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ASYMPTOTIC EXPANSIONS IN PERFORATED MEDIA WITH A PERIODIC STRUCTURE

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Dedicated to N. ARONSZAJN

0. Introduction. We consider elliptic equations

$$(1) \qquad Au_\varepsilon = f$$

in domains Ω_ε which consist of a perforated medium, with a "large" number of holes or of obstacles of "size" ε and which are arranged in a periodic manner, also with period ε. In (1) u_ε is subject to some boundary conditions, and we want to study the behaviour of u_ε as $\varepsilon \to 0$.

This problem has already been considered by L. Tartar [**14**], D. Cioranescu [**5**], and D. Ciroranescu and J. Saint Jean Paullin [**6**] by energy methods; one obtains in this manner the behaviour of u_ε as $\varepsilon \to 0$, and the periodic structure is *not* used in an essential manner. For situations where the "volume" occupied by the holes is "smaller" than in the present case, cf. V. A. Marcenko and E. Yu. Hruslov [**12**] and Rauch and Taylor [**13**].

In this paper we show that—by using this time the periodic structure in an (apparently) essential manner—one can obtain much more, that is, under suitable hyopthesis on f, one can obtain an expansion of any order in ε. We will construct functions $u_0, u_1, u_2, \ldots$ such that

$$u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots \varepsilon^m u_m)$$

is of order ε^m *in a Sobolev space on* Ω_ε. Actually, in the situations considered here $u_0 = 0, u_1 = 0$.

The method used here is a variant of the method of multi-scales as used in the book A. Bensoussan, J. L. Lions and G. Papanicolaou [**4**] (and as anticipated by J. Keller) for problems of homogenization arising in composite materials (We refer to the book just quoted for bibliographical references, in particular to the work of de Giorgi, Spagnolo and their associates, Bakhbalov, Babuska, Murat and Tartar.) The new part here is that in some case, boundary layer terms can be avoided. (The construction of boundary layer terms, when they are needed, is a largely open question in Composite Materials as well as in Perforated Media.) The structure of the expansion in perforated media has been briefly given in the lecture [**9**] of the Author in Poland and in lectures in the Collège de France, Fall 1977.

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The plan of the paper is as follows:
1. Setting of the problems.
2. Construction of the asymptotic expansion.
3. Error estimates.
4. Obstacles and rapidly varying coefficients (I).
5. Obstacles and rapidly varying coefficients (II).
6. Various remarks.

1. **Setting of the problems.** We define first in a precise manner the domains Ω_ε which consist of an open set $\Omega \subset R^n$ from which we take out a "large" number of "small" pieces arranged in a periodic manner.

Let us set

$$Y = \prod_{j=1}^{n}]0, y_j^0[,$$

and let us consider an open set $\mathcal{O}$ contained in Y; more precisely

$$\mathcal{O} \subset \bar{\mathcal{O}} \subset Y. \tag{1.1}$$

Let S denote the boundary of $\mathcal{O}$; we suppose that S is divided in two pieces

$$S = S_D \cup S_N \tag{1.2}$$

where the index D refers to Dirichlet and N to Neumann.

We define next $\varepsilon\bar{\mathcal{O}}$ and the set

$$\bar{\tau}(\varepsilon\bar{\mathcal{O}}) \tag{1.3}$$

where $\bar{\tau}$ denotes the set of all translations $\{\varepsilon k_1 y_1^0, \ldots, \varepsilon k_n y_n^0\}$ where the k_j's are integers.

If Ω is a bounded open set of R^n with boundary Γ, we define

$$\Omega_\varepsilon = \Omega \backslash (\Omega \cap \bar{\tau}(\varepsilon\bar{\mathcal{O}})). \tag{1.4}$$

We set

$$S_\varepsilon = \partial(\tau(\varepsilon\mathcal{O})) \cap \Omega \tag{1.5}$$

(this is the union of the portions of boundaries contained in Ω of all sets $\bar{\tau}(\varepsilon\mathcal{O})$ which intersect Ω).

