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THE C^*-ALGEBRA OF THE ELLIPTIC BOUNDARY PROBLEM

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0. **Introduction.** Let $\mathbf{R}_+^{n+1} = \{x = (x_0, \ldots, x_n): x_0 > 0\}$, and $\partial\mathbf{R}_+^{n+1} = \{x_0 = 0\}$. Consider unbounded differential operators L of $\mathfrak{H} = L^2(\mathbf{R}_+^{n+1})$ given by an expression $(a) = \sum_{|\alpha|\leqq N} a_\alpha D^\alpha$ over $\mathbf{R}_+^{n+1}$ and a set (b) of boundary expressions $(b_j) = \sum_{|\alpha|\leqq N_j} b_{j,\alpha} D^\alpha$, $N_j < N$, $j = 1, \ldots, m$. L is defined by (a), in $\operatorname{dom} L = \{u \in \mathfrak{H}_N: (b)u = 0\}$, with the L^2-Sobolev space $\mathfrak{H}_N = \mathfrak{H}_N(\mathbf{R}_+^{n+1})$. General assumptions: $a_\alpha^{(\beta)} \in CS(R_+^{n+1})$ $b_{j,\alpha}^{(\beta)} \in CS(\partial\mathbf{R}_+^{n+1})$, with the two C^*-function algebras over $\mathbf{R}_+^{n+1}$ and its boundary generated by $\lambda(x) = (1 + x^2)^{-1/2}$ and $s_j(x) = x_j\lambda(x)$, $j = 0, \ldots, n$, respectively.

Examples are the operators Δ_d and Δ_n, formed with the Laplace operator $(a) = \Delta$, and the Dirichlet and Neumann condition, $(b) = 1$, and $(b) = \partial/\partial x_0$, respectively. Δ_d and Δ_n are known to be negative self-adjoint operators of $\mathfrak{H}$, so that all operators of (0.1), below, are well defined bounded operators of $\mathfrak{H}$.

$$\begin{aligned} &\Lambda_d = (1 - \Delta_d)^{-1/2},\ \Lambda_n = (1 - \Delta_n)^{-1/2},\ S_d = D_0\Lambda_d, \\ &S_n = D_0\Lambda_n,\ S_{j,d} = D_j\Lambda_d,\ S_{n,j} = D_j\Lambda_n,\ j = 1, \ldots, n. \end{aligned} \tag{0.1}$$

The C^*-algebras generated by (taking operator norm closure in $\mathfrak{L}(\mathfrak{H})$ of the finitely generated algebra of the operators) (0.1), (or (0.1) together with the multiplication operators $a(M): \mathfrak{H} \to \mathfrak{H}$, defined by $(a(M)u)(x) = a(x)u(x)$, $x \in \mathbf{R}_+^{n+1}$, for $a \in CS(\mathbf{R}_+^{n+1})$) will be denoted by $\mathfrak{A}^\#$ and $\mathfrak{A}$, respectively. We shall refer to $\mathfrak{A}$ as of *the C^*-algebra of the elliptic boundary problem* in the half space $\mathbf{R}_+^{n+1}$. We believe this distinctive notation justified, because the algebra $\mathfrak{A}$ proves to be of interest for a variety of reasons, listed below. First, c.f. [**10**], $\mathfrak{A}$ contains (Fredholm) inverses L^{-1} of L generated by a general (Lopatinski—Shapiro type) variable coefficient boundary condition (b) and a suitable elliptic constant coefficient (a). Moreover we then even have $P_{L,\alpha} = D^\alpha L^{-1} \in \mathfrak{A}$, for all $|\alpha| \leqq N =$ order of L. Second, we shall make available good criteria for $A \in \mathfrak{A}$ to be Fredhom. Third, $\mathfrak{A}$ may be of interest as a type-1 C^*-algebra with a finite ideal chain

$$\mathfrak{A} \supset \mathfrak{E} \supset \mathfrak{C}, \tag{0.2}$$

where $\mathfrak{E}$ and $\mathfrak{C}$ denote the commutator ideal of $\mathfrak{A}$ and the compact ideal of $\mathfrak{H}$, respectively. In fact we get

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$$(0.3) \qquad \mathfrak{C}/\mathfrak{A} \cong C(\mathscr{M}), \quad \mathfrak{C}/\mathfrak{C} = C(\mathbf{B}^n, \mathfrak{C}(\mathfrak{h})),$$

with the compact ideal $\mathfrak{C}(\mathfrak{h})$ of another Hilbert space $\mathfrak{h}$, where the two spaces $\mathscr{M}$ and $\mathbf{B}^n$ will be explicitly characterized. Moreover, $U\mathfrak{C}U^* = \mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{A}_0^n$ with the algebra $\mathfrak{A}_0^n$ of singular integral operators over the boundary $\partial\mathbf{R}_+^{n+1}$, and a certain unitary operator U.

For a more detailed discussion of the second point we refer to **[4]** or **[10]**. The two main results were announced in **[4]** and in a lecture at the CIME conference at Stresa/Italy in September 1968. (C.f. also **[5]**). Let us note that these results imply general criteria for normal solvability of boundary problems (a), (b), with variable coefficients in both (a) and (b), for a non-compact domain differomorphic to a half-space (c.f. **[10]**). Clearly (a) will have to be elliptic (and md-elliptic (c.f. **[6]**)), and from the work of Aronszajn **[1]** it follows that (b) must be Lopatinski-Shapiro. The results may be new, however, since a non-compact domain, with non-compact boundary is involved.

Finally let us invite a comparison between our results here and the singular elliptic theory of **[12]**. **[13]**, which is much simpler in its structure. Clearly the two types of problems treated reflect the old alternative 'limit circle case' and 'limit point case' of Herman Weyl. The large class of intermediate cases here may be too difficult for an explicit discussion.

The algebra $\mathfrak{A}$ mainly consists of pseudo-differential operators (abbreviated 'ψdo's'). However, there also occurs another kind of singular integral operator-with singularity at the boundary $\partial\mathbf{R}_+^{n+1}$ only, and related to a Wiener-Hopf (or Mellin) convolution (c.f. **[7]**).

1. **Preparations.** We denote by $\mathbf{R}_+^{n+1}$ the set $\{x = (y, \mathbf{x}) \varepsilon \mathbf{R} \times \mathbf{R}^n: y > 0\}$. The Hilbert space on which our C^*-algebra of ψdo's will act is $L^2(\mathbf{R}_+^{n+1}) = \mathfrak{H}$. We will also have occasion to study various operators on several other Hilbert spaces, which we denote as follows: $L^2(\mathbf{R}_+) = \mathfrak{h}$, $L^2(\mathbf{R}^n) = \mathfrak{k}$, $L^2(\mathbf{R}^{n+1}) = \mathfrak{K}$. For any Hilbert space $\mathbf{H}$, we will denote by $\mathfrak{C}(\mathbf{H})$, or, when there is no risk of confusion, by $\mathfrak{C}$, the norm closed two-sided ideal of compact operators on $\mathbf{H}$.

The Laplacian Δ on $\mathfrak{K}$ is related to the operators Δ_d and Δ_n on $\mathfrak{H}$, the Laplacian with Dirichlet and Neumann boundary conditions, respectively, by means of even or odd reflections at $y = 0$. To describe this relationship let the two isometries E_0, E_e: $\mathfrak{H} \to \mathfrak{K}$ be defined by

$$(1.1) \qquad \begin{aligned} (E_o u)(y, \mathbf{x}) &= 2^{-1/2} u(y, \mathbf{x}), y \geqq 0, = -2^{-1/2} u(-y, \mathbf{x}), y < 0, \\ (E_e u)(y, \mathbf{x}) &= 2^{-1/2} u(y, \mathbf{x}), y \geqq 0, = 2^{-1/2} u(-y, \mathbf{x}), y < 0. \end{aligned}$$

The adjoints $E_o^*, E_e^*: \mathfrak{K} \to \mathfrak{H}$ are partial isometries, explicitely given by

$$(1.2) \qquad E_o^* v = 2^{1/2} v_o \big|_{\mathbf{R}_+^{n+1}}, \quad E_e^* v = 2^{1/2} v_e \big|_{\mathbf{R}_+^{n+1}},$$

where

$$v_o(y, \mathbf{x}) = 1/2(v(y, \mathbf{x}) - v(-y, \mathbf{x}))$$
(1.3)
$$v_e(y, \mathbf{x}) = 1/2(v(y, \mathbf{x}) + v(-y, \mathbf{x})), \ v \in \mathfrak{K}.$$

The isometries E_o, E_e satisfy $E_o^*E_o = E_e^*E_e = 1$. It is readily seen that

(1.4) $$E_eE_e^*v = v_e, \ E_oE_o^*v = v_0$$

The unbounded operator $\mathring{K} = 1 - \Delta$ of $\mathfrak{K}$, with domain $\mathrm{dom}(\mathring{K}) = C_0^\infty(\mathbf{R}^{n+1})$, is known to be essentially self-adjoint. We denote its closure by K. Similarly, let $\mathring{H}_d$, $\mathring{H}_n = 1 - \Delta$ on $\mathfrak{H}$, with

$$\mathrm{dom}(\mathring{H}_d) = \{u \varepsilon C_0^\infty(\mathbf{R}_+^{n+1}) \colon u = 0 \ \text{at}\ y = 0\},$$
(1.5)
$$\mathrm{dom}(\mathring{H}_n) = \{u \varepsilon C_0^\infty(\mathbf{R}_+^{n+1}) \colon \partial u/\partial y = 0 \ \text{at}\ y = 0\},$$

and we denote the closures by H_d and H_n. Clearly, $H_d \geqq 1$, $H_n \geqq 1$.

PROPOSITION 1.1. *H_d and H_n are both self-adjoint; we have*

(1.6) $$H_d = E_o^*KE_o, \ H_n = E_e^*KE_e.$$

Moreover, for any bounded continuous function f: $[1, \infty] \to \mathbf{R}$, we get

(1.7) $$f(H_d) = E_o^*f(K)E_o, f(H_n) = E_e^*f(K)E_e.$$

PROOF. The essential point is that the Fourier transform of $\mathbf{R}^{n+1}$ defines a unitary operator F: $\mathfrak{K} \to \mathfrak{K}$ which 'diagonalizes' K and leaves the spaces of even and odd functions invariant each. In details, we find that FKF^{-1} is the (unbounded) multiplication operator induced by the function $\lambda^{-2}(x) = 1 + x^2 = 1 + y^2 + \mathbf{x}^2$. This gives an explicit construction of the spectral family $P(\mu)$ of K: $FP(\mu)F^{-1}$ is the multiplication operator induced by $(\chi_\mu(\lambda^{-2}(x))$, with the characteristic function x_E of the interval $(-\infty, \mu]$. Then since F leaves $\mathrm{im}E_o$ and $\mathrm{im}E_e$ invariant, it becomes evident that also $P(\mu)$ leaves these spaces invariant. (It is natural to use the notation 'even' and 'odd' function for the functions of $\mathrm{im}E_e$ and $\mathrm{im}E_o$, respectively.) In other words, the self-adjoint operator K is reduced by each of these two spaces.

