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LIE DERIVATIVES IN DIFFERENTIABLE SPACES

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Dedicated to N. Aronszajn on the occasion of his seventieth birthday.

ABSTRACT. The Lie derivative $(\mathbf{L}_X t)p$ of a tensor field t at a point p on a differentiable space $\mathbf{S}$ may not be well defined. At each point $p \in S$ there is, however, a subspace $L_p\mathbf{S} \subseteq T_p\mathbf{S}$ such that $(\mathbf{L}_X t)p$ is well defined if and only if $X(p) \in L_p\mathbf{S}$. For any differentiable space, $L_p\mathbf{S} = T_p\mathbf{S}$ for every p in the complement of a nowhere dense subspace. In case $\mathbf{S}$ is either a coherent real-analytic space or a differentiable space of polyhedral type, then $L_p\mathbf{S} = T_p\mathbf{S}$ at every $p \in S$.

Introduction. On a differentiable manifold $\mathbf{M}$ the Lie derivative $\mathbf{L}_X t$ is well defined for any differentiable vector field X and any differentiable tensor field t. This need not be true, however, if M is more generally a differentiable space having singular points (and if the covariant rank of t is positive). The purpose of this note, then, is to characterize those vector fields X on a differentiable space such that $L_X t$ is well defined for every differentiable tensor field t. In fact, for each point p of a differentiable space $\mathbf{S}$ we identify a subspace $L_p\mathbf{S}$ of the tangent space $T_p\mathbf{S}$ such that $(\mathbf{L}_X t)p$ is well defined for every t if and only if $X(p) \in L_p\mathbf{S}$.

In § 1 we review some notions about differentiable spaces from a different point of view than that of [**4**]. In §2 we give examples showing that $\mathbf{L}_X t$ need not always be well defined, characterize $L_p\mathbf{S}$, and discuss the effects on $L_p\mathbf{S}$ of weakening the differentiable structure of $\mathbf{S}$. This characterization of $L_p\mathbf{S}$ together with a general result from §1 shows that the set of points p where $(\mathbf{L}_X t)p$ is possibly not well defined is always nowhere dense in $\mathbf{S}$. In §3 we apply these results to show that every C^∞ vector field on a coherent real analytic space gives well defined Lie derivatives. In §4 we explicitly calculate $L_p\mathbf{S}$ when $\mathbf{S}$ is locally diffeomorphic to polyhedral subsets of cartesian spaces. This calculation shows that every differentiable vector field on polyhedral spaces gives well defined Lie derivatives.

Throughout the paper we use the notion of smoothness category introduced by Palais in [**5**]. For the reader's convenience we have included a short appendix at the end of §1 recapitulating the definition and several results from [**5**]. Without mention to the contrary, "smooth" will mean $\mathscr{C}$-smooth, where $\mathscr{C}$ is some smoothness category. Finally, throughout the paper $\mathbf{R}^1$, $\mathbf{R}^n$ will denote the real cartesian spaces.

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1. **Differentiable Spaces.** Let S be an arbitrary subset of some $\mathbf{R}^n$. A real-valued function on S is *smooth* (of class $\mathscr{C}$) if it has a smooth local extension about each point $p \in S$. A (reduced) *subcartesian model* of class $\mathscr{C}$ is a set $S \subseteq \mathbf{R}^n$ together with its sheaf $\mathscr{C}_s$ of smooth functions. A *subcartesian differentiable space* of class $\mathscr{C}$ is a ringed space $\mathbf{R} = (R, \mathscr{F})$ with Hausdorff base space, that is locally isomorphic via value-preserving isomorphisms to subcartesian models of class $\mathscr{C}$. A *smooth mapping* $\mathbf{R} \to \mathbf{S}$ of differentiable spaces is a value-preserving ringed-space homomorphism. (This just means that $\mathscr{F}$ is a sheaf of functions and that smooth mappings map germs by composition $g \mapsto g \circ f$!) A mapping $\varphi\colon U_\varphi \to \mathbf{R}^{n_\varphi}$, U_φ open in $\mathbf{R}$, is a *chart* for $\mathbf{R}$ if it determines an isomorphism of $(U_\varphi, \mathscr{F}|_{U_\varphi})$ with the subcartesian model $(\mathrm{Im}_\varphi, \mathscr{C}_{\mathrm{Im}\varphi})$.

If $f\colon \mathbf{R} \to \mathbf{S}$ is a smooth mapping of differentiable spaces, φ is a chart about $p \in R$, and θ is a chart about $fp \in S$, then there is a smooth map (in the usual sense) $F\colon \mathbf{R}^{n_\varphi} \supseteq U \to \mathbf{R}^{n_\theta}$ extending $\theta \circ f \circ \varphi^{-1}$ in a neighborhood U of φp. For if x^i is a coordinate function on $\mathbf{R}^{n_\theta}$, then $x^i \circ \theta \circ f$ is smooth near p, and there is a smooth real-valued function F_i defined in a neighborhood of φp such that $F_i \circ \varphi = x^i \circ \theta \circ f$ near p. The smooth map $F := (F_1, \ldots, F_{n_\theta})$ then extends $\theta \circ f \circ \varphi^{-1}$ near φp. In particular, if φ and θ are two charts in a neighborhood of $p \in R$, then there are smooth connecting maps $f\colon \mathbf{R}^{n_\varphi} \supseteq U \to \mathbf{R}^{n_\theta}$ and $g\colon \mathbf{R}^{n_\theta} \supseteq U' \to \mathbf{R}^{n_\varphi}$ such that $\theta = f \circ \varphi$ and $\varphi = g \circ \theta$ near p. If $n_\varphi = n_\theta$, then f and g can be chosen so that $f|_U$ and $g|_{fU}$ are smooth diffeomorphisms for some neighborhood U of φp (see [**4**].)

At each point p of a differentiable space $\mathbf{S}$, the ring $\mathscr{F}_{\mathbf{S},p}$ is quasi-local, having $\mathfrak{m}_{\mathbf{S},p} := \{f \in \mathscr{F}_{\mathbf{S},p} / f(p) = 0\}$ for its maximal ideal. If $f\colon \mathbf{R} \to \mathbf{S}$ is a smooth map, then for each $p \in R$ the map $f^p\colon \mathscr{F}_{\mathbf{S},fp} \to \mathscr{F}_{\mathbf{R},p}$ induces a linear mapping of the vector spaces

$$f_p^*\colon \mathfrak{m}_{\mathbf{S},fp}/\mathfrak{m}^2_{\mathbf{S},fp} \to \mathfrak{m}_{\mathbf{R},p}/\mathfrak{m}^2_{\mathbf{R},p}.$$

Define the *algebraic tangent space* of $\mathbf{S}$ at q to be the space

$$\mathrm{t}_q\mathbf{S} := (\mathfrak{m}_{\mathbf{S},q}/\mathfrak{m}^2_{\mathbf{S},q})^*$$

of real-valued derivations on $\mathscr{F}_{\mathbf{S},p}$. For each $p \in R$ we then the dual mapping $f_{*p}\colon \mathrm{t}_p\mathbf{R} \to \mathrm{t}_{fp}\mathbf{S}$, and the correspondence $f \mapsto f_{*p}$ is a functor.

Let φ be a chart for $\mathbf{S}$ about p and define the *tangent space* of $\mathbf{S}$ at q to be

$$T_q\mathbf{S} := \varphi_{*q}^{-1}\, T_{\varphi q}\mathbf{R}^{n_\varphi},$$

where $T_{\varphi q}\mathbf{R}^{n_\varphi} \subseteq \mathrm{t}_{\varphi q}\mathbf{R}^{n_\varphi}$ is the usual tangent space. Using connecting maps, one sees easily that this definition of $T_q\mathbf{S}$ is independent of the chart chosen, and that $f_{*p}(T_p\mathbf{p}) \subseteq T_{fp}\mathbf{S}$.