With obvious notations we have

$$S_\varepsilon = S_{\varepsilon D} \cup S_{\varepsilon N}. \tag{1.6}$$

The boundary of Ω_ε contained in Γ is denoted by Γ_ε, so that

$$\partial\Omega_\varepsilon = \Gamma_\varepsilon \cup S_\varepsilon. \tag{1.7}$$

The basic problem we want to consider can now be stated as follows.

We are given a function f in Ω (a regularity hypothesis on f will be made later on), and we consider in Ω_ε the problem

$$-\Delta u_\varepsilon = f \text{ in } \Omega_\varepsilon, \tag{1.8}$$

$$u_\varepsilon = 0 \text{ on } \Gamma_\varepsilon, \tag{1.9}$$

$$u_\varepsilon = 0 \text{ on } S_{\varepsilon D}, \tag{1.10}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ on } S_{\varepsilon N} \tag{1.11}$$

(where $\partial/\partial\nu$ denotes the normal derivative to $S_{\varepsilon N}$, oriented toward the exterior of Ω_ε to fix ideas and $S_{\varepsilon N}$ is assumed regular). This problem admits a unique solution, at least in the Sobolev space $H_0^1(\Omega_\varepsilon)$ (We denote by $H_0^1(\Omega)$ the space of (real valued) functions which are in $L^2(\Omega)$ together with their first order derivatives (in the sense of distributions) and which are 0 on Γ. This space is also denoted by $\mathring{W}^{1,2}(\Omega)$ or $\mathring{P}^1(\Omega)$, etc. It has been studied as the completion of smooth functions for the norm

$$\left(\int_\Omega \left[v^2 + \left(\frac{\partial v}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial v}{\partial x_n}\right)^2\right] dx\right)^{\frac{1}{2}}$$

by N. Aronszajn and his associates in a series of papers (cf. Bibliography)). We want to study the behaviour of u_ε when $\varepsilon \to 0$.

2. **Construction of the asymptotic expansion.** We are going to look for u_ε in the form (this is a formal expansion for the time being, and such expansions (with technical differences) have been systematically used in A. Bensoussan, J. L. Lions and G. Papanicolaou [**4**])

$$u_\varepsilon = u_0 + \varepsilon u_1 + \cdots, \tag{2.1}$$

where

$$u_j = u_j(x, y), \qquad y = x/\varepsilon, \tag{2.2}$$

and where the functions u_j have the following properties:

$$\begin{aligned} &u_j(x, y) \text{ is defined for } x \in \Omega, y \in Y - \mathcal{O};\\ &u_j(x, y) \text{ is } Y\text{-periodic, that is, } u_j(x, y) \text{ admits the period}\\ &y_k^0 \text{ in the variable } y_k,\ k = 1, \ldots, n;\\ &u_j(x, y) = 0 \text{ for } x \in \Omega, y \in S_D. \end{aligned} \tag{2.3}$$

We remark that condition (1.10) will be satisfied by virtue of the structure of the functions u_j, provided the series converges.

We now make a formal identification. Let us consider (1.8) first. The operator $\partial/\partial x_k$ applied to a function $u_j(x, x/\varepsilon)$ becomes

$$\varepsilon^{-1}\frac{\partial}{\partial y_k}+\frac{\partial}{\partial x_k},$$

so that

$$\Delta=\varepsilon^{-2}\Delta_y+2\varepsilon^{-1}\Delta_{xy}+\varepsilon^0\Delta_x \tag{2.4}$$

where, using the summation convention,

$$\Delta_{xy}=\frac{\partial^2}{\partial x_i\partial y_i}. \tag{2.5}$$

Then (1.8) is equivalent to

$$-\Delta_y u_0=0, \tag{2.6}$$

$$-\Delta_y u_1-2\Delta_{xy}u_0=0, \tag{2.7}$$

$$-\Delta_y u_2-2\Delta_{xy}u_1-\Delta_x u_0=f, \tag{2.8}$$

etc.