Define

(1.8) $$K_e = K|(\mathrm{dom}K \cap \mathrm{im}E_e), \ K_o = K|(\mathrm{dom}K \cap \mathrm{im}E_o),$$

then K_e and K_o define self-adjoint operators of the Hilbert spaces $\mathfrak{K}_e = \mathrm{im}\ E_e$ and $\mathfrak{K}_o = \mathrm{im}\ E_o$, with spectral families $P_e(\mu) = P(\mu)\ \mathfrak{K}_e$ and $P_o(\mu) = P(\mu)\ \mathfrak{K}_o$. However, E_e acts as a unitary operator $\mathfrak{H} \to \mathfrak{K}_e$, for example, and $E_e^*|\mathfrak{K}_e$ is its inverse. Moreover, we get $H_n' = E_e^*KE_e = (E_e^*|\mathfrak{K}_e)K_eE_e$, trivially, which shows that H_n' is self-adjoint (in $\mathfrak{H}$), and that its spectral family is given by $Q_n(\mu) = E_e^*P(\mu)E_e$. This implies the proposition for H_n, if we can show that $H_n = H_n$. But by a simple calculation, $E_e^*\mathring{K}E_e \subset H_n \subset H_n$. (Note that for $u \in \mathrm{dom}H_n$) we need not have $E_eu \in \mathrm{dom}\mathring{K}$, because it may not be C^∞ at $y = 0$. However, since u satisfies the Neumann condition, $E_e u$ at least has continuous first derivatives and piecewiese continuous second derivatives, which implies $E_e u \in \operatorname{dom} K$, by a calculation). Also the fact that E_e is unitary from $\mathfrak{H}$ to $\mathfrak{K}_e$ then implies that $H_n' = H_n$. A similar argument will settle the proposition for H_0.

For $d = 1, 2, \ldots$, and a function $a \in L^\infty(\mathbf{R}^d)$ we define the *multiplication operator* $a(M)$, and the (formal) *Fourier multiplier* $a(D)$ as bounded operators of $L^2(\mathbf{R}^d)$ by

$$(1.9) \qquad (a(M)u)(z) = a(z)u(z),\ a(D) = F^{-1}a(M)F,$$

with the Fourier transform F of $\mathbf{R}^d$. It is known that $a(D)$ may be represented as a singular convolution operator. In particular, for $\lambda^\alpha(z) = (1 + z^2)^{-\alpha/2}$, $s_j(z) = z_j\lambda(z)$, the operators $\lambda^\alpha(D)$, $s_j(D)$ possess the explicit representations

$$(1.10) \qquad \begin{aligned} (\lambda^\alpha(D)u)(z) &= (2\pi)^{-d/2} \int G_{d,\alpha}(z - z')u(z')\,dz',\ \alpha > 0, \\ (s_j(D)u)\,(z) &= (2\pi)^{-d/2} \lim_{\varepsilon\to 0} \int_{|z-z'|\geqq\varepsilon} k_{d,j}(z - z')u(z')\,dz', \end{aligned}$$

where the kernels $G_{d,\alpha}$ and $k_{d,j}$ may be expressed in terms of modified Hankel functions ([**8**], formula (1.3), (1.4), or [**2**], [**16**]), $G_{d,\alpha}$ are known as Bessel potentials. They are C^∞-functions over $\mathbf{R}^d - \{0\}$, and satisfy (with some $c > 0$, and for all $\varepsilon > 0$, in fact, $\varepsilon = 0$ with exceptions)

$$(1.11) \qquad \begin{aligned} G_{d,\alpha}(z) &= O(e^{-c|z|}),\quad k_{d,j}(z) = O(e^{-c|z|}),\quad \text{as } |z| \to \infty, \\ G_{d,\alpha}(z) &= O(|z|^{-d+\alpha+\varepsilon}|),\quad \text{as } |z| \to 0. \end{aligned}$$

Actually, (1.11) and (1.12) below remain correct when differentiated. Then $k_{d,j}$, $j = 1, \ldots, d$ are kernels of Cauchy-type singular integral operators. In particular,

$$(1.12) \qquad k_{d,j}(z) = k^0_{d,j}(z) + O(|z|^{-d+\varepsilon}),\quad \text{as } |z| \to 0.$$

where $K^0_{d,j}$ is homogeneous of degree $= d$ in z, and has its integral over the unit sphere $= 0$, so that the Cauchy principal value in (1.10) exists for almost all z. Also, the $G_{d,\alpha}$ are $L^1(\mathbf{R}^d)$ so that the convolution integral in (1.10) exists for almost all z, assuming $u \in L^2(\mathbf{R}^d)$.

For $d = n + 1$ we also use the notation $a(M) = a(M_0, \mathbf{M})$, $a(D) = a(D_0, \mathbf{D})$. In particular we will have functions depending only on part of the variables and the corresponding operators, like $a(M_0)$, $b(\mathbf{D})$, etc. Note that $a(M_0)$ may either act on $L^2(\mathbf{R})$ or on $L^2(\mathbf{R}^{n+1}) = \mathfrak{K}$, etc.

We distinguish the operators $\Lambda = \lambda(D) = K^{-1/2}$, and $S_j = s_j(D) = D_j\Lambda$, $j = 0, \ldots, N$, acting on $\mathfrak{K}$, where $D_j = -i\partial/\partial x_j$. (We get $K =$

$F^{-1}(1 + M^2)F = (1 + D^2)$, as mentioned in the proof of proposition 1.1). Now, let us apply proposition 1.1, to obtain

$$(1.13)\qquad \begin{aligned} &H_d^{-1/2} = \Lambda_d = E_o^* \Lambda E_o,\ H_n^{-1/2} = \Lambda_n = E_e^* \Lambda E_e,\\ &S_d = D_0\Lambda_d = E_e^* S_0 E_o,\ S_n = D_0 \Lambda_n = E_o^* S_0 E_e,\\ &S_{d,j} = D_j \Lambda_d = E_o^* S_j E_o,\ S_{n,j} = D_j \Lambda_n = E_e^* S_j E_e,\\ &\qquad j = 1, \dots, n. \end{aligned}$$

(In (1.13) we have used the fact that $E_0^* D_0 u = D_0 E_e^* u$, but $E_o^* D_j u = D_j E_o^* u$, for appropriate functions u, and similarly for E_e^*.)

From (1.13) and (1.10) one derives integral representations for the operators Λ_d, Λ_n, S_d, S_n, $S_{d,j}$, $S_{d,n}$, using (1.1), (1.2), (1.3). These are conveniently written as

$$(1.14)\qquad \begin{aligned} &\Lambda_d = \Lambda_- - \Lambda_+,\ \Lambda_n = \Lambda_- + \Lambda_+,\ S_d = S_- - S_+,\\ &S_n = S_- + S_+,\ S_{d,j} = S_{j,-} - S_{j,+},\ S_{n,j} = S_{j,-} + S_{j,+}, \end{aligned}$$

with

$$(1.15)\quad (\Lambda_\pm u)(y, x) = \int_{\mathbf{R}_+^{n+1}} G_{n+1,1}(y \pm y', \mathbf{x} - \mathbf{x}')u(y', \mathbf{x}')\, d\mathbf{x}'\, dy'$$

and similar formulas for $S_\pm$, $S_{j,\pm}$, involving the kernels $k_{n+1,j}$.

For $n = 0$ we will denote $\Lambda_\pm = Q_\pm$, $S_\pm = P_\pm$. We note the explicit integral representations

$$(1.16)\qquad \begin{aligned} &(P_\pm u)(y) = i/\pi \int_{\mathbf{R}_+} K_1(y \pm y')\mathrm{sgn}(y \pm y')u(y')\, dy',\\ &(Q_\pm u)(y) = i/\pi \int_{\mathbf{R}_+} K_0(y \pm y')u(y')\, dy', \end{aligned}$$

with the modified Hankel functions ([15], [17]).

Next, we wish to review some facts about C^*-algebras. Let $\mathfrak{A}$ be a C^*-algebra, and let $\mathfrak{E}$ be the closed, self-adjoint, two-sided ideal generated by the commutators of $\mathfrak{A}$. Then $\mathfrak{A}/\mathfrak{E} = \mathfrak{A}^\wedge$ is a commutative C^*-algebra, and is isometrically isomorphic to the algebra $C(\mathcal{M})$ of continuous functions on the compact Hausdorff space $\mathcal{M}(\mathfrak{A})$, the set of all *-homomorphisms $m: \mathfrak{A}^\wedge \to \mathbf{C}$, with the relative w^*-topology. We define the symbol of $A \in \mathfrak{A}$, $\sigma_A: \mathcal{M}(\mathfrak{A}) \to \mathbf{C}$, by $\sigma_A(m) = m(A^\wedge)$, $A^\wedge = A + \mathfrak{E} \in \mathfrak{A}/\mathfrak{E}$. The induced map $\sigma: \mathfrak{A} \to C(\mathcal{M})$ is a^*-homomorphism of the two algebras.

LEMMA 1.2. *Let $\mathfrak{A}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ be C^*-algebras, $\mathfrak{A}_1$, $\mathfrak{A}_2 \subset \mathfrak{A}$, $\mathfrak{A}$ generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Then $\mathcal{M}(\mathfrak{A})$ is homeomorphic to a closed subspace of the Cartesian product $\mathcal{M}(\mathfrak{A}_1) \times \mathcal{M}(\mathfrak{A}_2)$. Moreover, a corresponding homeomorphism ι: $\mathcal{M}(\mathfrak{A}) \to \mathcal{M}(\mathfrak{A}_1) \times \mathcal{M}(\mathfrak{A}_2)$ can be constructed such that $\sigma_A(\iota^{-1}(m_1, m_2)) =$*

$\sigma^1_{A_1}(m_1)\cdot\sigma^2_{A_2}(m_2)$ for all $(m_1, m_2)\in\iota(\mathcal{M}(\mathfrak{A}))$, and $A = A_1A_2$, $A_j\in\mathfrak{A}_j$, and the symbols σ, σ^j of $\mathfrak{A}$, $\mathfrak{A}_j$.

PROOF. Denote the commutator ideals by $\mathfrak{C}$, $\mathfrak{C}_j$, respectively, and observe that $\mathfrak{C}_j\subset\mathfrak{C}$. Thus we have canonical homomorphisms $\pi_j\colon \mathfrak{A}_j^{\wedge}\to\mathfrak{A}^{\wedge}$, defined by $\pi_j(A_j+\mathfrak{C}_j) = A_j+\mathfrak{C}$. We define

$$(1.17)\qquad \iota = \pi_1^*\times\pi_2^*,$$

with the dual maps $\pi_j^*\colon \mathcal{M}(\mathfrak{A})\to\mathcal{M}(\mathfrak{A}_j)$, given by $\pi_j^*(m) = m\circ\pi_j$. The map ι clearly is continuous, and we need only show it is injective. If $\pi_j^*(m)A_j^{\wedge} = \pi_j^*(\bar m)A_j^{\wedge}$ for all $A_j^{\wedge}\in\mathfrak{A}_j^{\wedge}$, $j = 1, 2$, then $(m-\bar m)$ $(\Pi_{l=1}^N A^l+\mathfrak{C}) = \pi(m-\bar m)\,(A^l+\mathfrak{C}) = 0$, whenever all $A^l\in\mathfrak{A}_1\cup\mathfrak{A}_2$. Since operators of this form ΠA^l span the algebra $\mathfrak{A}$, by hypothesis, we get $m = \bar m$.