A straightforward application of the implicit function theorem (valid in all smoothness categories) shows that if $\dim T_p\mathbf{S} = n$, then $\mathbf{S}$ admits charts about p taking values in $\mathbf{R}^n$ but no charts taking values in $\mathbf{R}^m$, $m < n$. If $\mathbf{S}$ is a model in $\mathbf{R}^n$, define $\mathfrak{n}_s \subseteq \mathscr{C}_{\mathbf{R}^n}$ to be the ideal sheaf of functions vanishing on S. Then for each $p \in S$,

$$\iota_* T_p\mathbf{S} = \{v \in T_p\mathbf{R}^n | \ v \cdot f = 0, \text{ for all } f \in \mathfrak{n}_{\mathbf{S},p}\}, \text{ where } \iota\colon S \subsetneq \mathbf{R}^n.$$

Another characterization of the tangent spaces that is sometimes useful is the following. For each $p \in S$, and each chart φ at p, define $\mathfrak{M}(\varphi, p)$ to be the set of germs at φp of submanifolds $\mathbf{M}$ in $\mathbf{R}^{n_\varphi}$ such that for some neighborhood U in U_φ of p, φU is contained in M. Then for each $\mu \in \mathfrak{M}(\varphi, p)$, $T_{\varphi p}\,\mu$ has obvious meaning, and one verifies quite easily that

$$(*) \qquad \varphi_* T_p\mathbf{S} = \bigcap_{\mu \in \mathfrak{M}(\varphi,p)} T_{\varphi p}\mu \subseteq \mathfrak{t}_{\varphi p}\mathbf{R}^{n_\varphi}.$$

Applying φ_{*p}^{-1} to each side of $(*)$ gives the desired characterization.

It is well known that $T_p\mathbf{M} = \mathfrak{t}_p\mathbf{M}$ when $\mathbf{M}$ is a differentiable manifold of class C^∞ or C^ω. The usual proof **[10]** requires that the smoothness category under consideration be stable under differentiation and indefinite integration. The result holds, however, for many smoothness categories not having these stability properties.

PROPOSITION 1. *For any smoothness category $\mathscr{C} \subseteq C^\infty$ and any differentiable space $\mathbf{S}$ of class $\mathscr{C}$, $\mathfrak{t}_p\mathbf{S} = T_p\mathbf{S}$ for all $p \in S$.*

PROOF. First consider the special case $\mathbf{S} = (\mathbf{R}^n, \mathscr{C}_{\mathbf{R}^n})$. Let $\mathfrak{p}$, $\mathfrak{c}$, and $\mathfrak{e}$ denote the maximal ideals of $\mathbf{R}[x^1, \dots, x^n]_p$, $\mathscr{C}_{\mathbf{R}^n,p}$, and $C^\infty_{\mathbf{R}^n,p}$, respectively. Then

$$\mathbf{R}[x^1, \dots, x^n]_p \subseteq \mathscr{C}_{\mathbf{R}^n,p} \subseteq C^\infty_{\mathbf{R}^n,p}$$

and

$$\mathfrak{p}^k \subseteq \mathfrak{c}^k \subseteq \mathfrak{e}^k$$

for all powers k. The filtration $\{\mathfrak{e}^k/k \in \mathbf{N}\}$ on $C^\infty_{\mathbf{R}^n,p}$ induces the filtration $\{\mathfrak{e}^k \cap \mathscr{C}_{\mathbf{R}^n,p}/k \in \mathbf{N}\}$ on $\mathscr{C}_{\mathbf{R}^n,p}$, and these induce the filtration $\{\mathfrak{p}^k | \ k \in \mathbf{N}\}$ on $\mathbf{R}[x^1, \dots, x^n]_p$. Completing with respect to these filtrations we obtain

$$\hat{\mathfrak{p}} \subseteq \hat{\mathfrak{c}} \subseteq \hat{\mathfrak{e}}$$

and

$$\hat{\mathfrak{p}}^2 \subseteq \hat{\mathfrak{c}}^2 \subseteq \hat{\mathfrak{e}}^2,$$

where $\hat{\mathfrak{p}} = \hat{\mathfrak{e}}$ is the maximal ideal of $\mathbf{R}[\![x^1, \dots, x^n]\!]_p$, and $\hat{\mathfrak{p}}^2 = \hat{\mathfrak{p}}^2 = \hat{\mathfrak{e}}^2 = \mathfrak{e}^2$. Thus

$$\frac{\mathfrak{c}}{\mathfrak{c}^2} \approx \frac{\hat{\mathfrak{c}}}{\hat{\mathfrak{c}}^2} = \frac{\hat{\mathfrak{p}}}{\hat{\mathfrak{p}}^2} \approx \frac{\mathfrak{p}}{\mathfrak{p}^2} \approx (\mathbf{R}^n)^*.$$

Therefore $t_p\mathbf{R}^n = T_p\mathbf{R}^n$.

Now let φ be a chart for $\mathbf{S}$ about p satisfying $n_\varphi = \dim T_p\mathbf{S}$. Then $T_p\mathbf{S} = t_p\mathbf{S}$ follows from the commutativity of the diagram

$$\begin{array}{ccc} T_{\varphi p}\mathbf{R}^{n_\varphi} & = & t_{\varphi p}\mathbf{R}^{n_\varphi} \\ \wr\| & & \uparrow \varphi_{*p} \\ T_p\mathbf{S} & \xrightarrow{\subseteq} & t_p\mathbf{S}. \end{array}$$

Q.E.D.

The disjoint union $\bigcup_{p\in S} T_p\mathbf{S}$ inherits a differentiable structure (of class $\mathscr{C}$ when $\mathbf{S}$ is of class $\mathscr{C}^{+1}$) via the injections

$$\varphi_*: \bigcup_{p\in U_\varphi} T_p\mathbf{S} \to \bigcup_{p\in U_\varphi} T_{\varphi p}\mathbf{R}^{n_\varphi} \subseteq \mathbf{R}^{2n_\varphi}.$$

We denote this differentiable space by $T\mathbf{S}$, the *tangent pseudobundle* of $\mathbf{S}$, If $\mathbf{p}$, $\mathbf{S}$, and $f: \mathbf{p} \to \mathbf{S}$ are of class $\mathscr{C}^{+1}$, then the tangential map $f_*: T\mathbf{p} \to T\mathbf{S}$ is smooth (of class $\mathscr{C}$). The functor $f \mapsto f_*$ extends the classical tangent functor on differentiable manifolds.

The tensor powers $\otimes^n T\mathbf{S}$ are well defined and have the properties one expects (see [**4**]). The mapping f above induces the mapping $\otimes^n f_*$, which by abuse of notation we often denote f_*. Continuous, differentiable, and smooth (of any class $\mathscr{D} \supseteq \mathscr{C}$.) contravariant tensor fields are defined in the obvious way. If Y such a field, if $p \in S$, and if φ is a chart about p, then there exist contravariant tensor fields $\tilde{Y}$ of the same class as Y defined locally in $\mathbf{R}^{n_\varphi}$ such that $\tilde{Y}(\varphi q) = \varphi_*(Yq)$ for all q in a neighborhood of p. Any such $\tilde{Y}$ is called a *local representative* (*near* φp) *of* Y *relative to* φ.

