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LINEARIZED DECAY OF DISTURBANCES OF A VISCOUS LIQUID IN AN OPEN CONTAINER

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Dedicated to N. Aronszajn on the occasion
of his seventieth birthday

In this paper a particular linearized free surface problem of fluid mechanics is studied. We consider a small disturbance from equilibrium of a heavy incompressible viscous liquid in an open container. A constant external acceleration acts on the liquid. Surface tension is neglected which corresponds physically to the assumption of a strong external acceleration.

We study the Navier-Stokes equations linearized about the equilibrium state. This problem has been previously investigated by Kreĭn [17], Kreĭn and Laptev [18], and Askerov, Kreĭn, and Laptev [5]. Generalized solutions are obtained in these works, but time decay of the disturbance can be inferred only for special initial data. Work of Dussan V. and Davis [7] has shown the necessity of admitting initial data with strong singularities at the edge of the free surface, while Hocking [15] has investigated changes in the basic model to avoid these singularities. In Greenlee [14] existence was proven for singular initial configurations of the free surface. More importantly it was proven that initial perturbations of the flow tend to zero in time in the energy norm, while initial perturbations of the free surface decay in a weak sense appropriate to the singular initial data.

In the present work estimates of the rate of decay in time of initial perturbations of the flow and free surface are obtained. The rates of decay depend on the nature of the initial configuration of the free surface. The estimates are based on a detailed spectral analysis developed in Greenlee [12], [13] (cf. also Larionov [20]).

In §1 the problem is formulated and pertinent definitions and results of [14] are summarized. The rate of decay theorem is developed in §2 via three lemmas. Lemma 1 supplements the decay theorem of [14] for singular initial free surface data. In Lemma 2 more rapid rates of decay for smoother initial configurations of the free surface are derived from a theorem of [12], [20]. Lemma 3 provides additional rates of decay by

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applying quadratic interpolation (cf. Lions [**21**], Lions and Magenes [**24**], and Adams, Aronszajn and Hanna [**1**]) to the results of the previous two lemmas. The decay theorem then gives more concrete analytical statements implied by the interpolation norms.

A few observations are now in order. The Landau symbols o, O are always used to denote a limiting process as $t \to \infty$. Though the function spaces considered are obviously real, we use the corresponding complexifications whenever convenient. We study the problem via weak L^2 solution methods. It is well known that the pressure disappears in the resulting variational formulation, and is then determined via an orthogonal decomposition of the underlying Hilbert space. Finally, the author wishes to thank H. Brezis and S. H. Davis for enlightening comments on this work.

1. **Formulation of the Problem.** In this section we formulate the linearized free surface problem. Further details may be found in [**14**].

The open container occupied by the fluid in equilibrium is described as follows. Let Ω be a bounded Lipschitzian Graph (LG) domain in $\mathbf{R}^3$ (cf. Adams, Aronszajn, and Smith [**2**]), contained in the half-space $x_3 < 0$, where $\mathbf{x} = (x_1, x_2, x_3)$ denotes a point in $\mathbf{R}^3$. Identifying $x_3 = 0$ with $\mathbf{R}^2$, $\Gamma_s \subset \partial\Omega \cap \{x_3 = 0\}$ is assumed to be a nonempty LG domain in $\mathbf{R}^2$, which constitutes the equilibrium free surface. Now let $\Gamma_r = \partial\Omega \backslash \bar{\Gamma}_s$. Then $\partial\Omega$ is the disjoint union $\Gamma_s \cup \partial\Gamma_s \cup \Gamma_r$, and Γ_r serves as the container of the fluid in equilibrium. Observe that $\Gamma_r \cap \{x_3 = 0\}$ may be nonempty, in which case its closure is called an *overhanging dock*. If $\partial\Omega \cap \{x_3 = 0\}$ contains a neighborhood in $\mathbf{R}^2$ of $\bar{\Gamma}_s$, Γ_s is said to be *surrounded by an overhanging dock*.

We assume that a constant atmospheric pressure, $\bar{p}$, is given above the fluid. A constant acceleration, g, having direction perpendicular to $x_3 = 0$, maintains the fluid in the container. The fluid is assumed to have constant viscosity μ, constant density ρ, and $\nu = \mu/\rho$ is the kinematic viscosity. In this context we study the first-order linearized Navier-Stokes equations (cf. Wehausen and Laitone [**26**]), i.e., the perturbation equations obtained by linearization about the exact equilibrium solution given by zero flow and pressure $\bar{p} - \rho g x_3$.

Thus we consider the system of differential equations for $\mathbf{x} \in \Omega$ and $t \in (0, \infty)$

$$(1.1) \qquad \partial\mathbf{u}/\partial t = \nu\Delta\mathbf{u} - (1/\rho)\operatorname{grad} p, \quad \operatorname{div}\mathbf{u} = 0,$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity vector, Δ is the (spatial) Laplacian, and p is the pressure. We adopt the notation (cf. Duvaut and Lions [**8**])

$$u_{i,j} = \partial u_i/\partial x_j, \qquad i, j = 1, 2, 3,$$

and make use of the summation convention over repeated indices.

The linearized boundary conditions are as follows. First of all the viscous boundary condition on the wall of the container gives

$$\mathbf{u} = 0 \tag{1.2}$$

for $\mathbf{x} \in \Gamma_r$ and $t > 0$. Secondly, the disturbed free surface is given by an equation of the form

$$x_3 = \eta(\mathbf{x}', t),$$

where $\mathbf{x}' = (x_1, x_2) \in \Gamma_s$ and $t > 0$. Linearization (cf. **[26]**) gives the equations

$$\eta = (1/g)((p/\rho) - 2\nu u_{3,3}), \quad \partial\eta/\partial t = u_3 \tag{1.2$'$}$$

on Γ_s. Equations (1.2′) and absence of stress on the free surface yield

$$\begin{aligned} &u_{1,3} + u_{3,1} = 0, \quad u_{2,3} + u_{3,2} = 0, \\ &(\partial/\partial t)(p - 2\mu u_{3,3}) = \rho g u_3 \end{aligned} \tag{1.3}$$

for $\mathbf{x}' \in \Gamma_s$ and $t > 0$.