In the later application we will tend to identify $\mathcal{M}(\mathfrak{A})$ with its image $\iota\mathcal{M}(\mathfrak{A}))\subset\mathcal{M}(\mathfrak{A}_1)\times\mathcal{M}(\mathfrak{A}_2)$, because then $\mathfrak{A}$, $\mathfrak{A}_j$ will have a fixed analytical meaning. Then we may write the main formula of the lemma in the form

$$(1.17)\qquad \sigma_A(m_1, m_2) = \sigma^1_{A_1}(m_1)\cdot\sigma^2_{A_2}(m_2),\ (m_1, m_2)\in\mathcal{M}(\mathfrak{A}),$$

whenever $A = A_1A_2$, $A_j\in\mathfrak{A}_j$.

Later, we will be looking at the topological tensor product of C^*-algebras, so we also summarize some facts about them. For details c.f. [**3**]. For two Hilbert spaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ the algebraic tensor product is a pre-Hilbert space under the inner product $(f^1\otimes f^2, g^1\otimes g^2) = (f^1, g^1)\cdot(f^2, g^2)$, extended to $\mathfrak{H}_1\otimes\mathfrak{H}_2$ by linearity. Completing under the induced Hilbert norm we get the topological tensor product $\mathfrak{H}_1\mathbin{\hat\otimes}\mathfrak{H}_2$, which is a Hilbert space. For C^*-algebras $\mathfrak{A}_j$ on $\mathfrak{H}_j$ the algebraic tensor product $\mathfrak{A}_1\otimes\mathfrak{A}_2$ is a $*$-subalgebra of $\mathfrak{L}(\mathfrak{H}_1\mathbin{\hat\otimes}\mathfrak{H}_2)$ with $(A^1\otimes A^2)(f^1\otimes f^2) = (A^1f^1)\otimes(A^2f^2)$. We define the topological tensor product $\mathfrak{A}_1\mathbin{\hat\otimes}\mathfrak{A}_2$ as the closure of $\mathfrak{A}_1\otimes\mathfrak{A}_2$ in $\mathfrak{L}(\mathfrak{H}_1\mathbin{\hat\otimes}\mathfrak{H}_2)$. We mention without proof the following two lemmas (c.f. [**3**]).

LEMMA 1.3. *If $\mathfrak{A}_2$ is commutative and $\mathfrak{A}_1$ has commutator ideal $\mathfrak{C}(\mathfrak{H}_1)$, then the commutator ideal of $\mathfrak{A}_1\mathbin{\hat\otimes}\mathfrak{A}_2$ equals $\mathfrak{C}(\mathfrak{H}_1)\otimes\mathfrak{A}_2$, and we have*

$$(1.18)\qquad \mathcal{M}(\mathfrak{A}_1\mathbin{\hat\otimes}\mathfrak{A}_2) = \mathcal{M}(\mathfrak{A}_1)\times\mathcal{M}(\mathfrak{A}_2).$$

LEMMA 1.4. *Let $\mathfrak{F}(\mathfrak{H}_1)$ be the class of continuous operators of finite rank over $\mathfrak{H}_1$, then, for any C^*-algebra $\mathfrak{A}_2$ on $\mathfrak{H}_2$ we have $\mathfrak{F}(\mathfrak{H}_1)\otimes\mathfrak{A}_2$ dense in $\mathfrak{C}(\mathfrak{H}_1)\mathbin{\hat\otimes}\mathfrak{A}_2$.*

2. **The case** $n = 0$. The results of this section are essentially contained in [**7**]; we will summarized these results here, and make the connections to information concerning the algebras of ψdo's on $\mathbf{R}_+^{n+1}$ we wish to study. Let $\mathfrak{Q}$ be the C^*-algebra on $\mathfrak{h}$ generated by the ideal $\mathfrak{C}(\mathfrak{h})$ of compact

operators and three commutative C^*-algebras $\mathfrak{M}$, $\mathfrak{W}$, and $\mathfrak{Z}$: $\mathfrak{Z}$ is the algebra of multiplications $a(M)$, with $a \in C([0, \infty])$, a bounded continuous function over $[0, \infty)$, with limit at $+\infty$; $\mathfrak{W}$ is generated by the identity and the operators $E_e{}^*K_\varphi E_e$ with $K_\varphi u = \varphi * u = \int\varphi(y - y')u(y')dy'$, $u \in L^2(\mathbf{R})$, where φ is an even $L^1(\mathbf{R})$-function; and $\mathcal{M}$ is generated by the identity and the two operators $(K_\pm u)(y) = \int_0^\infty u(y')dy/(y' \pm y')$, where the Cauchy principal value is to be taken in defining the integral for K_-.

The operators of $\mathfrak{Z}$ already are in 'diagonal form', as multiplication operators. The other two algebras are explicitly diagonalizable via classical integral transforms. With the *Fourier cosine transform* $F_c: \mathfrak{h} \to \mathfrak{h}$ (defined as $F_c = E_e^* F E_e$), and the Mellin transform $M: \mathfrak{h} \to L^2(\mathbf{R})$,

$$(Mu)(t) = (FUu)(t) = (4\pi)^{-1/2}\int_0^\infty u(y)y^{-(1+it)}\,dy, \tag{2.1}$$

where

$$(Uu)(t) = 2^{1/2}e^t u(e^{2t}),\ U: \mathfrak{h} \to L^2(\mathbf{R}), \tag{2.2}$$

we get

$$\begin{aligned} F_c\mathfrak{W}F_c^* &= \{a(M) \in \mathfrak{L}(\mathfrak{h}) : a \in C([0, \infty])\} \\ M\mathfrak{M}M^* &= \{a(M) \in \mathfrak{L}(L^2(\mathbf{R})) : a \in C([-\infty, +\infty])\} \end{aligned} \tag{2.3}$$

so that $\mathfrak{Z}$, $\mathfrak{W}$ and $\mathfrak{M}$ are isometrically isomorphic to $C([0, \infty])$, $C([0, \infty])$, and $C([-\infty, +\infty])$, respectively. Repeated application of Lemma 1.2 yields an injective map $\iota: \mathcal{M}(\mathfrak{Q}) \to Q$ where

$$Q = \{(y, \xi, t) \in [0, \infty] \times [0, \infty] \times [-\infty, +\infty]\}. \tag{2.4}$$

Moreover, if ι is used to identify $\mathcal{M} = \mathcal{M}(\mathfrak{Q})$ with its image we have

$$\begin{aligned} &\sigma_{a(M)}(y, \xi, t) = a(x),\ \sigma_{H_\varphi^\sharp}(y, \xi, t) = \sqrt{2\pi}\,\tilde{\varphi}(\xi),\ \varphi \text{ even}, \\ &\sigma_{K_+}(y, \xi, t) = \pi \operatorname{sech}(\pi t/2),\ \sigma_{K_-}(y, \xi, t) = -i\pi \tanh(\pi t/2) \end{aligned} \tag{2.5}$$

where

$$H_\varphi^\sharp = E_e^* K_\varphi E_e, \text{ and } \tilde{\varphi} = F_c\varphi$$

THEOREM 2.1. (c.f. [7]). *$\iota(\mathcal{M}(\mathfrak{Q}))$ equals the union*

$$\{y = 0, \xi = \infty, t \in [-\infty, \infty]\} \cup \{y \in [0, \infty], \xi = \infty, t = \pm\infty\}$$
$$\cup \{y = \infty, \xi \in [0, \infty], t = \pm\infty\} \cup \{y = \infty, \xi = 0, t \in [-\infty, \infty].\}$$

It is convenient to rearrange the space $\mathcal{M}$ into a hexagon, as in fig. 2.1.

It should also be mentioned that the commutator ideal of equals $\mathfrak{C}(\mathfrak{h})$, and that the operators of $\mathfrak{C}(\mathfrak{h})$ are redundant as generators of $\mathfrak{Q}$, as follows from the fact that the C^*-algebra generated by $\mathfrak{M}$, $\mathfrak{W}$ and $\mathfrak{Z}$ only contains nontrivial compact operators and is irreducible (c.f. also Lemma 4.1).

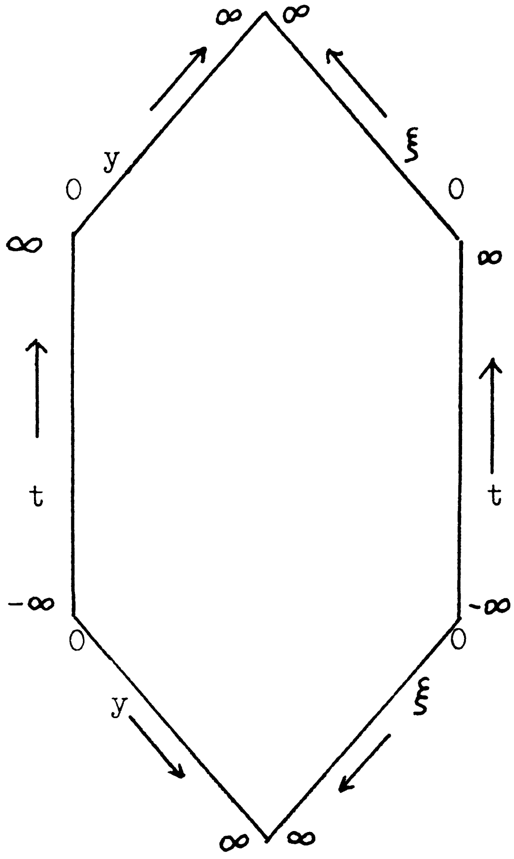

Fig. 2.1

The algebra $\mathfrak{Q}$ contains a variety of interesting operators: $\mathfrak{M}$ contains all 'Mellin convolutions' $S_\varphi: \mathfrak{h} \to \mathfrak{h}$, $\varphi \in L^1(\mathbf{R}_+, x^{-1/2}\, dx)$,

$$(S_\varphi u)(y) = \int_0^\infty \varphi(y/y')u(y')\, dy'/y', \tag{2.4}$$

and we have $MS_\varphi M^* = 2^{1/2}2\pi\, \varphi^\vee(M)$, with $\varphi^\vee = M_\varphi$, which gives the explicit diagonalization for the elements of $\mathfrak{M}$. S_φ is called a Mellin convolution operator because it is a convolution of the multiplicative group of positive reals $\mathbf{R}_+$, and because the Mellin transform acts as the Fourier transform for this group.

Next, $\mathfrak{Q}$ contains all Wiener-Hopf convolutions $H_\varphi: \mathfrak{h} \to \mathfrak{h}$, $\varphi \in L^1(\mathbf{R})$,

$$(H_\varphi u)(y) = \int_0^\infty \varphi(y - y')u(y')\, dy', \tag{2.5}$$

and we have (with $\varphi_e = E_e^* E_e \varphi$, $\varphi_o = E_o^* E_o \varphi$, $\varphi^{\wedge o} = F\varphi | R_+$)

$$\sigma_{H_\varphi}(y, \xi, t) = (2\pi)^{1/2}\{\varphi_e^{\wedge o}(\xi) + [\tanh(\pi t/2) + i \operatorname{sech}(\pi t/2)]\, \varphi_o^{\wedge o}(\xi)\}. \tag{2.6}$$

This fact is trivial for even φ, due to (2.5) and $H_\varphi^\# = H_\varphi + H_\varphi^+$, $(H_\varphi^+ u)(y) = \int_0^\infty \varphi(y + y')u(y')dy'$, where $H_\varphi^+ \in \mathfrak{C}(\mathfrak{h})$ for all $\varphi \in L^1(\mathbf{R})$. On the other hand, let ϕ be odd and let $Tu(y) = \int_{\mathbf{R}} u(y')dy'/(y - y')$ be the Hilbert transform. It is well known that $FTF^* = -i\pi \operatorname{sgn}(M)$, $\operatorname{sgn} y = y/|y|$.