A differentiable (smooth) covariant tensor field of rank m is a differentiable (smooth) map $\otimes^m T\mathbf{S} \to \mathbf{R}$ that is linear on fibers. A differentiable (smooth) mixed tensor field of type (m, n) is a differentiable (smooth) map $\otimes^m T\mathbf{S} \to \otimes^n T\mathbf{S}$ that preserves fibers and is linear along fibers. Making the convention $\otimes^0 T\mathbf{S} := \mathbf{S} \times \mathbf{R}$ and $\otimes^0 f_* := f \times \mathrm{Id}$, we can identify covariant tensors of rank m with mixed tensors of type $(m, 0)$, and contravariant tensors of rank m with mixed tensors of type $(0, \mathrm{m})$. We denote the bigraded module of differentiable (smooth) mixed tensor fields on $\mathbf{S}$ by $\mathscr{T}^{\mathrm{diff}}(\mathbf{S})$ ($\mathscr{T}(\mathbf{S})$) and the corresponding sheaf by $\mathscr{T}_{\mathbf{S}}{}^{\mathrm{diff}}$($\mathscr{T}_{\mathbf{S}}$). If $t \in \mathscr{T}^{\mathrm{diff}}(\mathbf{S})$ ($\mathscr{T}(\mathbf{S})$) is of type (m, n) and if φ is a chart about $p \in S$, then there exists a locally defined $\tilde{t} \in \mathscr{T}^{\mathrm{diff}}(\mathbf{R}^{n_\varphi})(\mathscr{T}(\mathbf{R}^{n_\varphi}))$ of the same type and a neighborhood $U \subseteq U_\varphi$ of p such that for all $q \in U$ and $Y \in \otimes^m T_q\mathbf{S}$, $\varphi_*(tY) = \tilde{t}(\varphi_* Y)$. Again, $\tilde{t}$ is called a local representative of t relative to φ.

DEFINITION. Assuming $\mathbf{S}$ to be of class $\mathscr{C}^{+1}$, denote the sheaf of vector fields of class $\mathscr{C}$ by $\mathscr{X}_{\mathbf{S}}$, and define

$$\check{T}_p\mathbf{S} := \{X(p)/X \in \mathscr{X}_{\mathbf{S},p}\}.$$

Other classes of vector fields will be appropriately denoted, for example the continuous fields by $\mathscr{X}_{\mathbf{S}}^{\mathrm{ctn}}$, the differentiable fields by $\mathscr{X}_{\mathbf{S}}^{\mathrm{diff}}$, the C^∞ fields by $\mathscr{X}_{\mathbf{S}}^{\mathscr{E}}$, etc., and the corresponding tangential subspaces will be denoted $\check{T}_p^{\mathrm{ctn}}\mathbf{S}$, $\check{T}_p^{\mathrm{diff}}\mathbf{S}$, $\check{T}_p^{\mathscr{E}}\mathbf{S}$, etc.

DEFINITION. A point $p \in S$ is said to be a *regular point* of $\mathbf{S}$ if $\dim T_q\mathbf{S}$ is constant in a neighborhood of p. The set of regular points in $\mathbf{S}$ is denoted Reg $\mathbf{S}$. Its complement in S, the set of *singular points* of $\mathbf{S}$, is denoted Sing $\mathbf{S}$.

PROPOSITION 2. *A point $p \in S$ is a regular point of* $\mathbf{S}$ *if and only if $\check{T}_p^{\mathrm{ctn}}\mathbf{S} = T_p\mathbf{S}$. The set of regular points* Reg $\mathbf{S}$ *is open and dense in S. If $p \in$* Reg $\mathbf{S}$, *then for any smoothness class $\mathscr{D} \supseteq \mathscr{C}$, $T_p\mathbf{S} = \check{T}_p^{\mathscr{D}}\mathbf{S}$.*

PROOF. The integer-valued functions $p \mapsto \dim T_p\mathbf{S}$ and $p \mapsto \dim \check{T}_p^{\mathrm{ctn}}\mathbf{S}$ are upper and lower semicontinuous, respectively, and $\dim T_p\mathbf{S} \geqq \dim \check{T}_p^{\mathrm{ctn}}\mathbf{S}$ for all $p \in S$. Thus if $T_p\mathbf{S} = \check{T}_p^{\mathrm{ctn}}\mathbf{S}$, then for all q in some neighborhood of p, $\dim T_p\mathbf{S} \geqq \dim T_q\mathbf{S} \geqq \dim \check{T}_q^{\mathrm{ctn}}\mathbf{S} \geqq \dim \check{T}_p^{\mathrm{ctn}}\mathbf{S} = \dim T_p\mathbf{S}$. Therefore $p \in$ Reg $\mathbf{S}$. On the other hand, if $p \in$ Reg $\mathbf{S}$, where $\dim T_p\mathbf{S} = n$, and if φ is a chart about p with $n_\varphi = n$, then for every q in some neighborhood of p, $\varphi_{*q}\colon T_q\mathbf{S} \to T_{\varphi q}\mathbf{R}^n$ is bijective. For these q, define

$$X_i(q) := \varphi_{*q}^{-1}((\partial/\partial x^i)\varphi q) \in T_q\mathbf{S},\ i = 1, \dots, n.$$

Each X_i is $\mathscr{D}$-smooth near p for any $\mathscr{D} \supseteq \mathscr{C}$, and

$$\check{T}_p^{\mathrm{ctn}}\mathbf{S} \supseteq \check{T}_p^{\mathscr{C}}\mathbf{S} \supseteq \operatorname{span}\{X_i(p)/\, i = 1, \dots, n\} = T_p\mathbf{S} \supseteq \check{T}_p^{\mathrm{ctn}}\mathbf{S}.$$

Clearly, then, Reg $\mathbf{S}$ is open in S. Suppose now that Reg $\mathbf{S}$ were not dense in S. Let p be an interior point of Sing $\mathbf{S}$, and let φ be a chart about p. Then there exists $q_1 \in (\text{Sing } \mathbf{S})^0 \cap U_\varphi$ such that $\dim T_{q_1}\mathbf{S} < \dim T_p\mathbf{S}$. Continuing inductively, we find a sequence $\{q_i\} \subseteq (\text{Sing } \mathbf{S})^0 \cap U_\varphi$ with $\dim T_{q_{i+1}}\mathbf{S} < \dim T_{q_i}\mathbf{S}$, which is absurd.

REMARKS. A detailed exposition in somewhat different language of part of the material in this section can be found in [**4**]. There, as in the classical definition of differentiable manifold, we make explicit use of connecting maps to define differentiable structures. The use of ringed spaces to define the notion of differentiable space, however, obviates the notions of maximal atlas or equivalence class of atlases, as well as the use *ad initio* of connecting maps. This theory of differentiable spaces can be broadened to include non-reduced spaces by employing non-reduced models (see [7]).

APPENDIX (Smoothness Categories). The following material is taken from [**5**]. Let $\mathscr{C}$ be a category whose objects are all open subsets of the cartesian spaces $\mathbf{R}^n$, $n \in \mathbf{N}$, whose morphisms are certain C^1-mappings, and whose binary operation is ordinary function composition. Then $\mathscr{C}$ is a *smoothness category* if the following conditions are satisfied.

(S1) For each U, $\mathscr{C}(U, \mathbf{R}^m)$ is a linear subspace of $C^1(U, \mathbf{R}^m)$ containing all constant maps.

(S2) $\mathscr{C}(\mathbf{R}^{n_1} \oplus \cdots \oplus \mathbf{R}^{n_k}, \mathbf{R}^m)$ contains all k-linear maps.