We now consider (1.11). We denote by $\nu_j(y)$ the jth component of ν. Then

$$\frac{\partial}{\partial\nu}=\varepsilon^{-1}\nu_j(y)\frac{\partial}{\partial y_j}+\nu_j(y)\frac{\partial}{\partial x_j}=\varepsilon^{-1}\frac{\partial}{\partial\nu(y)}+\nu_j(y)\frac{\partial}{\partial x_j}. \tag{2.9}$$

It follows that (1.11) is equivalent to

$$\frac{\partial u_0}{\partial\nu(y)}=0,\quad y\in S_N, \tag{2.10}$$

$$\frac{\partial u_1}{\partial\nu(y)}+\nu_j\frac{\partial u_0}{\partial x_j}=0,\quad y\in S_N, \tag{2.11}$$

$$\frac{\partial u_2}{\partial\nu(y)}+\nu_j\frac{\partial u_1}{\partial x_j}=0,\quad y\in S_N, \tag{2.12}$$

etc. For fixed $x\in\Omega$, $u_0(x,y)$ should satisfy in $Y-\mathcal{O}$ the equation (2.6), with boundary conditions (2.10) and $u_0(x,y)=0$ for $y\in S_D$ and u_0 being Y periodic. It follows that $u_0=0$. Then (2.7), (2.11) and $u_1(x,y)=0$ for $y\in S_D$ and Y periodic imply that $u_1=0$.

It follows that conditions on u_2 reduce to

$$\begin{gathered}-\Delta_y u_2=f(x)\text{ in }Y-\mathcal{O},\\ u_2(x,y)=0\text{ for }y\in S_D,\ \frac{\partial u_2}{\partial\nu(y)}=0\text{ for }y\in S_N,\\ u_2\text{ is }Y\text{-periodic.}\end{gathered} \tag{2.13}$$

In (2.13), x is a parameter. Therefore, let us introduce $w=w(y)$ as *the* solution of

$$(2.14)\qquad \begin{aligned} &-\Delta w(y) = 1 \text{ in } Y - \mathcal{O},\\ &w = 0 \text{ on } S_D,\ \frac{\partial w}{\partial \nu(y)} = 0 \text{ on } S_N,\\ &w \text{ is } Y\text{-periodic.}\end{aligned}$$

Then

$$(2.15)\qquad u_2 = u_2(x, y) = w(y)f(x).$$

Let us proceed with the computation. The equation "following" (2.8) is

$$(2.16)\qquad -\Delta_y u_3 - 2\Delta_{xy} u_2 = 0,$$

i.e., assuming f smooth,

$$(2.17)\qquad -\Delta_y u_3 = 2\frac{\partial w}{\partial y_i}\frac{\partial f}{\partial x_i},$$

with the conditions

$$(2.18)\qquad \begin{aligned} &u_3(x, y) = 0 \text{ for } y \in S_D,\\ &\frac{\partial u_3}{\partial \nu(y)} = -\nu_i(y)w(y)\frac{\partial f}{\partial x_i} \text{ for } y \in S_N,\\ &u_3 \text{ is } Y\text{-periodic.}\end{aligned}$$

Here again x is a parameter. We define $w^i(y)$ as the solution of

$$(2.19)\qquad \begin{aligned} &-\Delta_y w^i = 2\frac{\partial w}{\partial y_i} \text{ in } Y - \mathcal{O},\\ &w^i = 0 \text{ on } S_D,\ \frac{\partial w^i}{\partial \nu(y)} = -\nu_i(y)w(y) \text{ on } S_N,\\ &w^i \text{ is } Y\text{-periodic.}\end{aligned}$$

Then

$$(2.20)\qquad u_3 = w^i(y)\frac{\partial f}{\partial x_i}(x).$$

One easily obtains the general structure of u_m, $m \geqq 3$:

$$(2.21)\qquad \begin{aligned} &u_m = w^{(p)}(y)D^p f(x) \text{ (where the summation is extended}\\ &\text{to all } p,\ |p| = m - 2),\ w^{(p)} \in H^1(Y - \mathcal{O}).\end{aligned}$$

In (2.22) the $w^{(p)}$ can be recursively defined.