The following facts are simple calculations (for an odd $\phi \in L^1$).

$$\begin{aligned} E_o^* T E_e &= K_- + K_+, \quad -E_e^* T E_o E_o^* K_\phi E_e \\ &= -E_e^* T K_\phi E_e = i(2\pi^3)^{1/2} F_c^* \phi^{\wedge o}(M) F_c \in \mathfrak{W}, \\ -\pi^2 E_o^* K_\phi E_e &= E_o^* T R_e E_e^* T K_\phi E_e = (K_- + K_+)\, E_e^* T K_\phi E_e, \end{aligned} \tag{2.7}$$

(using the Hilbert inversion formula [15]). So it follows that $E_o^* K_\phi E_e = H_\phi^\# = H_\phi + H_\phi^+$, and that (by compactness of H_ϕ^+),

$$\sigma_{H_\phi}(y, \xi, t) = (2\pi)^{1/2}\{\tanh(t\pi/2) + i \operatorname{sech}(t\pi/2)\}\phi^{\wedge o}(\xi), \tag{2.8}$$

which implies (2.6).

The algebra $\mathfrak{Q}$ also contains the 'singular convolutions' $K_\pm^0 : \mathfrak{h} \to \mathfrak{h}$,

$$(K_\pm^0 u)(y) = \int_0^\infty u(y')\, e^{-|y \pm y'|}\, dy'/(y \pm y'), \tag{2.9}$$

and,

$$\begin{aligned} \sigma_{K_-^0}(y, \xi, t) &= -2i \arctan \xi \tanh(\pi t/2), \\ \sigma_{K_+^0}(y, \xi, t) &= 2 \arctan \xi \operatorname{sech}(\pi t/2), \end{aligned} \tag{2.10}$$

as proven in [7], Lemma 4.3. Our main interest is focused on the subalgebra $\mathfrak{P}$ of $\mathfrak{Q}$ generated by the operators (2.9) (and $\mathfrak{C}(\mathfrak{h})$). Again $\mathfrak{P}^\#/\mathfrak{C}$ is commutative, and naturally imbedded in $C(\mathcal{M}(\mathfrak{Q})) \cong \mathfrak{Q}/\mathfrak{C}$, and the commutator ideal is $\mathfrak{C}$. So we shall identify $\mathfrak{P}^\#/\mathfrak{C}$ with its image in $C(\mathcal{M}(\mathfrak{Q}))$. By looking at the symbols of the generators we see that $\mathfrak{P}^\#/\mathfrak{C}$ separates, in the sense of the Stone-Weierstrass theorem, precisely all the points of

$$\{y = \infty, \xi \in (0, \infty), t = \pm\infty\} \cup \{y = 0, \xi = \infty, t \in (-\infty, +\infty)\}, \tag{2.11}$$

while each of the three remaining segments in fig 2.1 collapses into a point. Thus it follows that $\mathcal{M}(\mathfrak{P}^\#)$ is homeomorphic to the triangle of Fig.2.2.

The algebra $\mathfrak{P}^\#$ still contains all Wiener-Hopf convolutions (2.5) with L^1-kernel, as follows from (2.6): The Fourier transforms of $\varphi_e, \varphi_o \in L^1$ are continuous and 0 at ∞, also $\varphi_o^\wedge$ is odd, hence $= 0$ at 0; hence $\sigma_{H_\varphi} = 0$ on the collapsing segments. Moreover, we claim, that another set of generators of $\mathfrak{P}^\#$ is given by the operators (1.16). Indeed, from the well known

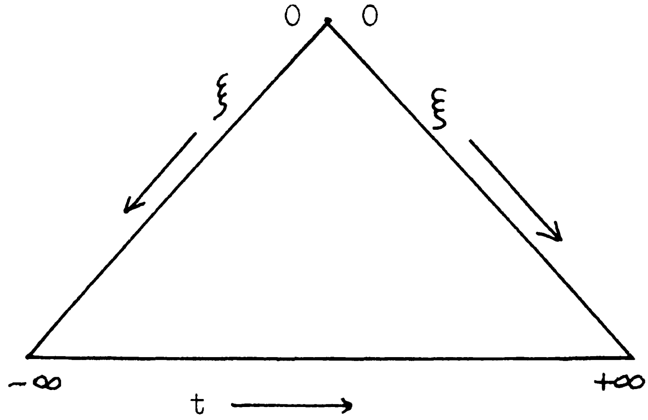

Fig. 2.2

asymptotic behaviour of the modified Hankel function K_1 at 0 and ∞ we conclude that (with a constant $c \neq 0$)

(2.12) $$\psi(y) = K_1(y) \operatorname{sgn} y - ce^{-|y|}/y \in L^1(\mathbf{R}),$$

so that $P_- - c_0 K_-^0$ is an L^1-Wiener-Hopf convolution, hence in $\mathfrak{P}^\#$. Similarly $P_+ - c_0 K_+^0 \in \mathfrak{C}(\mathfrak{h})$. Again Q_- is an L^1-Wiener-Hopf convolution, and Q_+ is compact. Hence $P_\pm$, $Q_\pm \in \mathfrak{P}^\#$. The algebra generated by them contains Q_+, a nontrivial compact operator, and is irreducible, hence contains $\mathfrak{C}(\mathfrak{h})$ (c.f. Dixmier [14]), also by Lemma 4.1. For a point of $\mathcal{M}(\mathfrak{P}^\#)$ with finite t we have $\xi = \infty$, hence $\sigma_{H_\psi} = 0$ at all points of that segment. Thus $\sigma_{P_-} = c\sigma_{K_-^0}$ separates all points of it. Similarly, a calculation shows that—up to a multiplicative non-vanishing constant— σ_{P_-} equals $\pm\, \xi\, \lambda(\xi)$ on the segments $t = \pm\infty$, using (2.6), (2.11), and that the kernel of K_-^0 in effect has Fourier transform arc tan ξ. Furthermore the symbol of Q_- is $\lambda(\xi)$ on $t = \pm\infty$, and zero elsewhere, hence separates the points at $t = \pm\infty$ from $|t| < \infty$. It follows that the symbols of $P_\pm$, $Q_\pm$ separate $\mathcal{M}(\mathfrak{P}^\#)$ so that these operators generate $\mathfrak{P}^\#$, as stated.

It is useful to convert the triangle $\mathcal{M}(\mathfrak{P}^\#)$ of fig.2.2 into the shape of fig.2.3 where we use the transformation of ξ-variables $\xi \to \xi \cdot \operatorname{sgn} t$ at the segments with $\xi \neq \infty$, because in these coordinates we get $\sigma_{H_\varphi}(x, \xi, t) = \varphi^\wedge(\xi)$ on $|t| = \infty$, for any $\varphi \in L^1$, and also for the singular Wiener-Hopf-convolution

$$K_-^0 = H_{\exp(-|y|)/y}.$$

Finally, let $\mathfrak{P}$ be the subalgebra of $\mathfrak{Q}$ with generators $\mathfrak{P}^\#$ and $\mathfrak{Z}$. For this algebra the Symbol space $\mathcal{M}(\mathfrak{P})$ is represented by fig. 2.4 The algebra $\mathfrak{P}$ is the 1-dimensional example of our algebra $\mathfrak{A}$ of ψdo's to be studied in

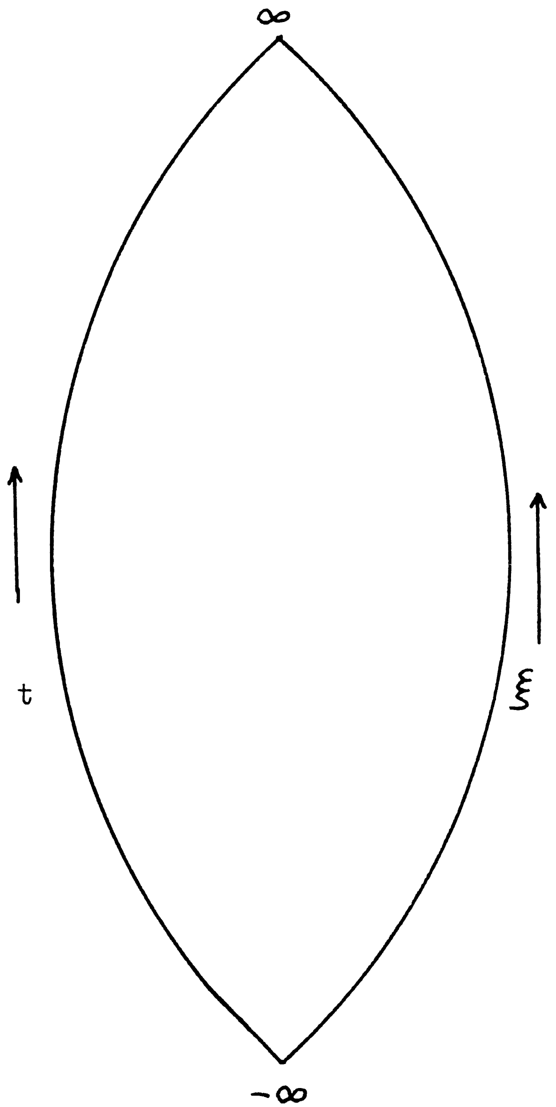

Fig. 2.3

section 3 ($n = 0$). We still have the commutator ideal of $\mathfrak{P}$ equal to $\mathfrak{C}$, in distinction from the cases $n \geqq 1$ studied in section 3. It may be observed that the 't-segment' in Fig. 2.4 is entirely over the point $x = 0$, reflecting the fact that the symbol of $A \in \mathfrak{P}$ on $|t| < \infty$ entirely is determined by the operator $\chi(M)A\chi(M)$, with any $\chi \in C_0^\infty([0, \infty])$, equal 1 near 0.

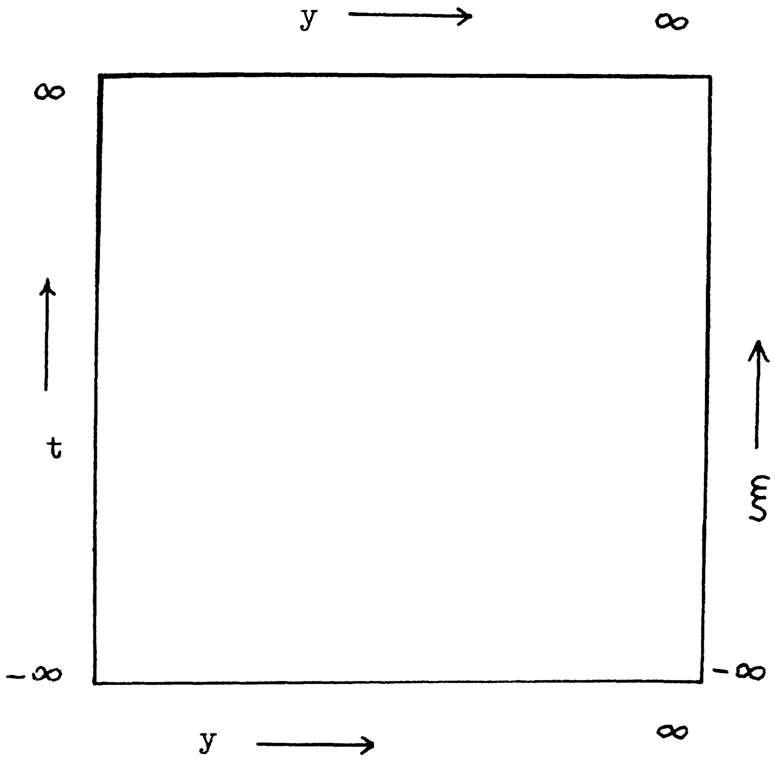

Fig. 2.4

We introduce the notation $\mathbf{L} = \mathcal{M}(\mathfrak{P}^{\#})$, for later use.