We wish to determine the motion of the liquid from a given disturbance, i.e., a given velocity distribution throughout the fluid, and a given initial shape of its free surface. The initial velocity distribution is given by setting

$$\begin{aligned} \mathbf{u}|_{t=0} &= \mathbf{u}_0 \text{ where } \operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega, \\ \mathbf{u}_0 &= 0 \text{ on } \Gamma_r. \end{aligned} \tag{1.4}$$

To specify the initial shape of the free surface, first note that by the transport theorem (cf. Serrin **[25]**), incompressibility and conservation of mass require that

$$\int_{\Gamma_s} \eta \equiv 0. \tag{1.4$'$}$$

Thus we impose

$$\eta|_{t=0} = \eta_0 \text{ where} \int_{\Gamma_s} \eta_0 = 0. \tag{1.5}$$

In order to obtain a precise formulation of a solvable, weak version of the initial-boundary value problem (1.1)–(1.5), certain Sobolev spaces of square-summable vector—and scalar-valued functions (cf. **[8, 24]**) and certain dual spaces will be introduced. We note that since Ω and Γ_s are LG, any element of $H^\alpha(\Omega)$ (respectively $H^\alpha(\Gamma_s)$), $\alpha \geqq 0$, is the restriction to Ω (respectively Γ_s) of an element of $H^\alpha(\mathbf{R}^3)$ (respectively $H^\alpha(\mathbf{R}^2)$). This follows from the simultaneous extension theorem for Bessel potentials and the fact that the Lebesgue correction of an element of H^α is a Bessel potential of order α (cf. **[2]**).

On Ω, we begin with the Hilbert space $(H^1(\Omega))^3$ with the usual inner product

$$(\mathbf{u}, \mathbf{v})_{1,\Omega} = (u_i, v_i)_{1,\Omega} = \int_\Omega (u_i v_i + u_{i,j} v_{i,j}).$$

The basic Hilbert space for the flow in Ω is constructed as follows. Denote by $\mathscr{C}(\Omega)$ the class of C^∞ vector-valued, solenoidal (i.e., divergence zero) functions on $\bar{\Omega}$ which vanish in a (variable) neighborhood of $\bar{\Gamma}_r$. $\mathscr{H}^1(\Omega)$ (respectively $\mathscr{H}^0(\Omega) \equiv \mathscr{L}^2(\Omega)$) is defined as the closure of $\mathscr{C}(\Omega)$ in $(H^1(\Omega))^3$ (respectively $(H^0(\Omega))^3 = (L^2(\Omega))^3$). Recall that the orthogonal complement of $\mathscr{L}^2(\Omega)$ in $(L^2(\Omega))^3$ consists of gradients of distributions (cf. Ladyzhenskaya **[19]**, Kopačevskiĭ **[16]**, and Lions **[22]**).

Let

$$\varepsilon_{i,j}(\mathbf{v}) = v_{i,j} + v_{j,i}, \; i, j = 1, 2, 3.$$

The symmetric bilinear form

$$E(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_\Omega \varepsilon_{i,j}(\mathbf{u}) \varepsilon_{i,j}(\mathbf{v})$$

is obviously continuous on $\mathscr{H}^1(\Omega)$. Moreover, there exists $c > 0$ such that

$$E(\mathbf{v}, \mathbf{v}) \geqq c \,\|\mathbf{v}\|^2_{1,\Omega}$$

for all $\mathbf{v} \in \mathscr{H}^1(\Omega)$, so that we may take $E(\mathbf{u}, \mathbf{v})$ as the inner product in $\mathscr{H}^1(\Omega)$ (cf. **[8]**, **[14]**, and Gobert **[9]**). The form $E(\mathbf{u}, \mathbf{v})$ arises in Green's formula for the Navier-Stokes equations (cf. **[19]**). For smooth solenoidal $\mathbf{u}$, smooth p and $\mathbf{v} \in \mathscr{C}(\Omega)$,

$$\int_\Omega (-\, \nu \Delta \mathbf{u} + (1/\rho) \operatorname{grad} p) \cdot \mathbf{v}$$
$$= \nu E(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_s} [\nu \varepsilon_{1,3}(\mathbf{u}) v_1 + \nu \varepsilon_{2,3}(\mathbf{u}) v_2 + ((-p/\rho) + 2\nu u_{3,3}) v_3].$$

This formula also holds for $\mathbf{v} \in \mathscr{H}^1(\Omega)$, since elements of $\mathscr{H}^1(\Omega)$ have a well-defined trace (cf. **[24]**) on Γ_s. In fact, there is a continuous linear trace mapping of $\mathscr{H}^1(\Omega)$ into $(H^{1/2}_{00}(\Gamma_s))^3$ (cf. **[24]** and Greenlee **[11]**) where the Hilbert space $H^{1/2}_{00}(\Gamma_s)$ is normed by

$$\|v\|^2_{1/2,\Gamma_s} = \int_{\Gamma_s} v^2 d^{-1} + \int_{\Gamma_s}\int_{\Gamma_s} (|v(\mathbf{x}') - v(\mathbf{y}')|^2/|\mathbf{x}' - \mathbf{y}'|^3) \mathbf{dx}'\mathbf{dy}'$$

with $d = d(\mathbf{x}') = \operatorname{dist}(\mathbf{x}', \partial\Gamma_s)$ for $\mathbf{x}' \in \Gamma_s$.

For $\mathbf{v} \in \mathscr{H}^1(\Omega)$, the trace of v_3 on Γ_s also has mean value zero, i.e.,

$$\int_{\Gamma_s} S\mathbf{v} = 0 \text{ for all } \mathbf{v} \in \mathscr{H}^1(\Omega),$$

where S: $\mathscr{H}^1(\Omega) \to H_{00}^{1/2}(\Gamma_s)$ is the continuous linear operation of taking the trace of $\mathbf{v}_3$ on Γ_s. Now let

$$\mathscr{H}^{1/2}(\Gamma_s) = \{v \in H_{00}^{1/2}(\Gamma_s): \int_{\Gamma_s} v = 0\},$$

provided with the norm of $H_{00}^{1/2}(\Gamma_s)$. Then $\mathscr{H}^{1/2}(\Gamma_s)$ is a Hilbert space and $S: \mathscr{H}^1(\Omega) \to \mathscr{H}^{1/2}(\Gamma_s)$ is continuous. In Lemma 2.1 of **[14]** it was proven that $S(\mathscr{H}^1(\Omega))$ is dense in