(S3) $f \in \mathscr{C}(U_1, U_2)$ if for each $p \in U_1$ there is a neighborhood $U' \subseteq U_1$ of p such that $f|_{U'} \in \mathscr{C}(U', U_2)$

(S4) If $f_i \in \mathscr{C}(U, R^{n_i})$, $i = 1, 2$, then $x \mapsto (f_1(x), f_2(x))$ belongs to $\mathscr{C}(U, R^{n_1} \oplus \mathbf{R}^{n_2})$.

(S5) If $f \in \mathscr{C}(U_1, U_2)$ and $f^{-1} \in C^1(U_2, U_1)$, then $f^{-1} \in \mathscr{C}(U_2, U_1)$.

Examples of smoothness categories are C^k for $k \in \mathbf{N}$, $\mathscr{E} := C^\infty$ and $\mathscr{A} := C^\omega$. The intersection of an arbitrary set of smoothness categories is again a smoothness category, and in particular, the intersection C^Ω of all smoothness categories is a smoothness category, the *Nash category.*

If $\mathscr{C}$ is a smoothness category, then $\mathscr{C}$ is uniquely determined among all smoothness categories by its real-valued functions, $\mathscr{C}(U, \mathbf{R})$, where U ranges over Ob $\mathscr{C}$. For each U, $C^\Omega(U, \mathbf{R})$ is the integral closure of the ring of polynomial functions on U in $C^\omega(U, \mathbf{R})$.

The mappings $f \in$ Map $\mathscr{C}$ such that $Df \in$ Map $\mathscr{C}$ constitute the maps of a smoothness category denoted $\mathscr{C}^{+1}$. If $\mathscr{C} \subseteq C^2$, then $\mathscr{C}$ together with the differentials of its maps generate a smoothness category $\mathscr{C}'$. A smoothness category $\mathscr{C}$ is said to be *differentiably stable* if $\mathscr{C} = \mathscr{C}^{+1}$, or equivalently, $\mathscr{C} = \mathscr{C}'$.

The implicit function theorem holds for any smoothness category $\mathscr{C}$: if $f \in \mathscr{C}(U_1 \times U_2, U_3)$ satisfies the usual hypotheses for the implicit function theorem, then the implicitly defined function $g\colon U_2 \to U_1$ is also $\mathscr{C}$-smooth.

2. **Characterization of Lie Vectors.** Given $X \in \mathscr{X}^{\text{diff}}_{\mathbf{S},p}$, $t \in \mathscr{T}^{\text{diff}}_{\mathbf{S},p}$ of type (l, k), and a chart φ about p, we may determine whether $((\otimes^k \varphi_*)^{-1} \circ \mathbf{L}_{\tilde{X}} \tilde{t} \circ \otimes^l \varphi_*)p$ is independent of the choices of φ, $\tilde{X}$, and $\tilde{t}$. If that is so, we define $(\mathbf{L}_X t)p$ to be this tensor. We showed in **[4]** that $(\mathbf{L}_X Y)p$ is well defined for any C^∞ vector field X and C^∞ contravariant tensor field Y (in this case $l = 0$ and $\otimes^0 \varphi_* := \varphi$), and the argument there remains valid for X and Y differentiable. We gave the following example to show, however, that $(\mathbf{L}_X t)p$ may not be well defined when t has positive covariant rank.

EXAMPLE 1. Let $S \subseteq \mathbf{R}^2$ be the union of the y-axis and all horizontal line segments of the form

$$\left[-\frac{1}{2^n}, \frac{1}{2^n}\right] \times \left\{\frac{k}{2^n}\right\}, \; n, k \in \mathbf{N},$$

and let $\iota\colon S \hookrightarrow \mathbf{R}^2$ denote the inclusion. The vector field $\partial/\partial x$ on $\mathbf{R}^2$ represents a smooth vector field on $\mathbf{S} := (S, C^\infty_S)$, and the differential form

$xdy \in \mathscr{T}(\mathbf{R}^2)$ represents the zero differential 1-form on $\mathbf{S}$. On $\mathbf{R}^2$, $\mathbf{L}_{\partial/\partial x} xdy = dy$. This represents, however, a non-zero 1-form on $\mathbf{S}$:

$$(\partial/\partial y)_{(0,0)} \in \iota_* T_{(0,0)}\mathbf{S}, \text{ and } \langle dy, \partial/\partial y\rangle_{(0,0)} = 1.$$

Definition. Let $\mathscr{L}^{\text{diff}}_{\mathbf{S},p} \subseteq \mathscr{X}^{\text{diff}}_{\mathbf{S},p}$ be the set of vector field germs X such that $(\mathbf{L}_X t)p$ is well defined for every $t \in \mathscr{T}^{\text{diff}}_{\mathbf{S},p}$, and let $\mathscr{L}_{\mathbf{S},p} = \mathscr{L}^{\text{diff}}_{\mathbf{S},p} \cap \mathscr{X}_{\mathbf{S},p}$. Define

$$L_p^{\text{diff}}\mathbf{S} := \{X(p) \mid X \in \mathscr{L}^{\text{diff}}_{\mathbf{S},p}\}$$

and

$$L_p\mathbf{S} := \{X(p) \mid X \in \mathscr{L}_{\mathbf{S},p}\}.$$

It is not difficult to prove the following (cf. [4]):

(i) $\mathscr{L}^{\text{diff}}_{\mathbf{S},p}$ is a submodule of $\mathscr{X}^{\text{diff}}_{\mathbf{S},p}$. If $\mathscr{C}$ is differentiably stable, then $\mathscr{L}_{\mathbf{S},p}$ is a Lie submodule of $\mathscr{X}_{\mathbf{S},p}$.

(ii) If $X \in \mathscr{X}^{\text{diff}}_{\mathbf{S},p}$ satisfies $X(p) = 0$, then $X \in \mathscr{L}^{\text{diff}}_{\mathbf{S},p}$.

One easily deduces from (i) and (ii) that if $X_1(p) = X_2(p)$ and $X_1 \in \mathscr{L}^{\text{diff}}_{\mathbf{S},p}$ then $X_2 \in \mathscr{L}^{\text{diff}}_{\mathbf{S},p}$, i.e., $X(p) \in L_p^{\text{diff}}\,\mathbf{S}$ implies $X \in \mathscr{L}^{\text{diff}}_{\mathbf{S},p}$. We now want to characterize the subspaces $L_p^{\text{diff}}\mathbf{S}$ and $L_p\mathbf{S}$ of $T_p\mathbf{S}$. For this we need the following.

Definition. Given a differentiable space $\mathbf{S}$ and $p \in S$, define

$$S^{(p)} := \{q \in S \mid \dim T_q\mathbf{S} = \dim T_p\mathbf{S}\}$$

and define $\mathbf{S}^{(p)}$ to be the resulting differentiable space with induced structure; for each n define

$$S^{(n)} := \{q \in S \mid \dim T_q\mathbf{S} = n\},$$

and let $\mathbf{S}^{(n)}$ be this subspace equipped with the structure induced from $\mathbf{S}$. Further, define

$$\hat{T}_p\mathbf{S} := \iota_* T_p\mathbf{S}^{(p)},$$

where $\iota: S^{(p)} \hookrightarrow S$ is the inclusion map.