3. **The algebra of ψdo's.** As a first step we examine the symbol space structure of $\mathfrak{A}^{\#}$, the C^*-algebra generated by the operators (1.13). Let $\mathbf{F}$ be the Fourier transform in the last n variables, i.e.,

$$(\mathbf{F}u)(y, \mathbf{x}) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-i\mathbf{x}\xi} u(y, \xi)\, d\xi. \tag{3.1}$$

$\mathbf{F}$ defines a unitary operator of both $\mathfrak{H}$ and $\mathfrak{K}$ and, in that sense, commutes with E_e and E_o: $E_e\mathbf{F}u = \mathbf{F}E_e u$, $E_o\mathbf{F}u = \mathbf{F}E_o u$, $u \in \mathfrak{H}$. Introduce $\tau(\mathbf{x}) = (1 + \mathbf{x}^2)^{1/2}$ and then the unitary substitution operator $T: \mathfrak{H} \to \mathfrak{H}$ (or $\mathfrak{K} \to \mathfrak{K}$)

$$(Tu)(y, \mathbf{x}) = \tau^{-1/2}(\mathbf{x}) u(y/\tau(\mathbf{x}), \mathbf{x}) \tag{3.2}$$

which commutes with E_e, E_o in a similar sense. Define $U = T\mathbf{F}$. Then

$$UAU^* = (1 + D_0^2)^{-1/2} \otimes \tau^{-2}(\mathbf{M}), \tag{3.3}$$

where $1 + D_0^2$ is the operator K for $n = 0$, acting on $L^2(\mathbf{R})$, and with respect to the tensor decomposition $\mathfrak{K} = L^2(\mathbf{R}) \hat{\otimes} \mathfrak{k}$. Indeed, we have $\mathbf{F} = G^*F$ with the 1-dimensional Furiero transform $G: \mathfrak{K} \to \mathfrak{K}$, acting on y. Also, $GT = T^*G$, $G^*T = T^*G^*$, by a calculation. It follows that

$$(3.4)\quad \begin{aligned} UAU^* &= TG^*(\tau^2(\mathbf{M}) + M_0^2)^{-1/2}GT^* = G^*T^*(\tau^2(\mathbf{M}) + M_0^2)^{-1/2}TG \\ &= \tau^{-1}(\mathbf{M})G^*(1 + M_0^2)^{-1/2}\, G, \end{aligned}$$

which proves (3.3) since the right hand side is only another way of writing the right hand side of (3.3).

Applying E_e or E_o we get

$$(3.5)\quad \begin{aligned} &UA_dU^* = Q_d \otimes \tau^{-1}(\mathbf{M}),\ UA_nU^* = Q_n \otimes \tau^{-1}(\mathbf{M}) \\ &US_nU^* = P_n \otimes 1,\ US_dU^* = P_d \otimes 1, \\ &US_{n,j}U^* = Q_d \otimes M_j\tau^{-1}(\mathbf{M}),\ US_{d,j}U^* = Q_n \otimes M_j\tau^{-1}(\mathbf{M}), \end{aligned}$$

with $P_n, P_d, Q_n, Q_d = P_- \pm P_+, Q_- \pm Q_+$, respectively.

PROPOSITION 3.1 *$U\mathfrak{A}^\# U^*$ coincides with the subalgebra of*

$$(3.6)\quad \mathfrak{P}^\# \hat{\otimes} \mathfrak{Z}_n,\ \mathfrak{Z}_n = \{a(M) \in \mathfrak{L}(\mathfrak{k}): a \in C(\mathbf{B}^n)\},$$

generated by the operators (3.5), *where $\mathbf{B}^n$ denotes the smallest compactification of $\mathbf{R}^n$ onto which s_j, $j = 1, \ldots, n$ and λ all extend as continuous functions. Moreover, if $\mathfrak{E}^\#$ is the commutator ideal of $\mathfrak{A}^\#$, then*

$$(3.7)\quad U\mathfrak{E}^\# U^* = \mathfrak{E}(\mathfrak{h}) \hat{\otimes} \mathfrak{Z}_n.$$

Proof. The first part was discussed above. Also, it is trivial that '$\subset$' holds in (3.7). By remark 4.2 we find that $U\mathfrak{E}^\# U^*$ contains $C \otimes \lambda(\mathbf{M})$, $C \otimes s_j(\mathbf{M})$, for all $C \in \mathfrak{E}(\mathfrak{h})$. Let $C = P$ be a 1-dimensional projection. Then the above operators generate the algebra of all $P \otimes a(\mathbf{M})$, $a \in C(\mathbf{B}^n)$, which is contained in $U\mathfrak{E}^\# U^*$. Then Lemma 1.4 may be used for the second statement.

Prosposition 3.2. *Let $\mathbf{L} = \mathbf{L}_t \cup \mathbf{L}_\xi = \mathcal{M}(\mathfrak{P}^\#)$, with the closed 't-segment' $\mathbf{L}_t$, and the open 'ξ-segment' $\mathbf{L}_\xi$, according to fig.* 2.3, *of* $\mathbf{L}$. *The C^*-algebra $\mathfrak{A}^*/\mathfrak{E}$ is isometrically isomorphic to a subalgebra of $C(\mathbf{L} \times \mathbf{B}^n)$. In fact, to the subalgebra generated by $\sigma_A(1) \cdot b(\mathbf{x})$, $1 \in \mathbf{L}$, $\mathbf{x} \in \mathbf{B}^n$, where $A \otimes b(M)$ is any one of the tensor products in* (3.5). *Moreover, the symbol space $\mathcal{M}(\mathfrak{A}^\#)$ is homeomorphic to the space obtained from $\mathbf{L} \times \mathbf{B}^n$ by collapsing each set $1 \times \mathbf{B}^n$ into a point, for all $1 \in \mathbf{L}_t$. Also the symbol of a generator $G = U^*((A \otimes b(M))U$ of the collection* (1.13) *then is equal to the function induced by $\sigma_A(l) \cdot b(\mathbf{x})$ on $\mathcal{M}(\mathfrak{A}^\#)$ as described by above homeomorphism.*

PROOF. By Lemma 1.3, $(\mathfrak{P}^\# \hat{\otimes} \mathfrak{Z}_n)/(\mathfrak{E}(\mathfrak{h}) \hat{\otimes} \mathfrak{Z}_n) \cong C(\mathbf{L} \times B^n)$; thus by proposition 3.1., $\mathfrak{A}^\#/\mathfrak{E}^\#$ is isometrically isomorphic to a subalgebra of

$C(\mathbf{L} \times B^n)$. The reminder of the proposition then is a matter of explicit calculations, involving Stone-Weierstrass, and the dual of the above injection. Note also the more direct proof of [6], Ch.V.

REMARK: It is immediate that (*i*) $\mathbf{B}^n$ is homeomorphic to the *n*-ball $\{x \in \mathbf{R}^n: |x| \leqq 1\}$, with $\{|x| = 1\}$ representing the infinite points of $\mathbf{B}^n$. Also that $\mathcal{M}^\# = \mathcal{M}(\mathfrak{A}^\#)$ is homeomorphic to an $n + 1$–ball

$$(3.8) \qquad \mathcal{M}_1^\# = \{(\xi_0, \boldsymbol{\xi}): \xi_0 \in \mathbf{R}, \boldsymbol{\xi} \in \mathbf{R}^n, \xi_0^2 + |\boldsymbol{\xi}|^2 \leqq 1\},$$

with a 1-dimensional segment $\mathcal{M}_2^\# = \{-1 \leqq \mu \leqq 1\}$ attached as a handle, with its endpoints $\mu = \pm 1$ identified with the north and south pole $\xi_0 = \pm 1$ of the ball, respectively. (fig 3.1),

Let $\mathfrak{A}$ be the C^*-algebra generated by $\mathfrak{A}^\#$ above, and the multiplication operators

$$(3.9) \qquad \lambda(M), s_j(M): \mathfrak{H} \to \mathfrak{H}, j = 1, \ldots, n + 1,$$

with the functions λ, s_j in $n + 1$-dimenions, as in section 1.
Fig. 3.1

The operators (3.9) generate a commutative C^*-algebra isometrically isomorphic to $C(\mathbf{H}^{n+1})$, with the compactification

$$(3.10) \qquad \mathbf{H}^{n+1} = \{x = (y, \mathbf{x}) \in B^{n+1}: y \geqq 0\}$$

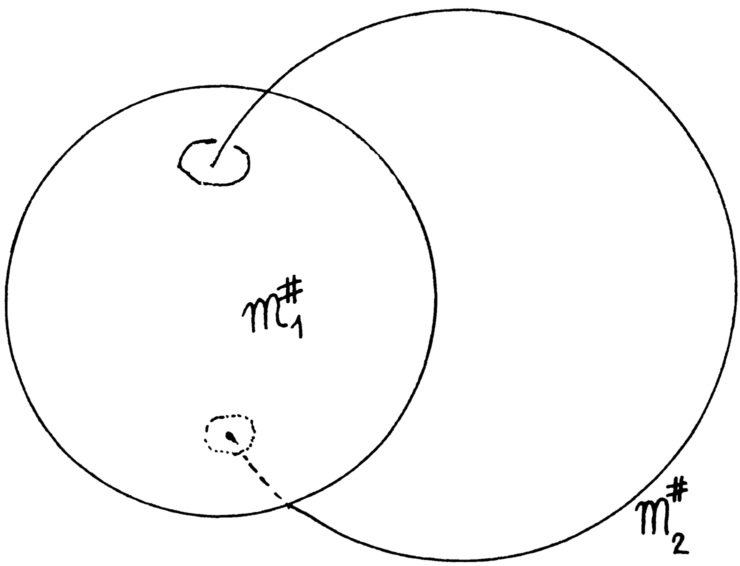

Fig. 3.1

of $\mathbf{R}_+^{n+1}$. Applying Lemma 1.2 we conclude that $\mathcal{M}(\mathfrak{A})$ is homeomorphic to a subspace of the product $\mathbf{H}^{n+1} \times \mathcal{M}^\sharp$.