$$\mathscr{H}^0(\Gamma_s) \equiv \mathscr{L}^2(\Gamma_s) \equiv \{v \in L^2(\Gamma_s): \int_{\Gamma_s} v = 0\}$$

with the obvious topology on $\mathscr{L}^2(\Gamma_s)$. The proof of Lemma 2.1 of **[14]** also shows that $S(\mathscr{H}^1(\Omega)) \cap H_0^1(\Gamma_s)$ is dense in

$$\{v \in H_0^1(\Gamma_s) : \int_{\Gamma_s} v = 0\}$$

where $H_0^1(\Gamma_s)$ denotes the closure of $C_0^\infty(\Gamma_s)$ in $H^1(\Gamma_s)$. It now follows from Chapter 5 of **[11]**, Theorem 13.3 of **[24]**, and quadratic interpolation that $S(\mathscr{H}^1(\Omega))$ is dense in $\mathscr{H}^{1/2}(\Gamma_s)$.

In order to formulate an evolution equation whose classical solutions are those of (1.1)–(1.5) (recall also (1.2′)), we first define weak solutions to two stationary problems. To simplify notation we now normalize the density, ρ, to unity.

PROBLEM 1. Given $\mathbf{f}$ on Ω, to solve

$$-\nu\Delta\mathbf{r} + \operatorname{grad} p_1 = \mathbf{f}, \operatorname{div} \mathbf{r} = 0$$

on Ω with the boundary conditions,

$$\mathbf{r} = 0 \text{ on } \Gamma_r,$$
$$\varepsilon_{1,3}(\mathbf{r}) = \varepsilon_{2,3}(\mathbf{r}) = -p_1 + 2\nu r_{3,3} = 0 \text{ on } \Gamma_s.$$

Let $\mathscr{H}^{-1}(\Omega)$ be the completion of $\mathscr{L}^2(\Omega)$ in the norm

$$\|\mathbf{u}\|_{-1,\Omega} = \sup\{|(\mathbf{u}, \mathbf{v})_{0,\Omega}| : \mathbf{v} \in \mathscr{H}^1(\Omega) \text{ and } \|\mathbf{v}\|_{1,\Omega} = 1\},$$

and denote the pairing between $\mathscr{H}^{-1}(\Omega)$ and $\mathscr{H}^1(\Omega)$ by $\langle \cdot, \cdot \rangle_\Omega$. Weak solutions of Problem 1 are defined as follows.

DEFINITION 1. Given $\mathbf{f} \in \mathscr{H}^{-1}(\Omega)$, $\mathbf{r} \in \mathscr{H}^1(\Omega)$ is a weak solution of Problem 1 if

$$\nu E(\mathbf{r}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega$$

for all $\mathbf{v} \in \mathscr{H}^1(\Omega)$.

For each $\mathbf{f} \in \mathscr{H}^{-1}(\Omega)$ there is a unique weak solution $\mathbf{r} \in \mathscr{H}^1(\Omega)$ of

Problem 1. If $\mathbf{f} \in \mathscr{L}^2(\Omega)$, $\mathbf{r}$ is the solution of the operator equation $\nu\mathscr{A}\mathbf{r} = \mathbf{f}$ where $\mathscr{A}$ is a positive definite self adjoint operator in $\mathscr{L}^2(\Omega)$, and $\mathscr{P} = \mathscr{A}^{-1}$ is a positive, compact self adjoint operator in $\mathscr{L}^2(\Omega)$. The operator $\mathscr{A}$ extends by continuity to a topological isomorphism of $\mathscr{H}^1(\Omega)$ onto $\mathscr{H}^{-1}(\Omega)$.

Problem 2. Given ϕ on Γ_s, to solve

$$-\nu\Delta \mathbf{s} + \operatorname{grad} p_2 = 0, \ \operatorname{div} \mathbf{s} = 0$$

on Ω with the boundary conditions,

$$\mathbf{s} = 0 \text{ on } \Gamma_r,$$
$$\varepsilon_{1,3}(\mathbf{s}) = \varepsilon_{2,3}(\mathbf{s}) = 0, \quad -p_2 + 2\nu s_{3,3} = \phi \text{ on } \Gamma_s.$$

In **[14]** a norm $\|\cdot\|_{R(S)}$ was defined on $R(S) \equiv S(\mathscr{H}^1(\Omega))$, with which $R(S)$ is a Hilbert space, and

$$R(S) \underset{c}{\subset} \mathscr{H}^{1/2}(\Gamma_s) \underset{c}{\subset} \mathscr{L}^2(\Gamma_s).$$

The notation $\underset{c}{\subset}$ is used to denote that the inclusions are algebraic and topological (cf. Aronszajn and Gagliardo **[4]**). Each inclusion is also dense. Now let $\mathscr{H}^-(\Gamma_s)$ be the dual of $R(S)$, with duality taken with respect to $\mathscr{L}^2(\Gamma_s)$, and denote the pairing between $\mathscr{H}^-(\Gamma_s)$ and $R(S)$ by $\langle \cdot, \cdot \rangle \Gamma_s$. Then weak solutions of Problem 2 are defined as follows.

Definition 2. Given $\phi \in \mathscr{H}^-(\Gamma_s)$, $\mathbf{s} \in \mathscr{H}^1(\Omega)$ is a weak solution of Problem 2 if

$$\nu E(\mathbf{s}, \mathbf{v}) = \langle \phi, S\mathbf{v} \rangle_{\Gamma_s}$$

for all $\mathbf{v} \in \mathscr{H}^1(\Omega)$.

For each $\phi \in \mathscr{H}^-(\Gamma_s)$ there is a unique weak solution $\mathbf{s} \in \mathscr{H}^1(\Omega)$ of Problem 2. If $\phi \in \mathscr{L}^2(\Gamma_s)$, $\mathbf{s}$ is given by the operator equation $\nu\mathbf{s} = T\phi$ where $T\colon \mathscr{L}^2(\Gamma_s) \to \mathscr{H}^1(\Omega)$ is continuous and $T\colon \mathscr{L}^2(\Gamma_s) \to \mathscr{L}^2(\Omega)$ is compact. Furthermore, $\mathscr{H}^1(\Omega)$ is the orthogonal direct sum of $N(S)$, the null space of S, and the closure of the range of T in $\mathscr{H}^1(\Omega)$: that is,

$$\mathscr{H}^1(\Omega) = N(S) \oplus \overline{R(T)}.$$

T extends by continuity to a topological isomorphism of $\mathscr{H}^-(\Gamma_s)$ onto $\overline{R(T)}$ in $\mathscr{H}^1(\Omega)$.