In the next section we shall justify the above construction.

3. **Error estimates.** We prove now the following Theorem.

THEOREM 3.1. *Let the functions* $u_2, u_3, \ldots$ *be defined by* (2.15), (2.20), ... *Then* u_ε *being the solution of* (1.8), ..., (1.11), *and assuming that*

(3.1) $$f \in \mathscr{D}(\Omega) = \text{space of functions with compact support in } \Omega,$$

one has

(3.2) $$\| u_\varepsilon - (\varepsilon^2 u_2 + \cdots + \varepsilon^m u_m) \|_{H^1(\Omega_\varepsilon)} \leqq C\varepsilon^m$$

where C does not depend on ε.

REMARK 3.1. Hypothesis (3.1) can be weakened (cf. Remark 3.3 below).

REMARK 3.2. Since $u_m = u_m(x, y) = u_m(x, x/\varepsilon)$, one has

$$\| \varepsilon^m u_m \|_{H^1(\Omega_\varepsilon)} = 0(\varepsilon^{m-1})$$

so that the term $\varepsilon^m u_m$ is indeed needed in (3.2).

REMARK 3.3. If we assume that $f \in C^{m-2}(\bar{\Omega})$ and that

(3.3) $$D^\alpha f = 0 \text{ on } \Gamma \text{ for } |p| \leqq m - 2,$$

then due to the structure of the formula (2.21), one has

(3.4) $$u_j(x, y) = 0 \text{ for } x \in \Gamma.$$

If we do not assume (3.3), then (3.2) is not correct, since $u_\varepsilon = 0$ on Γ_ε and $\varepsilon^2 u_2 + \cdots + \varepsilon^m u_m$ is not zero on Γ_ε. In such a case, one would need boundary layer correctors; the structure of these correctors is not known to the author. Of course (3.1) is unnecessarily strong; the estimate (3.2) is valid if one assumes that

(3.5) $$f \in C^{m+1}(\bar{\Omega}),\ D^p f = 0 \text{ on } \Gamma \text{ for every } p,\ |p| \leqq m - 1.$$

PROOF OF THEOREM 3.1. Let us introduce

(3.6) $$\varphi_\varepsilon = u_\varepsilon - (\varepsilon^2 u_2 + \cdots + \varepsilon^k u_k)$$

where k will be chosen later (and we shall then make precise the hypotheses on f which are sufficient to insure the validity of the argument). We have

(3.7) $$-\Delta \varphi_\varepsilon = \varepsilon^{k-1} g_\varepsilon$$

where

(3.8) $$g_\varepsilon = [\Delta_x u_{k-1} + 2\Delta_{xy} u_k] + \varepsilon \Delta_x u_k.$$

We assume that

(3.9) $$f \in C^k(\bar{\Omega}),\ D^p f = 0 \text{ on } \Gamma \text{ for every } p,\ |p| \leqq k - 2.$$

By virtue of (3.9) and of (2.21), we have

$$\varphi_\varepsilon = 0 \text{ on } \Gamma_\varepsilon, \tag{3.10}$$

since $u_\varepsilon = 0$ on $S_{\varepsilon D}$, and by construction of the u_j we have

$$\varphi_\varepsilon = 0 \text{ on } S_{\varepsilon D}. \tag{3.11}$$

On $S_{\varepsilon N}$ we have

$$\frac{\partial \varphi_\varepsilon}{\partial \nu} = \varepsilon^k h_\varepsilon, \quad h_\varepsilon = -\nu_j(y)\frac{\partial u_k}{\partial x_j}. \tag{3.12}$$

Multiplying (3.7) by φ_ε and using Green's formula, we obtain

$$\int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon|^2 dx = \int_{S_{\varepsilon N}} \varepsilon^k h_\varepsilon \varphi_\varepsilon dS_\varepsilon + \varepsilon^{k-1} \int_{\Omega_\varepsilon} g_\varepsilon \varphi_\varepsilon dx. \tag{3.13}$$

Let us verify that

$$\|g_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqq C \tag{3.14}$$

and

$$\|h_\varepsilon\|_{L^2(S_{\varepsilon N})} \leqq C\varepsilon^{-\frac{1}{2}}. \tag{3.15}$$

Here and in what follows, the C's denote various constants.