THEOREM 3.3 *$\mathcal{M}(\mathfrak{A})$ consists of the following union of subsets of* $\mathbf{H}^{n+1} \times \mathcal{M}^\sharp$:

$$\mathcal{M}(\mathfrak{A}) = ((\mathbf{H}^{n+1} - \overline{\mathbf{R}_+^{n+1}}) \times \mathcal{M}^\sharp) \cup (\overline{\mathbf{R}_+^{n+1}} \times \partial\mathcal{M}_1^\sharp) \cup (\overline{\partial\mathbf{R}_+^{n+1}} \times \mathcal{M}_2^\sharp) \cup (\partial\partial\mathbf{R}_+^{n+1} \times \mathcal{M}_2^\sharp), \tag{3.11}$$

with

$$\partial\mathcal{M}_1^\sharp,\ \partial\mathbf{R}_+^{n+1},\ \overline{\mathbf{R}_+^{n+1}},\ \partial\partial\mathbf{R}_+^{n+1}$$

denoting the boundary of the $n+1$-ball (3.8), *the boundary of* $\mathbf{R}_+^{n+1}$ *in* R^{n+1}, the closure of R_+^{n+1} and the boundary of ∂R_+^{n+1} in $\mathbf{H}^{n+1}$, *respectively.*

The proof is postponed to the end of section 3. To also get an insight into the structure of the commutator ideal $\mathfrak{E}$ of $\mathfrak{A}$ we introduce the C^*-algebra $\mathfrak{A}_0^d$, acting on $L^2(\mathbf{R}^d)$, and generated by

$$\lambda(D),\ \lambda(M),\ s_j(D),\ s_j(M),\ j = 1, \ldots, d. \tag{3.12}$$

As discussed in [11], $\mathfrak{A}_0^d$ has commutator ideal $\mathfrak{C}$, and we get

$$\mathcal{M}(\mathfrak{A}_0^d) = (\partial\mathbf{B}^d \times \mathbf{B}^d) \cup (\mathbf{B}^d \times \partial\mathbf{B}^d) = \mathbf{B}^d \times \mathbf{B}^d - \mathbf{R}^d \times \mathbf{R}^d \subset \mathbf{B}^d \times \mathbf{B}^d, \tag{3.13}$$

and

$$\sigma_{a(M)} = a(x),\ \sigma_{a(D)} = a(\xi),\ (x, \xi) \in \mathcal{M}(\mathfrak{A}_0^d). \tag{3.14}$$

THEOREM 3.4. *Let $W = \mathbf{F}^*T\mathbf{F}$ (as in* (3.1.) (3.2)); *We have* $\mathfrak{C}(\mathfrak{h}) \subset \mathfrak{E}$, *and*

$$W\mathfrak{E}W^* = \mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{A}_0^n, \text{ and } \mathfrak{E}/\mathfrak{C}(\mathfrak{H}) \cong C(\mathcal{M}(\mathfrak{A}_0^n), \mathfrak{C}(\mathfrak{h})), \tag{3.15}$$

with the class $C(\mathcal{M}, \mathfrak{C})$ of continuous functions from $\mathcal{M}$ to $\mathfrak{C}$.

We prove Theorem 3.4 in a series of lemmas.

LEMMA 3.5. *Let $a \in C(\mathbf{H}^{n+1})$, $A \in \mathfrak{A}^\sharp$, then $[a(M), A] = a(M)A - Aa(M) \in \mathfrak{C}(\mathfrak{h})$.*

PROOF. It suffices to show this for a generator of $\mathfrak{A}^\sharp$. Let $b\colon \mathbf{R}^{n+1} \to \mathbf{C}$ be the even extension of a. Clearly $E_\kappa a(M) = b(M)E_\kappa$ for $\kappa = e, o$. The generators (1.13) are all of the form $E_\kappa^* B\, E_\gamma$, $B = \Lambda, S_j$; $\kappa, \lambda = e, o$; we get $[a(M), E_\kappa^* B E_\gamma] = E_\kappa^*[b(M), B]E_\gamma = E_\kappa^* C E_\gamma$, $C \in \mathfrak{C}(\mathfrak{K}))$, by **[11]**, for example, which proves the lemma.

The following is an easy consequence of the lemma.

COROLLARY 3.6. *The set of all operators of the form*

(3.16) $$\sum_{i=1}^{N} a_i(M)E_i + C, \; a_i \in C(\mathbf{H}^{n+1}), \; E_i \in \mathfrak{E}^\#, \; C \in \mathfrak{C}(\mathfrak{H}),$$

is dense in $\mathfrak{E}$.

The following two lemmas essentially complete the proof of Theorem 3.4. We defer their proofs until the next section.

LEMMA 3.7. *For* $a \in C(\mathbf{H}^{n+1})$ *let* $\tilde{a} \in C(\mathbf{R}_+^{n+1}) \cap L^\infty(\mathbf{R}_+^{n+1})$ *be defined by* $\tilde{a}(y, \mathbf{x}) = a(0, \mathbf{x})$, $x = (y, \mathbf{x}) \in \mathbf{R}_+^{n+1}$. *Then, for all* $E \in \mathfrak{E}^\#$, $a(M)E - \tilde{a}(M)E \in \mathfrak{C}(\mathfrak{H})$.

By Corollary 3.6 and Lemma 3.7, $\mathfrak{E}$ is the closed linear span mod $\mathfrak{C}(\mathfrak{H})$ of $\{\tilde{a}(M)E, \; \tilde{a} \in C(\mathbf{B}^n), \; E \in \mathfrak{E}^\#$, which equals the closed linear span, mod $\mathfrak{C}$, of

(3.17) $$\{(1 \otimes a_0(\mathbf{M})) \cdot U^*C \otimes b_0(\mathbf{M})U: \; a_0, \; b_0 \in C(\mathbf{B}^n), \; C \in \mathfrak{C}(\mathfrak{h})\},$$

by Lemma 3.5 and Proposition 3.1. The following lemma says that the generators listed in (3.17) are equal, mod $\mathfrak{C}(\mathfrak{H})$ to $U^*(C \otimes (a_1(\mathbf{D})b_0(\mathbf{M}))U$, with $a_1(\mathbf{x}) = a_0(-\mathbf{x})$.

LEMMA 3.8. *Let* $a_0 \in C(\mathbf{B}^n)$, $C \in \mathfrak{C}(\mathfrak{h})$. *Then,*

(3.18) $$(T(1 \otimes a_0(\mathbf{D}))T^*)(C \otimes 1) - C \otimes a_0(\mathbf{D}) \in \mathfrak{C}(\mathfrak{H}).$$

Assuming Lemma 3.8, for the moment, we find that $W\mathfrak{E}W^*$ is the closed linear span, mod $\mathfrak{C}(\mathfrak{H})$, of

(3.19) $$\begin{aligned} WU^*(C \otimes a_1(\mathbf{D})b_0(\mathbf{M}))UW^* &= F^*(C \otimes a_1(\mathbf{D})b_0(\mathbf{M}))F \\ &= C \otimes a_0(\mathbf{M})b_0(\mathbf{D}), \end{aligned}$$

But this amounts to the first relation (3.15). Also $\mathfrak{A}_0 = \mathfrak{A}_0^n$ contains $\mathfrak{C}(\mathfrak{k})$ by [**6**]. Thus $W\mathfrak{E}W^*$ contains $\mathfrak{C}(\mathfrak{H}) = \mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{C}(\mathfrak{k})$, using [**3**], and hence also $\mathfrak{E} \supset \mathfrak{C}(\mathfrak{H})$. Furthermore, the second part of (3.15) then is a consequence of the investigation of $\mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{A}_0$ in [**3**]. This proves Theorem 3.4.

PROOF OF THEOREM 3.3. First, $\mathscr{M} = \mathscr{M}(\mathfrak{A})$ does not contain the points $(x, \xi) \in \mathbf{H}^{n+1} \times \mathscr{M}^\#$ with $x \in \mathbf{R}_+^{n+1}$, $\xi \in \mathring{\mathscr{M}}_1^\# = \mathscr{M}_1^\# - \partial\mathscr{M}_1^\#$ since, for $\psi \in C_0(\mathbf{R}_+^{n+1})$ we get $\psi(M)\Lambda_-\psi(M) \in \mathfrak{C}(\mathfrak{H})$, as is easily confirmed. Since $\mathfrak{C}(\mathfrak{H}) \subset \mathfrak{E}$, we must get the symbol $\psi(x)\sigma_{\Lambda_-}\psi(x) = 0$ at all points of $\mathscr{M}$. But σ_{Λ_-} is > 0 in $\mathscr{M}_1^\#$, by (3.5). For any of the above points ψ can be chosen such that the above product does not vanish. Hence the point cannot belong to $\mathscr{M}$.

Next let $\varepsilon > 0$, and $\mathbf{H}_\varepsilon^{n+1} = \{x \in \mathbf{H}^{n+1}: y \geqq \varepsilon\tau(\mathbf{x})\}$, $\mathbf{R}_\varepsilon^{n+1} = \mathbf{R}^{n+1} \cap \mathbf{H}_\varepsilon^{n+1}$, $\tau = (1 + \mathbf{x}^2)^{1/2}$, $\mathscr{M}_\varepsilon = \mathscr{M} \cap (\mathbf{H}_\varepsilon^{n+1} \times \mathscr{M}^\#)$. Then we can show that no (x, ξ) with $y > 0$, $\xi \in \mathscr{M}_2^\#$ is in $\mathscr{M}$, as follows. By a calculation we get

(3.20) $$\varphi(M)U^*(P_+ \otimes 1)U\varphi(M) = \varphi(M)U^*(\omega(M_0)P_+\omega(M_0) \otimes 1) \; U\varphi(M)$$

whenever $\varphi \in C_0(\mathbf{R}_\varepsilon^{n+1})$ and $\omega \in C_0^\infty((0, \infty])$ equals 1 for $y \geqq \varepsilon$. But the integral operator $\omega(M_0)P_+\omega(M_0)$ is in $\mathfrak{C}(\mathfrak{h})$, because its symbol in $\mathfrak{P}$ vanishes. Hence (3.20) is in $\mathfrak{C}$ which implies the statement, by an argument as above. So we get $\mathcal{M}_\varepsilon \subset \mathcal{M}'_\varepsilon$, with

$$(3.21)\quad \mathcal{M}'_\varepsilon = \{y \geqq \varepsilon\tau, |x| = \infty, \xi \in \mathcal{M}_1^\#\} \cup \{y \geq \varepsilon\tau, |x| < \infty, \xi \in \partial\mathcal{M}_1^\#\}.$$

We now show that $\mathcal{M}_\varepsilon = \mathcal{M}'_\varepsilon$. Let

$$(3.22)\quad \begin{aligned} \mathfrak{A}_\varepsilon &= \{A \in \mathfrak{A} : \operatorname{supp} \sigma_A \subset \mathcal{M}_\varepsilon\}, \\ \mathfrak{A}_{0,\varepsilon} &= \{A \in \mathfrak{A}_0^{n+1} : \operatorname{supp} \sigma_A \subset (\mathbf{H}^{n+1} \times \mathbf{B}^{n+1}) \cap \mathcal{M}(\mathfrak{A}_0^{n+1}) = \mathcal{M}'\}. \end{aligned}$$

Then $\mathfrak{A}_\varepsilon$ and $\mathfrak{A}_{0,\varepsilon}$ are both C^*-algebras, and $\mathfrak{A}_\varepsilon/\mathfrak{C}$ and $\mathfrak{A}_{0,\varepsilon}/\mathfrak{C}(\mathfrak{R}_{n+1})$ are both isomorphic to C^*-subalgebras of $C_\varepsilon(\mathbf{H}_\varepsilon^{n+1} \times \mathcal{M}_1^\#)$, the algebra of continuous functions on $\mathbf{H}^{n+1} \times \mathcal{M}_1^\#$ vanishing at $y = \varepsilon$. because the two balls $\mathbf{B}^{n+1}$ and $\mathcal{M}_1^\#$ are homeomorphic.