We remark that when Γ_s is surrounded by overhanging dock, the results of Cattabriga **[6]** imply that $R(S) = \mathscr{H}^{1/2}(\Gamma_s)$ up to equivalent norms (cf. **[14]**). A similar argument shows that if $\phi \in \mathscr{H}^{1/2}(\Gamma_s)$ and ϕ has compact support in Γ_s, then $\phi \in R(S)$. But it remains an open question whether $R(S)$ and $\mathscr{H}^{1/2}(\Gamma_s)$ are the same spaces in general.

Now following **[18]**, **[5]**, and **[14]**, weak solutions of (1.1) – (1.5) are obtained via the system of differential equations in $\mathscr{L}^2(\Omega)$

$$\nu \mathbf{r}(t) + \mathscr{A}^{-1}\mathbf{u}_t = 0, \quad \nu \mathbf{s}_t + gTS\mathbf{u}(t) = 0,$$
$$\mathbf{u}(t) = \mathbf{r}(t) + \mathbf{s}(t),$$

with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \mathbf{s}(0) = \nu^{-1}T(-g\eta_0),$$
$$\mathbf{r}(0) = \mathbf{u}(0) - \mathbf{s}(0),$$

where $\mathbf{u}_0 \in \mathscr{H}^1(\Omega)$ and $\eta_0 \in \mathscr{H}^-(\Gamma_s)$ are arbitrary. Writing $\mathbf{u} = \mathscr{A}^{-1/2}\,\mathbf{w}$, $\mathbf{r} = \mathscr{A}^{-1/2}\mathbf{y}$, and $\mathbf{s} = \mathscr{A}^{-1/2}\mathbf{z}$, weak solutions of (1.1)–(1.5) are defined by the initial value problem in $(\mathscr{L}^2(\Omega))^2$

$$\mathbf{X}_t + (\mathfrak{A} + \mathfrak{B})\mathbf{X} = 0, \quad \mathbf{X}(0) = \mathbf{X}_0, \tag{1.6}$$

where

$$\mathbf{X} = \begin{pmatrix}\mathbf{y}\\ \mathbf{z}\end{pmatrix}, \quad \mathfrak{A} = \begin{pmatrix}\nu\mathscr{A} & 0\\ 0 & 0\end{pmatrix}, \quad \mathfrak{B} = \frac{g}{\nu}\begin{pmatrix}-\mathscr{Q} & -\mathscr{Q}\\ \mathscr{Q} & \mathscr{Q}\end{pmatrix},$$
$$\mathbf{X}_0 = \begin{pmatrix}\mathbf{y}_0\\ \mathbf{z}_0\end{pmatrix} \in \mathscr{L}^2(\Omega) \times \overline{R(\mathscr{Q})},$$

and $\mathscr{Q} = \mathscr{A}^{1/2}TS\mathscr{A}^{-1/2}$ is a nonnegative self adjoint compact operator in $\mathscr{L}^2(\Omega)$.

Definition 3. A weak solution of (1.1)–(1.5) is a pair of functions $t \to \mathbf{u}(t)$, $\eta(t)$, $t \geqq 0$, with values in $\mathscr{H}^1(\Omega)$, $\mathscr{H}^-(\Gamma_s)$, respectively, defined as follows: $\mathbf{u} = \mathbf{r} + \mathbf{s} = \mathscr{A}^{-1/2}(\mathbf{y} + \mathbf{z})$, where $\mathbf{y}$, $\mathbf{z}$ satisfies (1.6); recalling that T^{-1} extends to a topological isomorphism of $\overline{R(T)}$ in $\mathscr{H}^1(\Omega)$ onto $\mathscr{H}^-(\Gamma_s)$, η is defined by applying this operator to $(-\nu/g)\mathbf{s} = (-\nu/g)\mathscr{A}^{-1/2}\mathbf{z}$.

The initial-boundary value problem (1.1)–(1.5) has a unique weak solution for each $\mathbf{u}_0 \in \mathscr{H}^1(\Omega)$ and $\eta_0 \in \mathscr{H}^-(\Gamma_s)$. The time decay results of §2 are obtained by spectral analysis of the operator $\mathfrak{A} + \mathfrak{B}$. This spectral problem is equivalent (cf. **[5]**, **[14]**) to the quadratic characteristic parameter problem $\mathbf{w} = (\lambda/\nu)\mathscr{P}\mathbf{w} + (g/\lambda\nu)\,\mathscr{Q}\mathbf{w}$. By means of this equivalence it follows that $\mathfrak{A} + \mathfrak{B}$ has an infinite discrete sequence of eigenvalues $\{\lambda_n\}$, all of whose real parts are positive. Zero and infinity are both limit points of $\{\lambda_n\}$, and are the only limit points of $\{\lambda_n\}$. Each eigenvalue λ_n has finite algebraic multiplicity. All nonreal eigenvalues of $\mathfrak{A} + \mathfrak{B}$ lie in the annulus

$$(\nu/2\,\|\mathscr{P}\|) < |\lambda| < (2g/\nu)\,\|\mathscr{Q}\|,$$

which is empty if $\nu^2 \geqq 4g\,\|\mathscr{P}\| \cdot \|\mathscr{Q}\|$. All eigenvalues of $\mathfrak{A} + \mathfrak{B}$ whose algebraic and geometric multiplicities do not agree are in the annulus

$$(\nu/2\|\mathscr{P}\|) \leqq |\lambda| \leqq (2g/\nu)\ \|\mathscr{Q}\|,$$

which is empty if $\nu^2 > 4g\,\|\mathscr{P}\| \cdot \|\mathscr{Q}\|$.

2. **The Decay Theorem.** We now proceed to the theorem on the rate of decay in time of initial disturbances of the flow and free surface. Three lemmas essential to the development are given first. For simplicity of notation let $P = \nu^{-1}\mathscr{P}$, $Q = (g/\nu)\mathscr{Q}$, and let Q_1 be the orthogonal projection onto $\overline{R(Q)}$ in $\mathscr{L}^2(\Omega)$.