Indeed by virtue of (3.9) and of (2.21), we have

$$\begin{aligned} |g_\varepsilon(x, y)| \leqq C &\sum_{|p|=k-2} [|w^{(p)}(y)| + |\nabla_y w^{(p)}(y)|] \\ &+ C \sum_{|p|=k-3} |w^{(p)}(y)|. \end{aligned} \tag{3.16}$$

But given a function $\Phi \in H^1(Y - \mathcal{O})$, we have

$$\int_{\Omega_\varepsilon} [\Phi^2(x/\varepsilon) + |\nabla_y \Phi|^2(x/\varepsilon)]\, dx \leqq C$$

which, together with (3.16), implies (3.14). Similarly,

$$|h_\varepsilon(x, y)| \leqq C \sum_{|p|=k-2} |w^{(p)}(y)|$$

and, given $\Phi \in H^1(Y - \mathcal{O})$, we have

$$\int_{S_{\varepsilon N}} \Phi^2(x/\varepsilon)\, dS_{\varepsilon N} \leqq C\varepsilon^{-1}.$$

Hence (3.15) follows. It follows from (3.13), (3.14), (3.15) that

$$\int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon|^2\, dx \leqq C\varepsilon^{k-1} \|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{k-1/2)} \|\varphi_\varepsilon\|_{L^2(S_{\varepsilon N})}. \tag{3.17}$$

But one can show (cf. D. Cioranescu **[5]**, D. Cioranescu and J. Saint Jean Paulin **[6]**) that

(3.18) $$\int_{\Omega_\varepsilon} \varphi^2 dx \leqq C \int_{\Omega_\varepsilon} |\nabla \varphi|^2 \, dx$$

for every $\varphi \in H^1(\Omega_\varepsilon)$ such that $\varphi = 0$ on $S_{\varepsilon D} \cup \Gamma_\varepsilon$ and with C independent of ε. Let us check that

(3.19) $$\int_{S_{\varepsilon N}} \varphi^2 dx \leqq \frac{c}{\varepsilon} \int_{\Omega_\varepsilon} [\varphi^2 + |\nabla \varphi|^2] \, dx$$

for all functions φ as in (3.18). Indeed if we introduce functions $c_j(y) \in C^1(\bar{Y})$ such that $c_j(y) = \nu_j(y)$ on S and $c_j = 0$ near the "boundary" of Y (considered as a parallelotope in R^n) (we need here that S be a C^1 variety), and if we extend c_j by periodicity, then

$$\int_{\Omega_\varepsilon} c_j(x/\varepsilon) \frac{\partial}{\partial x_j} (\varphi^2) \, dx = \int_{S_{\varepsilon N}} \varphi^2 dS_{\varepsilon N} - \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \frac{\partial c_j}{\partial y_j} (x/\varepsilon) \varphi^2 dx,$$

so that (3.19) follows.

Using (3.18) and (3.19), the estimate (3.17) gives

$$\left(\int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} \leqq C \varepsilon^{k-1},$$

and again using (3.18), we have

(3.20) $$\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqq C \varepsilon^{k-1}.$$

We now choose $k = m + 1$. Then the hypothesis (3.9) becomes (3.5). We have

$$u_\varepsilon - (\varepsilon^2 u_2 + \cdots + \varepsilon^m u_m) = \varphi_\varepsilon + \varepsilon^{m+1} \, u_{m+1}$$

so that using (3.20),

(3.21) $$\|u_\varepsilon - (\varepsilon^2 u_2 + \cdots + \varepsilon_m u_m)\|_{H^1(\Omega_\varepsilon)} \leqq C \varepsilon^m + \varepsilon^{m+1} \, \|u_{m+1}\|_{H^1(\Omega_\varepsilon)}.$$