Let us observe that the above homeomorphism between the balls may be chosen such that the symbols in $\mathfrak{A}$ and $\mathfrak{A}_0^{n+1}$ of corresponding operators of sets of generators of $\mathfrak{A}$ and $\mathfrak{A}_0^{n+1}$ agree, respectively. Recall, in that respect, that $\mathcal{M}_1^\#$ was constructed from the product $[-\infty, +\infty] \times \mathbf{B}^n$ by collapsing each of the two sets $\pm\infty \times B^n$ into a point. The generators (1.13) of $\mathfrak{A}^\#$ occur in pairs. In fact, these generators are derived from the 'convolution generators' of $\mathfrak{A}_0^{n+1}$ by applying E_e and E_o. Comparing the functions

$$(3.23)\quad \sigma_{A_d}|\mathcal{M}_1^\# = \sigma_{A_n}|\mathcal{M}_1^\#,\ \sigma_{S_d}|\mathcal{M}_1^\# = \sigma_{S_n}|\mathcal{M}_1^\#,\ \sigma_{S_{j,d}}|\mathcal{M}_1^\# = \sigma_{S_{j,n}}|\mathcal{M}^\#,$$

(where the symbols are taken in $\mathfrak{A}^\#$) with

$$(3.24)\quad \sigma_\Lambda, \sigma_{S_0}, \sigma_{S_j},$$

(with symbols of the commutative C^*-algebra $\mathfrak{A}^\#$ generated by these operators), then we find that the functions obtained agree on the interior of their balls of definition, after the transformation of variables

$$(3.25)\quad \xi = (\xi_0, \boldsymbol{\xi}) \to (\xi_0(1 + \boldsymbol{\xi}^2)^{1/2}, \boldsymbol{\xi}) = (\xi'_0, \boldsymbol{\xi}) = \xi'$$

has been carried out. Note that this is the transformation also underlying the unitary map T, above. Since the functions (3.23) and (3.24) are generators of $C(\mathcal{M}_1^\#)$ and $C(\mathbf{B}^{n+1})$, by Stone-Weierstrass, it follows that the homeomorphism (3.25) of the interiors extends into a homeomorphism $\mathcal{M}_1^\# \leftrightarrow \mathbf{B}^{n+1}$, as the dual map of the corresponding isomorphism of the function algebras. Henceforth we consider $\mathcal{M}_1^\#$ and $\mathbf{B}^{n+1}$ identified, by this homeomorphism. Then it follows that

$$(3.26)\quad \sigma_\Lambda = \sigma_{\Lambda_d} = \sigma_{\Lambda_n}, \sigma_S = \sigma_{S_d} = \sigma_{S_n}, \sigma_{S_j} = \sigma_{S_{j,d}} = \sigma_{S_{j,n}}, \text{ as } y > 0.$$

In particular (3.26) holds for $\mathcal{M}_\varepsilon \subset \mathcal{M}'_\varepsilon$. One obtains a map $\pi\colon C(\mathcal{M}'_\varepsilon) \to$

$C(\mathscr{M}_\varepsilon)$, defined by restriction $a \to a|\mathscr{M}_\varepsilon$ which is a $*$-homomorphism and such that (3.26) holds. The map π may be interpreted as a continuous $*$-homomorphism

$$\pi: \mathfrak{A}_{0,\varepsilon}/\mathfrak{C}(\mathfrak{H}) \to \mathfrak{A}_\varepsilon/\mathfrak{C}, \tag{3.27}$$

If we can show π to be an injection, then its dual must be surjective, which means that $\mathscr{M}_\varepsilon = \mathscr{M}'_\varepsilon$, as stated.

But the same map (3.27) may be obtained as follows. Let $\varphi \in C(\mathbf{H}^{n+1})$ be zero near $y = 0$ and $= 1$ near $\mathbf{H}^{n+1}_\varepsilon$. Or, alternately, regard φ extended to $\mathbf{B}^{n+1}$ by setting $\varphi = 0$ when $y < 0$. The operators

$$\varphi(M)\Lambda\varphi(M),\ \varphi(M)S_j\varphi(M),\ \varphi(M)\Lambda_\kappa\varphi(M),\ \varphi(M)S_\kappa\varphi(M),$$
$$\varphi(M)S_{\kappa j}\varphi(M),\ k = d,\ n;\ j = 0, \dots, n,$$

may be regarded as operators mapping either $\mathfrak{H} \to \mathfrak{H}$ or $\mathfrak{K} \to \mathfrak{K}$. In that sense a calculation confirms that

$$\varphi(M)\Lambda\varphi(M) \equiv \varphi(M)\Lambda_n\varphi(M) \equiv \varphi(M)\Lambda_d\varphi(M) \pmod{\mathfrak{C}(\mathfrak{H})}, \tag{3.28}$$

and similar congruences for the S_j etc. Note that the operators

$$\varphi(M)\Lambda\varphi(M),\ \varphi(M)S_j\varphi(M), j = 0, \dots, n, \tag{3.29}$$

generate a subalgebra of $\mathfrak{A}_0^{n+1}$, mod $\mathfrak{C}(\mathfrak{K})$, and a subalgebra of $\mathfrak{A}$, mod $\mathfrak{C}$, which contain $\mathfrak{A}_{0,\varepsilon}$ and $\mathfrak{A}_\varepsilon$, respectively, where the cosets of finitely generated elements correspond to each other by the map π. Note that, with the orthogonal projection $P_\delta: \mathfrak{H} \to L^2(\mathbf{R}^{n+1}_\delta)$.

$$\begin{aligned} \inf_{E\in\mathfrak{C}} \|A + E\| &= \inf_{E\in\mathfrak{C}} \|P_\delta(A + E) + (1 - P_\delta)E\| \geqq \inf_{E\in\mathfrak{C}} \|A + P_\delta E\| \\ &\geqq \inf_{C\in\mathfrak{C}(\mathfrak{C})} \|A + C\| \geqq \inf_{C\in\mathfrak{C}(\mathfrak{C})} \|A + C\| \end{aligned} \tag{3.30}$$

for sufficiently small δ, depending on φ, for any operator A finitely generated from (3.29). This shows that indeed π is injective, so that indeed $\mathscr{M}_\varepsilon = \mathscr{M}'_\varepsilon$. Letting $\varepsilon \to 0$, and taking closure in $\mathscr{M}$ we find that $\mathscr{M}$ contains all of the first and second set of the union (3.11).

Finally let us consider the last two sets of that union. In this connection let us look for an operator with symbol having support in the set $\{(x, \xi): \xi \in \mathscr{M}^\sharp_2\}$ which is not in $\mathfrak{C}$. Consider the algebra $\mathfrak{A}' = U\mathfrak{A}U^*$ which has $U\mathfrak{C}U^* = \mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{A}_0^n$ as a subalgebra, by (3.15). From (3.5) it is clear that $\mathfrak{A}'$ contains $P_\pm \otimes 1$, and thus also the C^*-algebra generated by the operators $P_+ \otimes 1$ and $P_+ \cdot P_- \otimes 1$. These two operators have symbol 0 on $\mathscr{M}^\sharp_1$ in the algebra $\mathfrak{A}^\sharp_-$, but separate interior points of the interval $\mathscr{M}^\sharp_2$. The existence of $\mathbf{B}_f \otimes 1 \in \mathfrak{A}'$ with $\sigma_{B_f} = f$ on $\mathscr{M}^\sharp_2$, $= 0$ on $\mathscr{M}^\sharp_1$ follows, where f may be any continuous function over $\mathscr{M}^\sharp_2$ vanishing at the boundary. Likewise, $\mathfrak{A}'$ contains $Ua(\mathbf{M})b(M_0)U^*$ whenever $a \in C_0^\infty(\mathbf{R}^n)$, $b \in C_0^\infty([0, \infty))$, since $ab \in C_0^\infty(\mathbf{R}^{n+1}_+) \subset \mathfrak{A}$. But,

$$A_f = Ua(\mathbf{M})b(M_0)U^*(B_f \otimes 1) = a(-\mathbf{D})b(M_0/\tau(\mathbf{M}))\,(B_f \otimes 1)$$

is never in $\mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{A}_0^n$, unless either $f = 0$ or $a = 0$ or $b(0) = 0$. Otherwise for every $\varphi \in C_0^\infty(\mathbf{R}^n)$ the operator $(\varphi, A_f\varphi) = C_f \in \mathfrak{L}(\mathfrak{h})$ defined by $(u, C_f v) = (u \otimes \varphi, A_f(v \otimes \varphi))$ would be compact. Indeed, the latter is not true, because a calculation shows that $C_f = B_f \cdot q$ with $q \in C((0, \infty))$ defined by $q(y) = (\varphi, a(-\mathbf{D})b(y/\tau(\mathbf{M}))\varphi)$ (with inner product in $\mathfrak{k}$). Taking symbols in $\mathfrak{P}$ it is clear that C_f is compact if and only if either $f = 0$ or $q(0) = 0$. But if $f \neq 0$, and $a \neq 0$ and $b(0) \neq 0$ then φ always may be chosen such that $q(0) = (\varphi, a(-D)\varphi)\, b(0) \neq 0$.

Suppose that some (x, ξ) with $y = 0$, $|\mathbf{x}| < \infty$ and $\xi \in \mathring{\mathcal{M}}_2$ is not in $\mathcal{M}$. Then $a \neq 0, f \neq 0$ may be chosen with supports in a sufficiently small neighbourhood of x and ξ such that the product still vanishes on $\mathcal{M}$ (because $\mathcal{M}$ is closed). It follows that $U^*A_fU \in \mathfrak{C}$, a contradiction if only $b(0) \neq 0$ is chosen. Hence $\mathcal{M}$ contains all such points. Since $\mathcal{M}$ is closed it therefore also contains the last two sets of the union (3.11). This completes the proof of Theorem 3.3.

4. **Proof of left-over auxiliary results.** PROOF OF LEMMA 3.7. We may restrict our attention to functions of the form $a(x) = P(x)/(1 + x^2)^{k/2}$, with a polynomial P of degree k, since $C(\mathbf{H}^{n+1})$ is the closed linear span of such functions. Then $a(x) - a(0, \mathbf{x}) = s_0(x)q(x)$, with a bounded function q, by a calculation. Also by Corollary 3.6 we may choose $E = P_{\varphi,\psi}$. Then, using Proposition 3.1, we find that it is sufficient to show compactness of $s_0(M)U^*(P_{\varphi,\psi} \otimes 1)U$ for any operator $P_{\varphi,\psi}\, u = \varphi \cdot (\psi, u)$, φ, ψ, $u \in \mathfrak{h}$. Or, equivalently, we must show that, for φ, ψ of the form $\varphi = 1$ in $[0, p]$, $= 0$ in (p, ∞),

$$(P_{\varphi,\psi} \otimes 1)U\, s_0^2(M)U^*(P_{\varphi,\psi} \otimes 1) = H \in \mathfrak{C}(\mathfrak{H}). \tag{4.1}$$

We may write $s_0^2(y, \mathbf{x}) = (\mu y)^2(1 + \mu^2\mathbf{x}^2)^{-1}$, $\mu = (1 + y^2)^{-1/2}$, so that

$$\begin{aligned}(\mathbf{F}s_0^2(M)\mathbf{F}^*u)(x) &= ((\mu y)^2\,(1 + \mu^2\mathbf{D}^2)^{-1}\,u)\,(x)\\ &= (2\pi)^{-n/2}y^2\mu^{2-n}\int G_{n,2}((\mathbf{x} - \mathbf{x}')/\mu)u(y, \mathbf{x}')\,d\mathbf{x}'.\end{aligned} \tag{4.2}$$

Then, by a calculation, $Hu = (2\pi)^{-n/2}(P_{\psi,\psi} \otimes Z)u$ with the integral operator

$$Zu(x) = \int_{\mathbf{R}_n} d\mathbf{x}' \int_0^\infty ds\varphi(s\tau)\varphi(s\tau')s^2\mu^{2-n}(s)G_{n,2}((\mathbf{x} - \mathbf{x}')/\mu(s))(\tau\tau')^{1/2}u(\mathbf{x}'). \tag{4.3}$$

Again it suffices to show that $Z \in \mathfrak{C}(\mathfrak{k})$. The interchange in integrations leading to (4.3) can be justified, since φ has compact support, from the properties of $G_{n,2}$, as listed in section 1. (Note that in (4.3) we introduced $\tau = \tau(\mathbf{x})$, $\tau' = \tau(\mathbf{x}')$.)