LEMMA 1. *Let* $\mathbf{u}_0 \in \mathscr{H}^1(\Omega)$ *and* $\eta_0 \in \mathscr{H}^-(\Gamma_s)$. *Then the corresponding weak solution* $\mathbf{u}$, η *of* (1.1)$-$(1.5) *satisfies* $\mathbf{u} = \mathrm{o}(1)$ *in* $\mathscr{H}^1(\Omega)$, $\eta = \mathrm{o}(1)$ *in* $\mathscr{H}^-(\Gamma_s)$, $\mathbf{u}_t = \mathrm{o}(1/t)$ *in* $\mathscr{H}^1(\Omega)$, $\eta_t = \mathrm{o}(1/t)$ *in* $\mathscr{H}^-(\Gamma_s)$, *and* $\eta_t = \mathrm{o}(1)$ *in* $R(S)$.

PROOF. The fact that $\eta_t \in R(S)$ and the three o(1) conclusions were proven in [**14**]. By hypothesis, $\mathbf{y}_0 \in \mathscr{L}^2(\Omega)$ and $\mathbf{z}_0 \in \overline{R(Q)} = R(Q_1)$ in (1.6). From [**14**],

$$\mathbf{X}(t) \equiv \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{z}(t) \end{bmatrix} = \sum_{n,k} d_{n,k} e^{-\lambda_n t} \begin{bmatrix} \sum_{j=0}^{k} ((-1)^j/j!) t^j \mathbf{y}_{n,k-j} \\ \sum_{j=0}^{k} ((-1)^j/j!) t^j \mathbf{z}_{n,k-j} \end{bmatrix}$$

belongs to $\mathscr{L}^2(\Omega) \times R(Q_1)$, where

$$\mathbf{X}_{n,k} \equiv \begin{bmatrix} \mathbf{y}_{n,k} \\ \mathbf{z}_{n,k} \end{bmatrix}$$

is for $k = 0$ an eigenvector, and for $k > 0$ a generalized eigenvector of $\mathfrak{A} + \mathfrak{B}$ (cf. Agmon [**3**], p. 180.) This expansion is derived from a Riesz basis (cf. Gohberg and Krein [**10**], pp. 310–311) of characteristic and associated vectors of the characteristic parameter problem $\mathbf{w} = \lambda P\mathbf{w} + (1/\lambda)Q\mathbf{w}$ (cf. [**13**], [**14**]). Hence

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{z}(t) \end{bmatrix} = \sum_{n} d_{n,0}\, e^{-\lambda_n t} \begin{bmatrix} \mathbf{y}_{n,0} \\ \mathbf{z}_{n,0} \end{bmatrix} + \mathrm{O}(e^{-\ell t})$$

where $\ell = (1/2)\ \|P\|$ and the displayed λ_n's are positive and tend to zero. Differentiation of $\mathbf{X}(t)$ yields

$$\begin{bmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{bmatrix} = \frac{1}{t} \sum_{n} d_{n,0}(-\lambda_n t) e^{-\lambda_n t} \begin{bmatrix} \mathbf{y}_{n,0} \\ \mathbf{z}_{n,0} \end{bmatrix} + \mathrm{O}(e^{-\ell t}).$$

Since $|(-\lambda_n t)e^{-\lambda_n t}| \leqq e^{-1}$ for all $t \geqq 0$ and $\mathbf{w} = \mathbf{y} + \mathbf{z}$, Lemma 5.1 of **[14]** and Lebesgue's dominated convergence theorem yield

$$\left\| \begin{matrix} \mathbf{w}_t \\ \\ \mathbf{z}_t \end{matrix} \right\|^2 = \mathrm{o}(1/t^2).$$

So $\mathbf{w}_t$, $\mathbf{z}_t$ and $\mathbf{y}_t$ are each $\mathrm{o}(1/t)$ in $\mathscr{L}^2(\Omega)$ and $\mathbf{u}_t = \mathscr{A}^{-1/2}(\mathbf{y}_t + \mathbf{z}_t) = \mathrm{o}(1/t)$ in $\mathscr{H}^1(\Omega)$.

Now the differential equations of (1.6) are,

$$\begin{aligned} &\mathbf{y}_t + \nu\mathscr{A}\mathbf{y} - (g/\nu)\,\mathscr{Q}\mathbf{y} - (g/\nu)\mathscr{Q}\mathbf{z} = 0, \\ &\mathbf{z}_t + (g/\nu)\mathscr{Q}\mathbf{y} + (g/\nu)\mathscr{Q}\mathbf{z} = 0. \end{aligned} \tag{2.1}$$

Since $\mathscr{A}$ is positive definite, addition of equations (2.1) yields $\mathbf{y} = \mathrm{o}(1/t)$ in $\mathscr{L}^2(\Omega)$. Compactness of $\mathscr{Q}$ and the second of equations (2.1) thus give $\mathscr{Q}\mathbf{z} = \mathrm{o}(1/t)$ in $\mathscr{L}^2(\Omega)$. Hence $\mathscr{A}^{-1/2}\,\mathbf{y} = \mathbf{r} = \mathrm{o}(1/t)$ in $\mathscr{H}^1(\Omega)$ which implies that $S\mathscr{A}^{-1/2}\,\mathbf{y} = \mathrm{o}(1/t)$ in $R(S)$, and by the definition of $\mathscr{C}$, $S\mathscr{A}^{-1/2}\,\mathbf{z} = \mathrm{o}(1/t)$ in $\mathscr{H}^-(\Gamma_s)$. Now by **[14]** (recall the second of equations (1.2′)), $\eta_t = S\mathbf{u}$, and also $\mathbf{u} = \mathscr{A}^{-1/2}(\mathbf{y} + \mathbf{z})$. Thus $\eta_t = S\mathscr{A}^{-1/2}(\mathbf{y} + \mathbf{z}) = \mathrm{o}(1/t)$ in $\mathscr{H}^-(\Gamma_s)$, which concludes the proof.