But

$$\frac{\partial}{\partial x_i} u_{m+1} = w^{(p)}(y) \frac{\partial}{\partial x_i} D^p f(x) + \varepsilon^{-1} \frac{\partial w^{(p)}}{\partial y_i} (y) D^p f(x),$$
$$|p| = m - 1,$$

so that

$$\left| \frac{\partial}{\partial x_i} u_{m+1}(x, y) \right| \leqq C \sum_{|p|=m-1} |w^{(p)}(y)| + \frac{c}{\varepsilon} \sum_{|p|=m-1} \left| \frac{\partial w^{(p)}}{\partial y_i} (y) \right|$$

and therefore $\|u^{m+1}\|_{H^1(\Omega_\varepsilon)} \leqq C/\varepsilon$ so that (3.21) implies (3.2), and the proof is completed.

4. **Obstacles and rapidly varying coefficients (I).** We consider now an

extension of the above situations. The geometrical data are the same but in Ω_ε instead of considering the operator $-\Delta$ we consider the operator

$$A^\varepsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}(x/\varepsilon)\frac{\partial}{\partial x_j}\right) \tag{4.1}$$

where

$$\begin{aligned} &a_{ij}(y) \in L^\infty(R^n),\ a_{ij} \text{ is } Y\text{-periodic},\\ &a_{ij}(y)\,\xi_i\xi_j \geqq \alpha\xi_i\xi_i,\ \alpha > 0 \text{ a.e. in } y. \end{aligned} \tag{4.2}$$

We consider the problem

$$A^\varepsilon u_\varepsilon = f \text{ in } \Omega_\varepsilon, \tag{4.3}$$

$$u_\varepsilon = 0 \text{ on } \Gamma_\varepsilon, \tag{4.4}$$

$$u_\varepsilon = 0 \text{ on } S_{\varepsilon D}, \tag{4.5}$$

$$\frac{\partial u_\varepsilon}{\partial \nu_{A_\varepsilon}} = 0 \text{ on } S_{\varepsilon N}. \tag{4.6}$$

in (4.6) $$\frac{\partial}{\partial \nu_{A^\varepsilon}} = a_{ij}(y)\nu_i(y)\frac{\partial}{\partial x_j}, \qquad y = x/\varepsilon.$$

Physically, problem (4.3) ... (4.6) corresponds to a composite material with a periodic structure (the period being εY) and which is perforated (the "holes" or "obstacles" being $\mathcal{O}_\varepsilon$) with the same periodic structure. We want to study the asymptotic behavior of u_ε as $\varepsilon \to 0$.

Asymptotic expansion. We use for u_ε the same "ansatz" than in (2.1), (2.2), (2.3). We now have

$$\begin{aligned} A^\varepsilon &= \varepsilon^{-2}A_1 + \varepsilon^{-1}A_2 + \varepsilon^0 A_3,\\ A_1 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right),\\ A_2 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right),\\ A_3 &= -\frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \frac{\partial}{\partial \nu_{A^\varepsilon}} &= \varepsilon^{-1} a_{ij}(y)\nu_i(y)\frac{\partial}{\partial y_j} + \varepsilon^0 a_{ij}(y)\nu_i(y)\frac{\partial}{\partial x_j}\\ &= \varepsilon^{-1}\frac{\partial}{\partial \nu_{A_1}} + a_{ij}(y)\nu_i(y)\frac{\partial}{\partial x_j}. \end{aligned} \tag{4.8}$$