Theorem 1. *Let $X \in \mathscr{X}^{\text{diff}}_{\mathbf{S},p}$. Then $(\mathbf{L}_X t)p$ is well defined for every $t \in \mathscr{T}^{\text{diff}}_{\mathbf{S},p}$ if and only if $X(p) \in \hat{T}_p\mathbf{S}$. Thus*

$$L_p\mathbf{S} = \check{T}_p^{\mathscr{C}}\mathbf{S} \cap \hat{T}_p\mathbf{S}$$

and

$$L_p^{\text{diff}}\mathbf{S} = \check{T}_p^{\text{diff}}\mathbf{S} \cap \hat{T}_p\mathbf{S}.$$

Proof. In [4] (Lemma 5.5) we established the following: If for some chart φ and some local representative X_φ of X, $(L_{X_\varphi} t_\varphi) \circ \otimes^k \varphi_*$ is independent of the choice of the local representative t_φ of t, then for any other chart θ about p, any local representative t_θ of t relative to θ, any local representative X_θ of X, and any connecting map f with $f \circ \varphi = \theta$ near p, we have

$$(\otimes^l f_* \circ \mathbf{L}_{X_\varphi} t_\varphi \circ \otimes^k \varphi_*)p = (L_{X_\theta} t_\theta \circ \otimes^k \theta_*)p \in \theta_* \otimes^l T_p\mathbf{S}.$$

We now employ this result. Let φ satisfy $n_\varphi = \dim T_p\mathbf{S}$, and choose some X_φ and t_φ relative to φ. Using coordinates to write

$$X_\varphi = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}$$

and

$$t_\varphi = \sum_{\substack{|\alpha|=l \\ |\beta|=k}} g_\alpha^\beta dx^\alpha \otimes \frac{\partial^k}{\partial x^\beta},$$

we have

$$(2) \qquad (\mathbf{L}_{X_\varphi} t_\varphi)\varphi p = \sum_{\alpha,\beta} (X_\varphi(\varphi p) \cdot g_\alpha^\beta)\left(dx^\alpha \otimes \frac{\partial^k}{\partial x^\beta}\right)p + \sum_{\alpha,\beta} g_\alpha^\beta(p) \mathbf{L}_{X_\varphi}\left(dx^\alpha \otimes \frac{\partial^k}{\partial x^\beta}\right)p.$$

Now assume $t = 0$ in some neighborhood of p, and $X(p) \in \hat{T}_p\mathbf{S}$. At each $q \in S^{(p)}$ the set $\{(dx^\alpha \otimes (\partial/\partial x^\beta))q \mid |\alpha| = l, |\beta| = k\}$ is a basis for the space of $\mathbf{R}$-linear maps $\otimes^l T_q\mathbf{S} \to \otimes^k T_q\mathbf{S}$, and this implies that each coefficient function g_α^β vanishes on $\varphi(S^{(p)})$. Thus $g_\alpha^\beta(\varphi p) = 0$ and $v \cdot g_\alpha^\beta = 0$ for each $v \in \varphi_* \hat{T}_p\mathbf{S}$. Both terms on the right of (2) then vanish and it follows from the linearity of $\mathbf{L}_X$ that $(\mathbf{L}_X t)p$ is well defined for each t. Thus $\hat{T}_p\mathbf{S} \subseteq L_p^{\mathrm{diff}}\mathbf{S}$.

Now let $v \in \check{T}_p^{\mathrm{diff}}\mathbf{S} \backslash \hat{T}_p\mathbf{S}$, and let $X \in \mathscr{X}_{\mathbf{S},p}^{\mathrm{diff}}$ satisfy $X(p) = v$. Since $\dim \hat{T}_p\mathbf{S} < \dim T_p\mathbf{S}$, there is a neighborhood $U \subseteq U_\varphi$ of p suh that $\varphi(S^{(p)} \cap U)$ lies in some proper smooth subvariety of $\mathbf{R}^n$ which we may assume to be $\{q \in \mathbf{R}^n / x^n(q) = 0\}$. Since $\dim T_q\mathbf{S} = n$ only when $q \in S^{(p)}$, $t_\varphi := x^n dx^l \wedge \dots \wedge dx^n$ is a local representative of the zero n-form on $\mathbf{S}$. Let $X_\varphi := \sum a^i(\partial/\partial x^i)$ be a local representative for X in a neighborhood of φp. Since $v \notin \hat{T}_p\mathbf{S}$, $a^n(\varphi p) \neq 0$. Thus

$$\begin{aligned}(\mathbf{L}_{X_\varphi} t_\varphi)\varphi p &= \mathbf{L}_{X_\varphi}(x^n\, dx^1 \wedge \dots \wedge dx^n)\varphi p \\ &= (a^n\, dx^1 \wedge \dots \wedge dx^n)\varphi p.\end{aligned}$$

Since $0 \neq (\partial/\partial x^l)(\varphi p) \wedge \dots \wedge (\partial/\partial x^n)(\varphi p) \in \otimes^n \varphi_* T_p\mathbf{S}$, then $\mathbf{L}_X t$ does not represent the zero n-form at p, and so $X \notin \mathscr{L}_{\mathbf{S},p}^{\mathrm{diff}}$. It follows that $v \notin L_p^{\mathrm{diff}}\mathbf{S}$.

Since $L_p\mathbf{S} = L_p^{\mathrm{diff}}\mathbf{S} \cap \check{T}_p\mathbf{S}$, and $\check{T}_p\mathbf{S} \subseteq \check{T}_p^{\mathrm{diff}}\mathbf{S}$, the theorem is proved.

COROLLARY. *If $p \in \mathrm{Reg}\, \mathbf{S}$, then for every $X \in \mathscr{X}_{\mathbf{S},p}^{\mathrm{diff}}$ and for every $t \in \mathscr{T}_{\mathbf{S},p}^{\mathrm{diff}}$, the Lie derivative $\mathbf{L}_X t(p)$ is well defined. In particular, the set of points in S at which some Lie derivative could be multivalued is nowhere dense in S.*

In Example 1 the various spaces just defined are as follows:

$$\check{T}_p\mathbf{S} = \langle(\partial/\partial x)p\rangle \text{ for each } p \in S$$

and

$$S^{(p)} = \begin{cases} y\text{-axis} & \text{if } x(p) = 0 \\ S\backslash y\text{-axis} & \text{if } x(p) \neq 0. \end{cases}$$

Thus

$$\hat{T}_p\mathbf{S} = \begin{cases} \langle(\partial/\partial y)p\rangle & \text{if } x(p) = 0 \\ \langle(\partial/\partial x)p\rangle & \text{if } x(p) \neq 0 \end{cases}$$

and

$$L_p\mathbf{S} = \begin{cases} \langle 0\rangle & \text{if } x(p) = 0 \\ \langle(\partial/\partial x)p\rangle & \text{if } x(p) \neq 0. \end{cases}$$

REMARKS ON THE WEAKENING OF STRUCTURE. (i) If $\mathscr{D}$ is some smoothness category containing $\mathscr{C}$, then it can happen that vector fields X of class $\mathscr{D}$ exist with $X(p) \notin \hat{T}_p\mathbf{S}$ even when there are no such fields of class $\mathscr{C}$.

EXAMPLE 2. Let $g \in C^2(\mathbf{R}^1)$ be the mapping $x \mapsto x^{7/3}$. Define $\mathbf{S}$ to be the graph of g equipped with the C^ω-structure induced from $\mathbf{R}^2$. Then $\check{T}_{(0,0)}\mathbf{S} = \hat{T}_{(0,0)}\mathbf{S} = 0$. The C^1-vector field $\tilde{X}(x, y) := (\partial/\partial x)(x, y) + g'(x)(\partial/\partial y)(x, y)$ on $\mathbf{R}^2$ restricts to a C^1-vector field on $\mathbf{S}$, and $X(0, 0) \notin L_{(0,0)}\mathbf{S}$. The 2-form $xdx \wedge dy$ on $\mathbf{R}^2$ represents the zero 2-form on $\mathbf{S}$, and $L_X(xdx \wedge dy)$ represents a non-zero element of $\mathscr{T}(\mathbf{S})$.