LEMMA 2. *Let $\mathbf{u}_0 \in \mathscr{H}^1(\Omega)$ and $\eta_0 \in R(S)$. Then the corresponding weak solution $\mathbf{u}$, η of (1.1)–(1.5) satisfies $\mathbf{u} = \mathrm{o}(1/t)$ in $\mathscr{H}^1(\Omega)$, $\eta = \mathrm{o}(1/t)$ in $\mathscr{H}^-(\Gamma_s)$, $\mathbf{u}_t = \mathrm{o}(1/t^2)$ in $\mathscr{H}^1(\Omega)$, and $\eta_t = \mathrm{o}(1/t)$ in $R(S)$.*

PROOF. Since $\eta_0 \in R(S)$, $\mathbf{z}_0 \in R(Q)$ and there exists a unique $\zeta_0 \in \overline{R(Q)} = R(Q_1)$ such that $\mathbf{z}_0 = Q\zeta_0$. Also $\mathbf{y}_0 \in \mathscr{L}^2(\Omega)$ and, as usual, $\mathbf{w}_0 = \mathbf{y}_0 + \mathbf{z}_0$. By Theorem 2.2 of **[13]** (cf. also **[20]**, **[12]**), there exists a unique system of coefficients $\{c_{n,k}\}$ such that

$$\mathbf{w}_0 = \sum_{n,k} c_{n,k}\mathbf{x}_{n,k}$$

and

$$\zeta_0 = \sum_{n,k} c_{n,k} \sum_{j=0}^{k} ((-1)^j/\lambda_n^{j+1})Q_1\mathbf{x}_{n,k-j},$$

where $\{\mathbf{x}_{n,k}\}$ is the system of characteristic and associated vectors corresponding to the characteristic values $\{\lambda_n\}$ of $\mathbf{w} = \lambda P\mathbf{w} + (1/\lambda)Q\mathbf{w}$. Furthermore this expansion is in terms of a Riesz basis for $\mathscr{L}^2(\Omega) \times R(Q_1)$, so there is an equivalent norm $\|\cdot\|_*$ on $\mathscr{L}^2(\Omega) \times R(Q_1)$ with respect to which the expansion is orthonormal, i.e.,

$$\left\| \begin{matrix} \mathbf{w}_0 \\ \\ \zeta_0 \end{matrix} \right\|_*^2 = \sum_{n,k} |c_{n,k}|^2$$

(cf. [10], [13], [14]). Since Q is compact,

$$\begin{bmatrix} \mathbf{w}_0 \\ \mathbf{z}_0 \end{bmatrix} = \begin{bmatrix} \sum_{n,k} c_{n,k}\mathbf{x}_{n,k} \\ \sum_{n,k} c_{n,k} \sum_{j=0}^{k} ((-1)^j/\lambda_n^{j+1})\, Q\mathbf{x}_{n,k-j} \end{bmatrix}$$

and

$$\left\| \begin{matrix} \mathbf{w}_0 \\ \mathbf{z}_0 \end{matrix} \right\|_*^2 \leqq K^2 \sum_{n,k} |c_{n,k}|^2,$$

where $K = \max[1, \|Q\|_*]$. Now $\mathbf{y}_0 = \mathbf{w}_0 - \mathbf{z}_0$ and so (cf. [14])

$$\begin{bmatrix} \mathbf{y}_0 \\ \mathbf{z}_0 \end{bmatrix} = \begin{bmatrix} \sum_{n,k} c_{n,k}(\lambda_n P\mathbf{x}_{n,k} + P\mathbf{x}_{n,k-1}) \\ \sum_{n,k} c_{n,k} \sum_{j=0}^{k} ((-1)^j/\lambda_n^{j+1}) Q\mathbf{x}_{n,k-j} \end{bmatrix} \equiv \sum_{n,k} c_{n,k} \begin{bmatrix} \mathbf{y}_{n,k} \\ \mathbf{z}_{n,k} \end{bmatrix}.$$

As in the proof of Lemma 1,

$$\mathbf{X}_{n,k} = \begin{bmatrix} \mathbf{y}_{n,k} \\ \mathbf{z}_{n,k} \end{bmatrix}$$

is for $k = 0$ an eigenvector and for $k > 0$ a generalized eigenvector of $\mathfrak{A} + \mathfrak{B}$. The unique solution of (1.6) is

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{z}(t) \end{bmatrix} = \sum_{n,k} c_{n,k} e^{-\lambda_n t} \begin{bmatrix} \sum_{j=0}^{k} ((-1)^j/j!) t^j \mathbf{y}_{n,k-j} \\ \sum_{j=0}^{k} ((-1)^j/j!) t^j \mathbf{z}_{n,k-j} \end{bmatrix}.$$

Since $\mathbf{z}_{n,k} \in R(Q)$, there is for each n, k a unique element $\zeta_{n,k} \in R(Q_1)$ such that $\mathbf{z}_{n,k} = Q\zeta_{n,k}$. Define $\zeta(t) : [0, \infty) \to R(Q_1)$ by

$$\zeta(t) = \sum_{n,k} c_{n,k} e^{-\lambda_n t} \sum_{j=0}^{k} ((-1)^j/j!) t^j \zeta_{n,k-j},$$

so that $\mathbf{z}(t) = Q\zeta(t)$, and define $\mathbf{w}(t)$: $[0, \infty) \to \mathscr{L}^2(\Omega)$ by

$$\mathbf{w}(t) = \sum_{n,k} c_{n,k} e^{-\lambda_n t} \sum_{j=0}^{k} ((-1)^j/j!) j^j \mathbf{x}_{n,k-j}.$$

(Note that $\mathbf{w}(t) = \mathbf{y}(t) + \mathbf{z}(t)$.) Then $\mathbf{w}(0) = \mathbf{w}_0$, $\zeta(0) = \zeta_0$, and it follows as in the technique of the corresponding part of the proof of Lemma 1 (by use of $\|\cdot\|_*$) that $\mathbf{w}_t$ and ζ_t are $o(1/t)$ in $\mathscr{L}^2(\Omega)$. The second of equations (2.1) is $\mathbf{z}_t + Q\mathbf{y} + Q\mathbf{z} = 0$, so the definition of $\zeta(t)$ yields $\zeta_t + Q_1\mathbf{y} + Q_1\mathbf{z} = 0$. Since the proof of Lemma 1 gives $Q_1\mathbf{y} = o(1/t)$ in $\mathscr{L}^2(\Omega)$, it now follows that $Q_1\mathbf{z} = \mathbf{z} = o(1/t)$ in $\mathscr{L}^2(\Omega)$. Thus $\mathbf{u} = \mathscr{A}^{-1/2}(\mathbf{y} + \mathbf{z}) = o(1/t)$

in $\mathscr{H}^1(\Omega)$ and the definition of η implies that $\eta = o(1/t)$ in $\mathscr{H}^-(\Gamma_s)$. Continuity of $S\colon \mathscr{H}^1(\Omega) \to R(S)$ gives $\eta_t = S\mathbf{u} = o(1/t)$ in $R(S)$.