We obtain by identification

$$(4.9)\qquad \begin{aligned} &A_1 u_0 = 0 \text{ in } Y - \mathcal{O}, \\ &u_0 = 0 \text{ on } S_D, \frac{\partial u_0}{\partial \nu_{A_1}} = 0 \text{ on } S_N, \\ &u_0 \text{ is } Y\text{-periodic} \end{aligned}$$

so that

$$u_0 = 0$$

and, in the same manner, $u_1 = 0$. Then

$$(4.10)\qquad \begin{aligned} &A_1 u_2 = f, \\ &u_2 = 0 \text{ on } S_D, \frac{\partial u_2}{\partial \nu_{A_1}} = 0 \text{ on } S_N, \\ &u_2 \text{ is } Y\text{-periodic.} \end{aligned}$$

This is the analogue of (2.13), with $-\Delta_y$ replaced by A_1. We introduce $w(y)$ as the solution of

$$(4.11)\qquad \begin{aligned} &A_1 w = 1 \text{ in } Y - \mathcal{O}, \\ &w = 0 \text{ on } S_D, \frac{\partial w}{\partial \nu_{A_1}} = 0 \text{ on } S_N, \\ &w \text{ is } Y\text{-periodic} \end{aligned}$$

and we obtain

$$(4.12)\qquad u_2(x, y) = w(y) f(x).$$

We proceed as in §2 and we find (2.21) for the general structure of $u_m(x, y)$, where the $w^{(p)}$ are computed as in §2 but using A_1, A_2, A_3 instead of $-\Delta_y$, $-2\Delta_{xy}$, $-\Delta_x$. The error estimates are unchanged.

5. **Obstacles and rapidly varying coefficients (II).** We consider now a situation somewhat analogous to that of §4, but where the coefficients a_{ij} have a "much smaller" period than the period of the holes. Let us set $Z = \prod_{j=1}^{n}]0, z_j^0[$ and let us consider functions $a_{ij}(z)$, $a_0(z)$ such that

$$(5.1)\qquad \begin{aligned} &a_{ij}(z), a_0(z) \in L^\infty(R^n), a_{ij} \text{ and } a_0 \text{ are } Z\text{-periodic}, \\ &a_{ij}(z)\xi_i\xi_j \geqq \alpha\xi_i\xi_i, \; a_0(z) \geqq \alpha, \; \alpha > 0 \text{ a.e. in } z. \end{aligned}$$

We set now

$$(5.2)\qquad A^\varepsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon^2}\right)\frac{\partial}{\partial x_j}\right) + a_0\left(\frac{x}{\varepsilon^2}\right)$$

and we consider the problem

(5.3) $$A^\varepsilon u_\varepsilon = f \text{ in } \Omega_\varepsilon \text{(where } \Omega_\varepsilon \text{ is defined as in §1)},$$

(5.4) $$u_\varepsilon = 0 \text{ on } \Gamma_\varepsilon,$$

(5.5) $$u_\varepsilon = 0 \text{ on } S.$$

(We suppose that $S_D = S$; if S_N is of positive measure on S, there are some technical difficulties that we want to avoid here.) We want to study the asymptotic behavior of u_ε as $\varepsilon \to 0$.

Asymptotic expansion. We look for u_ε in the form

(5.6) $$u_\varepsilon = u_0(x, y, z) + \varepsilon u_1(x, y, z) + \cdots, \; y = x/\varepsilon, \; z = x/\varepsilon^2,$$

where

(5.7) $$\begin{aligned} &u_j(x, y, z) \text{ is defined for } x \in \Omega, \; y \in Y - \mathcal{O}, \; z \in Z, \\ &u_j \text{ is } Y\text{-periodic in } y, \; Z\text{-periodic in } z, \\ &u_j(x, y, z) = 0 \text{ if } x \in \Omega, \; z \in Z \text{ and } y \in S. \end{aligned}$$