(ii) For any smoothness category $\mathscr{D} \supseteq \mathscr{C}$, a differentiable space $\mathbf{S}$ of class $\mathscr{C}$ gives rise to a canonical space of class $\mathscr{D}$, *the space obtained from* $\mathbf{S}$ *by weakening of structure*. If $\mathbf{R}$ is the resulting differentiable space of class $\mathscr{D}$, then it can happen that $\dim T_p\mathbf{R} < \dim T_p\mathbf{S}$. In the space constructed above, for example, $\dim T_{(0,0)}\mathbf{S} = 2$. If $\mathscr{D} = C^2$, then $\dim T_{(0,0)}\mathbf{R} = 1$. Similar examples show that $\dim \check{T}_p\mathbf{S}$ and $\dim \hat{T}_p\mathbf{S}$ may increase or decrease upon weakening the structure.

(iii) The following modification of Example 1 shows that weakening of structure may increase $\check{T}_p\mathbf{S}/L_p\mathbf{S}$. Let $f \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ be nowhere analytic and define $S \subseteq \mathbf{R}^2$ to be the closure in $\mathbf{R}^2$ of

$$\bigcup_{k, n \in N}\left(\text{Graph}\left(f + \frac{k}{2^n}\right)\right) \cap \left(\left[-\frac{1}{2^n}, \frac{1}{2^n}\right] \times R\right).$$

Equip S with the C^ω-structure inherited from $\mathbf{R}^2$. Then for every $p \in S$, $\dim T_p\mathbf{S} = 2$, that is, $\mathbf{S}$ is regular, and so $L\mathbf{S} = \check{T}\mathbf{S}$. Weakening the structure to C^∞ gives a differentiable space that is C^∞-diffeomorphic with that of Example 1, where $L\mathbf{S} \subsetneq \check{T}\mathbf{S}$.

3. **Analytic Spaces.** General references for this section are [3], [8], and [9].

Let $S \subseteq \mathbf{R}^n$ be a real analytic set, let $\mathfrak{n}$ be the defining ideal subsheaf of S in $\mathscr{A}_{\mathbf{R}^n} := C^{\omega}_{\mathbf{R}^n}$, and let $\mathbf{S} := (S, \mathscr{A}_S) = (S, \mathscr{A}_{\mathbf{R}^n}/\mathfrak{n})$.

PROPOSITION 3. *For each $p \in \mathbf{R}^n$, $p \in S^{(n)}$ if and only if the Jacobian extension $J_1(\mathfrak{n}_p)$ (i.e., the ideal generated by $\mathfrak{n}$ and all first derivatives of germs in $\mathfrak{n}$) is properly contained in $\mathscr{A}_{\mathbf{R}^n, p}$. In other words, $S^{(n)}$ is the real analytic set defined by the sheaf of ideals $J_1(\mathfrak{n})$.*

PROOF. Clearly $df(p) = 0$ for every $p \in S^{(n)}$ and every $f \in \mathfrak{n}_p$. Thus $J_1(\mathfrak{n}_p) \subsetneqq \mathscr{A}_{\mathbf{R}^n, p}$ for every $p \in S^{(n)}$. Conversely, if $p \notin S^{(n)}$, then $\dim T_p\mathbf{S} < n$ and there is a germ $f \in \mathfrak{n}_p$ such that $df(p) \neq 0$. It follows that $1 \in J_1(\mathfrak{n}_p)$ and therefore that $J_1(\mathfrak{n}_p) = \mathscr{A}_{\mathbf{R}^n, p}$.

PROPOSITION 4. *Let $(S, \mathscr{E}_S)$ be the space obtained from $\mathbf{S}$ by weakening of structure. Then for every $p \in S$,*

(i) $T_p\mathbf{S} = T_p(S, \mathscr{E}_S)$
(ii) $(S, \mathscr{A}_S)^{(p)} = (S, \mathscr{E}_S)^{(p)}$
(iii) $\hat{T}_p\mathbf{S} = \hat{T}_p(S, \mathscr{E}_S)$
(iv) $\check{T}^{\mathscr{E}}_p\mathbf{S} = \check{T}_p(S, \mathscr{E}_S)$.

PROOF. (i) As in Proposition 1, let $\mathfrak{a}_p$ and $\mathfrak{e}_p$ denote the maximal ideals of $\mathscr{A}_{\mathbf{R}^n, p}$ and $\mathscr{E}_{\mathbf{R}^n, p}$, respectively, and let $\mathfrak{n}_{\mathscr{E}}$ be the ideal subsheaf in $\mathscr{E}_{\mathbf{R}^n}$ that defines S. Then

$$T_p\mathbf{S} \cong \frac{\mathfrak{a}_S}{\mathfrak{a}_S^2} \cong \frac{\mathfrak{a}/\mathfrak{n}}{(\mathfrak{a}^2 + \mathfrak{n})/\mathfrak{n}} \cong \frac{\mathfrak{a}}{\mathfrak{a}^2 + \mathfrak{n}} \cong \frac{\hat{\mathfrak{a}}}{\hat{\mathfrak{a}}^2 + \hat{\mathfrak{n}}}.$$

Similarly,

$$T_p(S, \mathscr{E}_S) = \frac{\mathfrak{e}}{\mathfrak{e}^2 + \mathfrak{n}_{\mathscr{E}}}.$$

But from Theorem 3.5 and Remark 3.8 of Malgrange [3], we have $\hat{\mathfrak{n}} + \hat{\mathfrak{n}}_{\mathscr{E}}$. Since $T_p(S, \mathscr{E}_S) \subseteq T_p\mathbf{S}$, (i) is proven.

(ii) Immediate from (i).

(iii) It follows from Proposition 3 that $(S, \mathscr{A}_S)^{(p)} := \mathbf{S}^{(p)}$ is also a real analytic set, and so from part (i), $T_p(S^{(p)}, \mathscr{A}_S(p)) = T_p(S^{(p)}, \mathscr{E}_S(p)) = T_p(S, \mathscr{E}_S)^{(p)}$, that is, $\hat{T}_p\mathbf{S} = \hat{T}_p(S, \mathscr{E}_S)$.

(iv) Trivial.

THEOREM 2. *Let $\mathbf{S} = (S, \mathscr{A}_S)$ be a coherent real analytic set. Then $\check{T}^{\mathscr{E}}\mathbf{S} \subseteq \hat{T}\mathbf{S}$. (From Theorem 1 it then follows that $(\mathbf{L}_X t)p$ is well defined for every $p \in S$, $t \in \mathscr{T}^{\mathrm{diff}}_{\mathbf{S}, p}$, and $X \in \mathscr{X}^{\mathscr{E}}_{\mathbf{S}, p}$).*

PROOF. Since $\mathscr{A}_S = \mathscr{A}^{\mathbf{R}^n/\mathfrak{n}}$ is coherent by hypothesis, so is $\mathfrak{n}$. Thus there exist a neighborhood U of p in $\mathbf{R}^n$ and functions $g_i \in \mathscr{A}(U)$, i $= 1, \ldots, m$, such that for every $q \in U$, the germs $[g_1]_q, \ldots, [g_m]_q$ generate $\mathfrak{n}_q$. Without loss of generality we may assume that $\dim T_p\mathbf{S} = n$, so that $S^{(p)} = S^{(n)} = V(J_1(\mathfrak{n}))$. Then of course $g_i(q) = 0$ for all i and for all $q \in S^{(n)} \cap U$, and for all $v \in T_q\mathbf{R}^n$, $v \cdot g_i = 0$.