Substituting φ in details we get Z to have the kernel

$$
\begin{aligned}
z(\mathbf{x}, \mathbf{x}') &= (\tau\tau')^{1/2} \int_0^{p\gamma} s^2(1 + s^2)^{n/2-1} G_{n,2}((\mathbf{x} - \mathbf{x}')(1 + s^2)^{1/2})\, ds \\
&= (\tau\tau')^{1/2} \int_0^{p\gamma} (p\gamma - s) G(s, \mathbf{x} - \mathbf{x}')\, ds,
\end{aligned} \tag{4.4}
$$

with $\gamma = \gamma(\mathbf{x}, \mathbf{x}') = \mathrm{Min}\{\tau^{-1}, \tau'^{-1}\}$, and with

$$
G(s, \zeta) = \partial/\partial s\{s^2(1 + s^2)^{n/2-1} G_{n,2}(\zeta(1 + s^2)^{1/2}) \tag{4.5}
$$

by a partial integration. Using the estimates (1.11) and the fact that $(\tau\tau')^{1/2} \leqq 1/\gamma$, one finds that for every $\varepsilon > 0$, $z = z_{1,\varepsilon} + z_{2,\varepsilon}$, with $|z_{1,\varepsilon}(\mathbf{x}, \mathbf{x}')| \leqq \lambda(\mathbf{x}) f_\varepsilon(\mathbf{x} - \mathbf{x}')$, $z_{2,\varepsilon}(\mathbf{x}, \mathbf{x}') \leqq g_\varepsilon(\mathbf{x} - \mathbf{x}')$, $\|g_\varepsilon\|_{L^1} < \varepsilon$, with an $f_\varepsilon \in L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ and $\lambda(\mathbf{x}) = (1 + \mathbf{x}^2)^{-1/2}$. If χ_q denotes the characteristic function of the ball $\{|\mathbf{x}| < q\}$, then it follows that $\chi_q(\mathbf{x}) z_{1,\varepsilon}(\mathbf{x}, \mathbf{x}') = 0(\chi_q(\mathbf{x}) f_\varepsilon(\mathbf{x} - \mathbf{x}')) \in L^2(\mathbf{R}^{2n})$, so that the operator $\chi_q(M) Z_{1,\varepsilon}$ is a Hilbert Schmidt operator and therefore compact. On the other hand we get $(1 - \chi_q(M))Z \to 0$ in norm convergence of $\mathfrak{k}$, by Schur's criterion. This implies compactness of Z.

Proof of Lemma 3.8. It suffices to consider the assertion for the case where a_0 is either $\lambda(\mathbf{x})$ or $s_j(\mathbf{x})$, $j = 1, \ldots, n$. Also we may assume $C = P_{\varphi,\psi}$ again, as in the preceding proof, and with the same choice of φ, even with $p = 1$. With these simplification we are reduced to showing that $K = K_1 + K_2$ is compact, with

$$
\begin{aligned}
(K_1 u)(x) &= \int (\tau'/\tau)^{1/2}\{\varphi(y\tau'/\tau) - \varphi(y)\}\, k(\mathbf{x} - \mathbf{x}')\psi(t) u(t, \mathbf{x}')\, dt d\mathbf{x}', \\
(K_2 u)(x) &= \int ((\tau'/\tau)^{1/2} - 1)\varphi(y) k(\mathbf{x} - \mathbf{x}')\psi(t) u(t, \mathbf{x}')\, dt d\mathbf{x}',
\end{aligned} \tag{4.6}
$$

where k is the convolution kernel of $a_0(\mathbf{D})$, as in (1.11).

Observe that $K_2 = P_{\varphi,\psi} \otimes V$, with an integral operator $V: \mathfrak{k} \to \mathfrak{k}$ having kernel $((\tau'/\tau)^{1/2} - 1) k(\mathbf{x} - \mathbf{x}')$. The compactness of V may be proven with the method used for Z in the preceding proof.

We turn now to K_1. Let $\varphi_\alpha(y) = \varphi(\alpha y)$ and write, for $\eta > 1$,

$$
K_1 = \varphi_\eta(M_0) K_1 + (1 - \varphi_{1/\eta}(M_0)) K_1 + (\varphi_{1/\eta}(M_0) - \varphi_\eta(M_0)) K_1. \tag{4.7}
$$

We shall show that the first two operators at right of (4.7) are Hilbert Schmidt, and that the last one tends to zero in norm, as $\eta \searrow 1$. The kernel of K_1 is non-zero only if $y\tau'/\tau > 1$ as $y \leqq 1$ (case (a)), or, $y\tau'/\tau \leqq 1$, as $y > 1$ (case(b)). In case (a) we get $1 + \mathbf{x}^2 \leqq y^2(1 + \mathbf{x}'^2)$, $y < 1$, or, $0 \leqq (1 - y^2)(1 + \mathbf{x}'^2) \leqq \mathbf{x}'^2 - \mathbf{x}^2 \leqq 2|\mathbf{x}'|\, |\mathbf{x} - \mathbf{x}'|$, so

$$
|\mathbf{x} - \mathbf{x}'|^{-1} \leqq 2|\mathbf{x}'|((1 - y^2)(1 + \mathbf{x}'^2))^{-1} \leqq c\lambda(\mathbf{x}')(1 - y^2)^{-1}. \tag{4.8}
$$

Similarly in case (*b*) we get

$$|\mathbf{x} - \mathbf{x}'|^{-1} \leq c(1 - y^{-2})^{-1} \lambda(\mathbf{x}). \tag{4.9}$$

Let $\eta > 1$, then $\varphi_\eta(M_0)K_1$ has kernel zero for case (*a*), otherwise,

$$\kappa_1(y, t, \mathbf{x}, \mathbf{x}') = \varphi_\eta(y)k(\mathbf{x} - \mathbf{x}')\psi(t)(\tau'/\tau)^{1/2}. \tag{4.10}$$

For arbitrary y, t, $\mathbf{x}$, $\mathbf{x}'$ we get, using $k(z) = 0(|z|^{-r} \lambda^r(z))$, $r \geqq n$, (c.f. (1.11)),

$$\begin{aligned} &\kappa_1(y, t, \mathbf{x}, \mathbf{x}') \\ &= 0(\varphi_\eta(y)(1 - y^2)^{-r}\psi(t)\lambda^{1/2}(\mathbf{x})\lambda^{r-1/2}(\mathbf{x}')\lambda^r(\mathbf{x} - \mathbf{x}')) \end{aligned} \tag{4.11}$$

for all $r \geqq n$. This implies that $\kappa_1 \in L^2(\mathbf{R}^{2n+2})$. i.e., is a Schmidt kernel. Accordingly $\varphi_\eta(M_0)K_1 \in \mathfrak{C}(\mathfrak{H})$.

Similarly for $L_\eta = (1 - \varphi_{1/\eta}(M_0))K_1$ the kernel is $\neq 0$ only in case (*b*), and we get its kernel estimated by

$$\begin{aligned} &\kappa_2(y, t, \mathbf{x}, \mathbf{x}') \\ &= 0((1 - \varphi_{1/\eta}(y))(1 - y^{-2})^{-r} \psi(t)\lambda^r(\mathbf{x} - \mathbf{x}')\lambda^{r-1/2}(\mathbf{x})\lambda^{1/2}(\mathbf{x}')) \end{aligned} \tag{4.12}$$

which again implies L_η to be a Schmidt-operator, hence compact.

Finally, regarding the third term in (4.7) —with kernel χ_3— we note that $\kappa_3 \neq 0$ only as $1/\eta < y < \eta$, which is a small interval as $\eta \searrow 1$. For such y we may use either (4.8) or (4.9) for the estimate $|\mathbf{x} - \mathbf{x}'|^{-1} \leqq c|y - 1|^{-1}$. The we may estimate, with (1.11) again,

$$\kappa_3 = 0(\psi(t)(\varphi_{1/\eta}(y)) - \varphi_\eta(y))|\mathbf{x} - \mathbf{x}'|^{-n+\varepsilon}|y - 1|^{-\varepsilon} e^{-\varepsilon|\mathbf{x}-\mathbf{x}'|}).$$

where also Peetre's inequality was used, and where $0 < \varepsilon < 1$. It then is a consequence of Schur's Lemma that this operator tends to zero in operator norm, as $\eta \searrow 1$. This proves Lemma 3.8.

LEMMA 4.1. *The C*-algebra, generated by the operators* (1.16) *only, is equal to* $\mathfrak{P}^\#$*, hence contains* $\mathfrak{C}(\mathfrak{h})$.

PROOF. We have $0 \neq Q_+ \in \mathfrak{C}(\mathfrak{h}) \cap \mathfrak{P}^\#$. Using a resolvent integral one shows that $\mathfrak{P}^\#$ also contains all the orthogonal projections Q onto the (finite dimensional) eigenspaces to eigenvalues $\neq 0$ of the compact self-adjoint operator Q_+. Any eigenfunction $\varphi(y)$ to an eigen value $\lambda \neq 0$ is in $\mathfrak{h}$, is analytic in $(0, \infty)$, and is of exponential decay in a complex neighbourhood of $(1, \infty)$. Therefore the Fourier cosine transform also is analytic. In particular this implies that $\tilde{\varphi}$ will not vanish identically anywhere in $(0, \infty)$. Also, the Fourier cosine transform diagonalizes the operator $H^\#_\lambda = Q_- + Q_+ = Q_n$; we get $F_c H^\#_\lambda F_c = \lambda(M_0)$. If there exists a 1-dimensional nonvanishing eigenvalue it follows that $\mathfrak{P}^{\#}$ contains $P_{\omega,\psi}$ arbitrarily close to any operator of rank 1, hence contains $\mathfrak{C}(\mathfrak{h})$. If no such eigenvalue exists one will iterate the procedure starting from $f(Q_n)Qf(Q_n)$, with a suitable real-valued function f, to obtains projections in $\mathfrak{P}^{\#}$ of lower and lower dimension, finally of dimension 1, which will complete the proof of Lemma 4.1.

REMARK 4.2. By a similar technique one proves that $U\mathfrak{C}^{\#}U^*$ contains $C \otimes \lambda(\mathbf{M})$ and $C \otimes s_j(\mathbf{M})$ for all $C \in \mathfrak{C}(\mathfrak{h})$ and $j = 1, \ldots, n$.

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