') \times \mathscr{H}(\mathscr{D}'')$. This extension having been made, consider for any z in $\mathscr{D}'$ and w in $\mathscr{D}''$ the number

$$B(K'_z, K''_w)$$

and note that since the map $z \to K_z$ is conjugate analytic, as we have earlier observed, the function

$$F(z, w) = \overline{B(K'_z, K''_w)}$$

is an analytic function defined throughout $(\mathscr{D}' \times \mathscr{D}'')$.

Let us check the value of $F(z, w)$ at some pair (z, w) in $M' \times M''$; from the definition of $B(u, v)$ we find that

$$F(z, w) = f(z, w)$$

and so $F(z, w)$ is in fact an extension of the initial $f(z, w)$. The uniqueness of this extension is an immediate consequence of Lemma 1.

We pass to a further, similar theorem, due essentially to Bergman and Schiffer. [**3**].

THEOREM 9. *Let $\mathscr{D}$ be a domain in $\mathbf{C}^n$ and M a set of uniqueness in $\mathscr{D}$; let $K(z, w)$ be sesqui-analytic in $\mathscr{D} \times \mathscr{D}$ and let $f(z, w)$ be a symmetric function defined on $M \times M$ so that $f(z, w) = f(w, z)$. Suppose that for all finite sets $\{z_i\}$ in M and equally many complex numbers $\{a_i\}$ we have*

$$\left|\sum\sum a_i a_j f(z_i, z_j)\right| \leqq \sum\sum a_i \bar{a}_j K(z_i, z_j).$$

Then there exists a uniquely determined function $F(z, w)$, analytic in $\mathscr{D} \times \mathscr{D}$ and coinciding with $f(z, w)$ on $M \times M$

PROOF. The sesqui-analytic function $K(z, w)$ is clearly a positive matrix on $M \times M$ and so in view of Theorem 5 is also one on $\mathscr{D} \times \mathscr{D}$. Let $\mathscr{H}(\mathscr{D})$ be the associated reproducing kernel space, and let u be an element of that space which is a finite sum of the form

$$u = \sum \bar{a}_i K_{z_i}$$

where the points $\{z_i\}$ are in M. The set of all such u forms a linear space $\mathscr{U}$, dense in $\mathscr{H}(\mathscr{D})$ in view of the fact that M is a set of uniqueness for $\mathscr{D}$. We define a quadratic form $Q(u)$ on that subspace as follows:

$$Q(u) = \sum\sum \bar{a}_i \bar{a}_j \overline{f(z_i, z_j)}.$$

We then note that our inequality reads $|Q(u)| \leqq \|u\|^2$. From this it follows that Q is in fact extendable by continuity to a quadratic form over $\mathscr{H}(\mathscr{D})$. The assertion that $Q(u)$ is a quadratic form means that for any pair of vectors u and v and corresponding complex coefficients a and b we have

$$Q(au + bv) = a^2A + b^2C + 2abB$$

for appropriate coefficients A, B and C. Thus there is determined a well-defined function $B(u, v)$ from the formula

$$Q(au + bv) = a^2Q(u) + b^2Q(v) + 2abB(u, v)$$

and an easy computation then shows that

$$B(u, v) = (1/4)[Q(u + v) - Q(u - v)].$$

It is obvious that $B(u, v)$ is symmetric, i.e., $B(u, v) = B(v, u)$.

Suppose the vectors u and v are of norm at most 1; it follows that $\|u + v\|^2 \leqq 4$ and $\|u - v\|^2 \leqq 4$ and so

$$|B(u, v)| \leqq (1/4)[|Q(u + v)| + |Q(u - v)|] \leqq 2.$$

More generally, then, for vectors u and v not 0 in $\mathscr{H}(\mathscr{D})$,

$$|B(u, v) = \left|B\left(\frac{u}{\|u\|}, \frac{v}{\|v\|}\right)\|u\| \; \|v\| \; \right| \leqq 2 \; \|u\| \; \|v\|.$$

It should also be clear that $B(u, v)$ is a bilinear form on $\mathscr{M}$. This we would see, either from the known correspondence between quadratic and symmetric bilinear forms, as explained, for example, in Aronszajn [**2**], or here, from the fact that we have an explicit formula for $B(u, v)$, viz.

$$B(u, v) = \sum\sum \bar{a}_i\bar{b}_j\overline{f(z_i, z_j)}.$$

As in the proof of the previous theorem, then, the bilinear form B admits a unique continuation to a bilinear form defined on $\mathscr{H}(\mathscr{D}) \times \mathscr{H}(\mathscr{D})$ and which we denote by the same letter.

Next we consider the analytic function

$$F(z, w) = \overline{B(K_z, K_w)}$$

defined on $\mathscr{D} \times \mathscr{D}$ and compute its value for some pair (z, w) in $M \times M$. We find $F(z, w) = f(z, w)$ from the definition of $B(u, v)$. The uniqueness of $F(z, w)$ is of course a consequence of Lemma 1.

References

1. N. Aronszajn, *Theory of Reproducing Kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

2. N. Aronszajn, *Introduction to Hilbert Spaces*, Stillwater, Oklahoma, 1950.

3. S. Bergman and M. Schiffer, *Kernel Functions and Conformal Mappings*, Compositio Math. **8** (1951), 205–249.

4. W. F. Donoghue, Jr., *Monotone Matrix Functions and Analytic Continuation*, Grundlehren der Matematischen Wissenschaften, Band 207, Springer-Verlag, Berlin, 1974.

5. Carl H. FitzGerald, *Quadratic Inequalities and Analytic Continuation* (preliminary report).

6. H. Meschowski, *Hilbertsche Raume mit Kernfunktion*, Grundlehren der Mathematischen Wissenschaften, Band 113, Springer-Verlag, Berlin, 1962.

7. D. Widder, *The Laplace Transform*, Princeton University Press, 1947.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CA 92717