Now returning to the expansion in the proof of Lemma 1,

$$\begin{bmatrix} \mathbf{y}_{tt} \\ \\ \mathbf{z}_{tt} \end{bmatrix} = \frac{1}{t^2} \sum_n d_{n,0}(\lambda_n t)^2 e^{-\lambda_n t} \begin{bmatrix} \mathbf{y}_{n,0} \\ \\ \mathbf{z}_{n,0} \end{bmatrix} + O(e^{-\ell t}),$$

which implies that $\mathbf{y}_{tt}$ and $\mathbf{z}_{tt}$ are $o(1/t^2)$ in $\mathscr{L}^2(\Omega)$. Differentiation and addition of equations (2.1) gives $\mathbf{y}_{tt} + \mathbf{z}_{tt} = \mathscr{A}\mathbf{y}_t = o(1/t^2)$ in $\mathscr{L}^2(\Omega)$. So, since $\mathscr{A}$ is positive definite, $\mathbf{y}_t = o(1/t^2)$ in $\mathscr{L}^2(\Omega)$. Similarly, the expansion of $\zeta(t)$ yields $\zeta_{tt} = o(1/t^2)$ in $\mathscr{L}^2(\Omega)$. Hence $\mathbf{z}_t = Q_1\mathbf{z}_t = -\zeta_{tt} - Q_1\mathbf{y}_t = o(1/t^2)$ in $\mathscr{L}^2(\Omega)$, and $\mathbf{u}_t = \mathscr{A}^{-1/2}(\mathbf{y}_t + \mathbf{z}_t) = o(1/t^2)$ in $\mathscr{H}^1(\Omega)$, which concludes the proof.

Now let $W_1 = R(S)$, $W_0 = \mathscr{H}^-(\Gamma_s)$, and denote by W_τ, $0 < \tau < 1$, the spaces obtained by quadratic interpolation between W_1 and W_0. It follows that $W_{1/2} = \mathscr{L}^2(\Gamma_s)$. Further let $V_1 = \mathscr{H}^1(\Omega) \times R(S)$, $V_0 = \mathscr{H}^1(\Omega) \times \mathscr{H}^-(\Gamma_s)$, and denote by V_τ, $0 < \tau < 1$, the spaces obtained by quadratic interpolation between V_1 and V_0. Then (cf. §2.6.6 of [24]) $V_\tau = \mathscr{H}^1(\Omega) \times W_\tau$, $0 < \tau < 1$, and in particular $V_{1/2} = \mathscr{H}^1(\Omega) \times \mathscr{L}^2(\Gamma_s)$.

LEMMA 3. *Let $\mathbf{u}_0 \in \mathscr{H}^1(\Omega)$ and $\eta_0 \in W_\tau$ where $0 < \tau < 1$. Then the corresponding weak solution $\mathbf{u}$, η of (1.1)–(1.5) satisfies $\mathbf{u} = O(1/t^\tau)$ in $\mathscr{H}^1(\Omega)$, $\eta = O(1/t^\tau)$ in $\mathscr{H}^-(\Gamma_s)$, $\mathbf{u}_t = O(1/t^{1+\tau})$ in $\mathscr{H}^1(\Omega)$, $\eta_t = O(1/t)$ in W_τ, and $\eta_t = O(1/t^\tau)$ in $R(S)$.*

PROOF. The mapping $T(t)$ defined by

$$T(t)\begin{bmatrix} \mathbf{u}_0 \\ \\ \eta_0 \end{bmatrix} = \begin{bmatrix} \mathbf{u}(t) \\ \\ \eta(t) \end{bmatrix}$$

is linear and the proof of Lemma 1 shows that the norm of $T(t)\colon V_0 \to V_0$ is $O(1)$. Moreover, the proof of Lemma 2 easily implies that the norm of $T(t)\colon V_1 \to V_0$ is $O(1/t)$. Since $V_\tau = \mathscr{H}^1(\Omega) \times W_\tau$, the quadratic interpolation theorem of **[21]** shows that the norm of $T(t)\colon \mathscr{H}^1(\Omega) \times W_\tau \to V_0$ is $O(1/t^\tau)$. So

$$\left\| \begin{bmatrix} \mathbf{u}(t) \\ \\ \eta(t) \end{bmatrix} \right\|_{V_0} = \left\| T(t) \begin{bmatrix} \mathbf{u}_0 \\ \\ \eta_0 \end{bmatrix} \right\|_{V_0} = O(1/t^\tau),$$

which proves the first two conclusions. The remaining conclusions are verified similarly by considering the linear mappings

$$\begin{bmatrix} \mathbf{u}_0 \\ \\ \eta_0 \end{bmatrix} \to \mathbf{u}_t \text{ and } \begin{bmatrix} \mathbf{u}_0 \\ \\ \eta_0 \end{bmatrix} \to \eta_t.$$

Now let $\mathscr{H}^{-1/2}(\Gamma_s)$ denote the dual of $\mathscr{H}^{1/2}(\Gamma_s)$, where duality is taken with respect to $\mathscr{L}^2(\Gamma_s)$. Then

$$R(S) \underset{c}{\subseteq} \mathscr{H}^{1/2}(\Gamma_s) \underset{c}{\subseteq} \mathscr{L}^2(\Gamma_s) \underset{c}{\subseteq} \mathscr{H}^{-1/2}(\Gamma_s) \underset{c}{\subseteq} \mathscr{H}^{-}(\Gamma_s),$$

and each inclusion is dense. For $0 < \alpha < 1/2$, let

$$\mathscr{H}^{\alpha}(\Gamma_s) = \{v \in H_0^{\alpha}(\Gamma_s): \int_{\Gamma_s} v = 0\} = \{v \in H^{\alpha}(\Gamma_s): \int_{\Gamma_s} v = 0\}$$