Given pointwise linearly independent vector fields $Y_1, \ldots, Y_n$ on U of class C^∞, define

$$G_{(Y_1,\ldots,Y_n)} = \sum_{i=1}^{m} (g_i)^2 + \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_j \cdot g_i)^2.$$

Then for any such $(Y_1, \ldots, Y_n)$, $G_{(Y_1,\ldots,Y_n)}S^{(p)} \cap U = 0$. Conversely, if $q \in U \backslash S^{(p)}$, then (1) some $f \in \mathfrak{n}^p$ satisfies $f(p) \neq 0$ if $p \in S \cap U$, in which case $g_i(q) \neq 0$ for some i, or (2) there is a germ $f \in \mathfrak{n}_q$ such that $df(q) \neq 0$ in case $q \in U \cap (S \backslash S^{(p)})$, whereupon $(Y_j \cdot g_i)q \neq 0$ for some i and j. In either case, $G_{(Y_1,\ldots,Y_n)}(q) \neq 0$.

Thus $S^{(p)} \cap U$ is precisely the set of zeros in U of $G_{(Y_1,\ldots Y_n)}$, independent of the choice of $Y_1, \ldots, Y_n$.

Now if $Y_1, \ldots, Y_n$ are chosen to be analytic, then $G_{(Y_1,\ldots,Y_n)}$ is analytic. Thus for each compact neighborhood K of p in U, $G_{(Y_1,\ldots,Y_n)}$ satisfies some Łojasiewicz inequality:

$$G_{(Y_1,\ldots,Y_n)}(q) \geqq c_{K,G}(\text{distance}(q, S^{(p)} \cap U))^k, \text{ some } k \in \mathbf{N}, c_{K,G} > 0$$

This implies in particular that $(Y)^k G_{(Y_1,\ldots,Y_n)}(p) \neq 0$ for any C^∞ vector field germ Y at p satisfying $Y(p) \notin \hat{T}_p\mathbf{S}$. Clearly then, for any $Z_1, \ldots, Z_n \in \mathscr{X}^{\mathscr{E}}_{\mathbf{R}^n, p}$ whose k-jets are sufficiently close to the k-jets of $Y_1, \ldots, Y_n$, respectively, the same condition is maintained: $(Y)^k G_{(Z_1,\ldots,Z_n)}(p) \neq 0$. But of course the sets of jets satisfy $\{j^kX / X \in \mathscr{X}^{\mathscr{A}}_{\mathbf{R}^n, p}\} = \{j^kX \mid X \in \mathscr{X}^{\mathscr{A}}_{\mathbf{R}^n, p}\}$ for any $k \in \mathbf{N}$. Thus for every choice of $Z_1, \ldots, Z_n \in \mathscr{X}^{\mathscr{A}}_{\mathbf{R}^n, p}$ with the $Z_i(p)$ linearly independent, $(Y)^k F_{(Z_1,\ldots,Z_n)}(p) \neq 0$.

Now assume contrary to our assertion that there exists $X \in \mathscr{X}^{\mathscr{E}}_{\mathbf{S}, p}$ with $X(p) \notin \hat{T}_p\mathbf{S}$. Since $X(p) \neq 0$, there is a neighborhood U' of p in U with a C^∞ coordinate system $x^1, \ldots, x^n$ such that $X|_{U'} = [\partial/\partial x^1|_{S \cap U'}]_p$ and $U' \cap S^{(p)} \subseteq \{q \in U' | x^1(q) = 0\}$. Define $Y_j = \partial/\partial x^j \in \mathscr{X}^{\mathscr{E}}_{\mathbf{R}^n}(U')$, $j = 1, \ldots, n$.

Then

$$(*) \qquad \left(\frac{\partial}{\partial x^1}\right)^k g_i \bigg|_{S \cap U'} = 0, \text{ for all } k \in \mathbf{N} \text{ and all } j.$$

Of course

$$\left(\frac{\partial}{\partial x^1}\right)^k Y_j g_i(p) = Y_j \left(\frac{\partial}{\partial x^1}\right)^k g_i(p).$$

Since $T^{\mathscr{E}}_p(S, \mathscr{E}_S) = T_p\mathbf{S}$ $(= \mathbf{R}^n)$ by Proposition 4, $Y_j(\partial/\partial x^1)^k g_i(p) = 0$, and it follows that $(\partial/\partial x^1)^k G_{(Y_1,\ldots,Y_n)} = 0$ for all $k \in \mathbf{N}$, a contradiction. We conclude that there is no $X \in \mathscr{X}^{\mathscr{E}}_{\mathbf{S}, p}$ with $X(p) \notin \hat{T}_p\mathbf{S}$.

COROLLARY. *Let $S \subseteq \mathbf{R}^n$ be a coherent real algebraic set. Then for any differentiably stable smoothness category $\mathscr{C}$, $\check{T}^{\mathscr{E}}_p(S, \mathscr{C}_S) \supseteq \hat{T}_p(S, \mathscr{C}_S) = \hat{T}_p(S, \mathscr{E}_S)$.*

PROOF. If we write $\mathbf{S} = (S, \mathscr{C}_S)$, then Proposition 4 holds for this $\mathbf{S}$ because $\mathfrak{n}_{\mathscr{E}} = \mathfrak{n}_{\mathscr{C}} \cdot \mathscr{E}_{\mathbf{R}^n}$, so that $\hat{\mathfrak{n}}_{\mathscr{E}} = \hat{\mathfrak{n}}_{\mathscr{C}}$. Observe that $(S, \mathscr{A}_S)$ is also coherent, so applying the theorem to $(S, \mathscr{A}_S)$ and applying Proposition 4 again to the pair $(S, \mathscr{A}_S)$, $(S, \mathscr{E}_S)$ gives the desired result.

REMARKS. The hypothesis of coherence in the above propositions seems somewhat artificial. An attractive conjecture is that for arbitrary Whitney-regular stratified spaces of class C^∞, any C^∞ vector field germ X satisfies $X(p) \in \hat{T}_p\mathbf{S}$. The results of the next section also support this conjecture for a substantially different type of Whitney-regular stratified spaces. An example of a stratified space that is not Whitney-regular and where not all vector fields act in a single-valued fashion is the following:

$$S \coloneqq \{(0, 0)\} \cup \bigcup_{n=1}^{\infty} ([0, 1/n] \times \{1/n\}) \subseteq \mathbf{R}^2$$

equipped with the induced C^∞ structure.

4. **Differentiable Polyhedra.** In this section we establish the well-definedness of Lie derivatives on spaces locally diffeomorphic with polyhedral subsets of cartesian spaces. These spaces have played an instrumental role in the theory of Bessel potentials [**1**].

DEFINITION. *A polyhedral model* of class $\mathscr{C}$ is a polyhedral subset $P \subseteq \mathbf{R}^n$ together with its structure sheaf $\mathscr{C}_P$. *A differentiable polyhedron* (or *differentiable space of polyhedral type*) of class $\mathscr{C}$ is a differentiable space that is locally isomorphic to polyhedral models of class $\mathscr{C}$.

The theorem stated below is practically obvious if the polyhedral models can be chosen to be in general position. In this case one sees clearly that $\check{T}_p\mathbf{S} = \hat{T}_p\mathbf{S}$ for every $p \in S$. If the models are however not in general position, then this equality may not hold. For example, consider the set

$$S \coloneqq (x\text{-axis}) \cup (\text{closed left half-plane}) \subseteq \mathbf{R}^2$$

equipped with the induced C^∞-structure. Then $\hat{T}_0\mathbf{S} = \operatorname{span}\{\partial/\partial x|_0, \partial/\partial y|_0\}$ while $\check{T}_0\mathbf{S} = \operatorname{span}\{\partial/\partial x|_0\}$.

The notion of differentiable polyhedron defined here is apparently more restrictive than the notion of triangulated space employed, for instance, in [**2**]. Therefore the results of the previous section do not seem to follow via the theorem of Łojasiewicz from the theorem below.