(cf. **[11]**, **[24]**, and Lions and Magenes **[23]**). In §1 we have used the fact that $\mathscr{H}^{1/2}(\Gamma_s)$ is an interpolation space by quadratic interpolation between the closed subspace of $H_0^1(\Gamma_s)$ given by $\{v \in H_0^1(\Gamma_s): \int_{\Gamma_s} v = 0\}$, and $\mathscr{L}^2(\Gamma_s)$. By the reiteration property of quadratic interpolation, the spaces $\mathscr{H}^{\alpha}(\Gamma_s)$ are the spaces obtained by quadratic interpolation between $\mathscr{H}^{1/2}(\Gamma_s)$ and $\mathscr{L}^2(\Gamma_s)$. It now follows (cf. **[23]**, **[2]**, and **[11]**) that for each α such that $0 \leqq \alpha \leqq 1/2$, the Hilbert space

$$_{\alpha}\mathscr{L}^2(\Gamma_s) = \{v \in \mathscr{L}^2(\Gamma_s): \int_{\Gamma_s} v^2 d^{-2\alpha} < \infty\},$$

where for $\mathbf{x}' \in \Gamma_s$, $d = d(\mathbf{x}') = \operatorname{dist}(\mathbf{x}', \partial\Gamma_s)$ and the obvious norm is employed, satisfies

$$\mathscr{H}^{\alpha}(\Gamma_s) \underset{c}{\subseteq} {}_{\alpha}\mathscr{L}^2(\Gamma_s),$$

the inclusion being dense.

Now define $\mathscr{H}^{-\alpha}(\Gamma_s)$ for $0 < \alpha < 1/2$ as the dual of $\mathscr{H}^{\alpha}(\Gamma_s)$ with respect to $\mathscr{L}^2(\Gamma_s)$. By duality, the spaces $\mathscr{H}^{-\alpha}(\Gamma_s)$ are the spaces obtained by quadratic interpolation betwen $\mathscr{L}^2(\Gamma_s)$ and $\mathscr{H}^{-1/2}(\Gamma_s)$. Next observe that if $\alpha < 1/2$ and v is a measurable function on Γ_s such that $\int_{\Gamma_s} v^2 d^{2\alpha} < \infty$, the Cauchy-Schwarz inequality and $\int_{\Gamma_s} d^{-2\alpha} < \infty$ (since Γ_s is LG) imply that $v \in L^1(\Gamma_s)$. Thus for $0 < \alpha < 1/2$, the Hilbert space $_{-\alpha}\mathscr{L}^2(\Gamma_s)$ of all measurable functions v on Γ_s such that

$$\int_{\Gamma_s} v^2 d^{2\alpha} < \infty \text{ and } \int_{\Gamma_s} v = 0,$$

with the obvious norm, satisfies

$$_{-\alpha}\mathscr{L}^2(\Gamma_s) \underset{c}{\subseteq} \mathscr{H}^{-\alpha}(\Gamma_s),$$

the inclusion being dense.

THEOREM. 1. *Let* $0 \leqq \alpha < 1/2$, $\mathbf{u}_0 \in \mathscr{H}^1(\Omega)$, *and* $\eta_0 \in {}_{-\alpha}\mathscr{L}^2(\Gamma_s)$. *Then the corresponding weak solution* $\mathbf{u}, \eta$ *of* (1.1)–(1.5) *satisfies* $\mathbf{u} = O(t^{\alpha-(1/2)})$ *in* $\mathscr{H}^1(\Omega)$, $\eta = O(t^{\alpha-(1/2)})$ *in* $\mathscr{H}^{-}(\Gamma_s)$, $\mathbf{u}_t = O(t^{\alpha-(3/2)})$ *in* $\mathscr{H}^1(\Omega)$, and $\eta_t = o(t^{-1})$ *in* $\mathscr{H}^{-}(\Gamma_s)$. *In addition, if* $\alpha = 0$, $\eta_t = O(t^{-1/2})$ *in* $\mathscr{L}^2(\Gamma_s)$.

2. *Let* $0 < \alpha \leqq 1/2$, $\mathbf{u}_0 \in \mathscr{H}^1(\Omega)$, *and* $\eta_0 \in \mathscr{H}^{\alpha}(\Gamma_s)$. *Further assume that* Γ_s *is surrounded by an overhanging dock, or that* η_0 *has compact support in*

Γ_s. Then the corresponding weak solution $\mathbf{u}$, η *of* (1.1)–(1.5) *satisfies*, $\mathbf{u} = O(1/t^{\alpha+(1/2)})$ *in* $\mathscr{H}^1(\Omega)$, $\eta = O(1/t^{\alpha+(1/2)})$ *in* $\mathscr{H}^-(\Gamma_s)$, $\mathbf{u}_t = O(1/t^{\alpha+(3/2)})$ *in* $\mathscr{H}^1(\Omega)$, $\eta_t = O(1/t)$ *in* $\mathscr{H}^\alpha(\Gamma_s)$, *and* $\eta_t = O(1/t^{\alpha+(1/2)})$ *in* $R(S)$. *In addition, if* $\alpha = 1/2$, O *can be replaced by* o.

3. *If* $\mathbf{u}_0 \in \mathscr{H}^1(\Omega)$ *and* $\eta_0 = 0$, *which is true for submerged initial disturbances, then the corresponding weak solution* $\mathbf{u}$, η *of* (1.1)–(1.5) *satisfies* $\mathbf{u} = o(1/t)$ *in* $\mathscr{H}^1(\Omega)$, $\eta = o(1/t)$ *in* $\mathscr{H}^-(\Gamma_s)$, $\mathbf{u}_t = o(1/t^2)$ *in* $\mathscr{H}^1(\Omega)$, *and* $\eta_t = o(1/t)$ *in* $R(S)$.

PROOF. Part 1 is a direct consequence of the preceding remarks and Lemma 3. Part 2 follows similarly by noting that the hypotheses imply that $\eta_0 \in \mathscr{H}^\alpha(\Gamma_s)$ if and only if $\eta_0 \in W_{\alpha+(1/2)}$. Part 3 is a special case of Lemma 2.

REMARK. Part 1 of the Theorem with $\alpha = 0$ yields rates of decay for smooth initial disturbances, without invoking the restrictive assumption that η_0 vanish at the boundary of Γ_s. For such initial disturbances, Part 2 of the Theorem applies for every $\alpha < 1/2$ when Γ_s is surrounded by an overhanging dock. We conjecture, but have not proven, that the latter holds without the overhanging dock assumption.

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