THEOREM 3. *On differentiable polyhedra, Lie differentiation*

$$\mathbf{L}\big|_p : \mathscr{X}^{\text{diff}}_{\mathbf{S},p} \times \mathscr{T}^{\text{diff}}_{\mathbf{S},p} \to \otimes\, T_p\mathbf{S}$$

is well defined at every $p \in S$

Proof. We shall show that $\check{T}_p\mathbf{S} \subseteq \hat{T}_p\mathbf{S}$ for every $p \in S$. The theorem is of a purely local nature, and so we restrict attention to polyhedral models $\mathbf{P} \subseteq \mathbf{R}^n$ with $p \in P$ and $\dim T_p\mathbf{P} = n$. If $n = 0$, then obviously $\check{T}_p\mathbf{P} = \hat{T}_p\mathbf{P}$. We now proceed by induction on n. Let $n \in \mathbf{N}$ and suppose the theorem holds for all differentiable polyhedra $\mathbf{S}$ at points $p \in S$ with $\dim T_p\mathbf{S} < n$. Choose a triangulation of P, and let B be a closed ball centered at p with radius small enough so that $B \cap (P \backslash \text{star}(p)) = \varnothing$. Then $\mathbf{P} \cap \partial\mathbf{B}$ is a differentiable polyhedron of class $\mathscr{C}$ with maximum tangential dimension $n - 1$, and $P \cap B$ is the cone over $P \cap \partial B$ with vertex at p.

We now establish two formulae from which the theorem immediately follows:

(i) $\hat{T}_p\mathbf{P} = \langle \{p\} \cup (P \cap \partial B)^{(n-1)} \rangle_p$, where $\langle S \rangle_p$ denotes the affine span of $S \subseteq \mathbf{R}^n$ equipped with the vector space structure centered at p, and where $(P \cap \partial B)^{(n-1)}$ is the set of $q \in P \cap \partial B$ such that $\dim T_q(\mathbf{P} \cap \partial\mathbf{B}) = n - 1$.

(ii) $$\check{T}_p\mathbf{P} = \bigcap_{q \in P \cap \partial B} \langle \{p\} \cup \check{T}_q(\mathbf{P} \cap \partial\mathbf{B}) \rangle_p.$$

For convenience we have made the natural identification between $T_q\mathbf{R}^n$ and the n-dimensional vector space centered at q.

To prove the theorem, let $q \in (P \cap \partial B)^{(n-1)}$. Then $\hat{T}_q(\mathbf{P} \cap \partial\mathbf{B}) = T_q(\mathbf{P} \cap \partial\mathbf{B})^{(n-1)}$, and by the induction hypothesis, $\check{T}_q(\mathbf{P} \cap \partial\mathbf{B}) \subseteq \hat{T}_q(\mathbf{P} \cap \partial\mathbf{B})$. Thus

$$\begin{aligned}
\check{T}_p\mathbf{P} &\subseteq \bigcap_{q \in (P \cap \partial B)^{(n-1)}} \langle \{p\} \cup \check{T}_q(\mathbf{P} \cap \partial\mathbf{B}) \rangle_p && \text{(from (ii))}\\
&\subseteq \bigcap_{q \in (P \cap \partial B)^{(n-1)}} \langle \{p\} \cup \hat{T}_q(\mathbf{P} \cap \partial\mathbf{B} \rangle_p \\
&\subseteq \langle \{p\} \cup (P \cap \partial B)^{(n-1)} \rangle_p \\
&= \hat{T}_p\mathbf{P} && \text{(from (i))}
\end{aligned}$$

Thus $L_p\mathbf{P} = \check{T}_p\mathbf{P}$.

To prove (i), let $\rho\colon \mathbf{R}^n\backslash\{p\} \to \partial B$ be the radial projection, and note that ρ is of class $\mathscr{C}$. For any $q \in B\backslash\{p\}$ there is a neighborhood U of q in P that is $\mathscr{C}$-diffeomorphic to $\mathbf{R} \times \rho(U)$. Since for any two differentiable spaces $\mathbf{R}$ and $\mathbf{S}$ we have a natural isomorphism (**[4]**)

(*) $$T_{(p,q)}(\mathbf{R} \times \mathbf{S}) \approx T_p\mathbf{R} \oplus T_q\mathbf{S},$$

then

$$\dim\ T_q\mathbf{P} = 1 + \dim\ T_{\rho q}(\mathbf{P} \cap \partial\mathbf{B}).$$

Thus

$$P^{(p)} = \mathrm{cone}_p((P \cap \partial B)^{(n-1)})$$

and so

$$\hat{T}_p\mathbf{P} = T_p\mathbf{P}^{(p)} = \langle \{p\} \cup (P \cap \partial B)^{(n-1)} \rangle_p.$$

To prove (ii), let $\tau_{q,t}$ be the translation in $\mathbf{R}^n$ that sends q into $tq + (1 - t)p$, $q \in \mathbf{R}^n$, $t \in \mathbf{R}^1$. Note that (*) implies

$$(\substack{*\\*}) \qquad \check{T}_{(p,q)}(\mathbf{R} \times \mathbf{S}) \approx \check{T}_p\mathbf{R} \oplus \check{T}_q\mathbf{S}.$$

Then for all $q \in P \cap \partial B$ and $t \in (0, 1]$,

$$\begin{aligned}\check{T}_{tq+(1-t)p}\mathbf{P} &= (\tau_{q,t})_*\check{T}_q(\mathbf{P} \cap \partial\mathbf{B}) + \text{ kern } \rho_{*(tq+(1-t)p)} \\ &= \langle \{p\} \cup (\tau_{q,t})_*\, \check{T}_q(\mathbf{P} \cap \partial\mathbf{B})\rangle_{tq+(1-t)p} \\ &= \langle \{p\} \cup \check{T}_q(\mathbf{P} \cap \partial\mathbf{B})\rangle_{tq+(1-t)p}.\end{aligned}$$

For any differentiable space $\mathbf{S}$ and any net $\{q\}$ in $\mathbf{S}$ converging to p, $\check{T}_p\mathbf{S} \subseteq \lim_{q\to p} \check{T}_q\mathbf{S}$ whenever this limit exists. Thus

$$\check{T}_p\mathbf{P} \subseteq \langle \{p\} \cup \check{T}_q(\mathbf{P} \cap \partial\mathbf{B})\rangle_p \text{ for every } q \in P \cap \partial\mathbf{B},$$

i.e.,

$$\check{T}_p\mathbf{P} \subseteq \bigcap_{q \in P \cap \partial B} \langle \{p\} \cup \check{T}_q(\mathbf{P} \cap \partial\mathbf{B})\rangle_p.$$

On the other hand, if $X(p) \in \bigcap_{q \in P \cap \partial B} \langle \{p\} \cup \check{T}_q(\mathbf{P} \cap \partial\mathbf{B}\,)\rangle_p$,

then for every $q \in P \cap \partial B$

$$X(q) := (\tau_{q,0}^{-1})_*X(p) \in \langle \{p\} \cup \check{T}_q(\mathbf{P} \cap \partial\mathbf{B})\rangle_p,$$

hence for every $t \in [0, 1]$

$$X(tq + (1 - t)p) := (\tau_{q,t}^{-1})_*X(p) \in \langle \{p\} \cup T_q(\mathbf{P} \cap \partial\mathbf{B})\rangle_p.$$

Then the constant vector field on $\mathbf{R}^n$

$$\tilde{X}(q) := (\tau_{q,t}^{-1})_*X(p)$$

extends X, proving that $X(p) \in \check{T}_p\mathbf{P}$.

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