With these notations, we find that

(5.8) $$\begin{aligned} A^\varepsilon &= \varepsilon^{-4}A_1 + \varepsilon^{-3}A_2 + \varepsilon^{-2}A_3 + \varepsilon^{-1}A_4 + \varepsilon^0 A_5. \\ A_1 &= -\frac{\partial}{\partial z_i}\left(a_{ij}(z)\frac{\partial}{\partial z_j}\right), \\ A_2 &= -\frac{\partial}{\partial z_i}\left(a_{ij}(z)\frac{\partial}{\partial y_j}\right) - \frac{\partial}{\partial y_i}\left(a_{ij}(z)\frac{\partial}{\partial z_j}\right), \\ A_3 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(z)\frac{\partial}{\partial y_j}\right) - \frac{\partial}{\partial z_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial_j z}\right), \\ A_4 &= -\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial y_j}\right), \\ A_5 &= -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + a_0(z). \end{aligned}$$

We use (5.6) and (5.8) in (5.3). We find that

$$A_1 u_0 = 0$$

which, since u_0 should be Z periodic, implies that

(5.9) $$u_0 = u_0(x, y).$$

The term in ε^{-3} gives

$$A_1 u_1 + A_2 u_0 = 0,$$

i.e.,

(5.10) $$A_1 u_1 = \frac{\partial a_{ij}}{\partial z_i} \frac{\partial u_0}{\partial y_j}.$$

We introduce $\chi^j(z)$ by

(5.11) $$A_1 \chi^j = -\frac{\partial a_{ij}}{\partial z_i}, \ \chi^j \text{ is } Z\text{-periodic}$$

(which defines χ^j up to an additive constant). Then the general solution of (5.10) is

(5.12) $$u_1 = -\chi^j(z) \frac{\partial u_0}{\partial y_j} + \tilde{u}_1(x, y).$$

The term in ε^{-2} gives

(5.13) $$A_1 u_2 + A_2 u_1 + A_3 u_0 = 0$$

which admits a Z-periodic solution iff

(5.14) $$\int_Z (A_2 u_1 + A_3 u_0)\, dz = 0.$$

We replace in (5.14) u_1 by its value (5.12) and we use (5.9); we obtain

(5.15) $$\mathscr{A} u_0 = 0 \text{ for } x \in \Omega, \ y \in Y - \mathcal{O},$$

where

(5.16) $$\mathscr{A} = -q_{ij} \cdot \frac{\partial^2}{\partial y_i \partial y_j},$$
$$q_{ij} = \frac{1}{|Z|} \int_Z \left[a_{ij}(z) - a_{ik}(z) \frac{\partial \chi^j}{\partial z_k}(z) \right] dz, \ |Z| = \prod_j z_j^0.$$

The operator $\mathscr{A}$ is the *homogenized operator* corresponding to the ε^2 Z-periodic structure; it is an *elliptic operator* (cf. A. Bensoussan, J. L. Lions and G. Papanicolaou [**4**] and the bibliography therein). Therefore as a function of y, u_0 should satisfy (5.15) together with the boundary conditions

$$u_0 = 0 \text{ on } S, \ u_0 \text{ is } Y\text{-periodic}.$$

Therefore

$$u_0 = 0$$

and (5.12) reduces to $u_1 = \tilde{u}_1(x, y)$ Then (5.13) reduces to

$$A_1 u_2 + A_2 \tilde{u}_1 = 0,$$

so that

(5.17) $$u_2 = -\chi^j(z)\frac{\partial \tilde{u}_1}{\partial y_j} + \tilde{u}_2(x, y).$$

The term in ε^{-1} gives

(5.18) $$A_1 u_3 + A_2 u_2 + A_3 \tilde{u}_1 = 0,$$

which admits a Z-periodic solution iff

$$\int_Z (A_2 u_2 + A_3 \tilde{u}_1)\, dz = 0;$$

hence

$$\mathscr{A}\tilde{u}_1 = 0$$

and we conclude as for u_0 that $\tilde{u}_1 = 0$. Therefore $u_2 = \tilde{u}_2(x, y)$ and (5.18) gives

(5.19) $$u_3 = -\chi^j(z)\frac{\partial \tilde{u}_2}{\partial y_j} + \tilde{u}_3(x, y).$$

The term in ε^0 gives

(5.20) $$A_1 u_4 + A_2 u_3 + A_3 u_2 = f$$

and (5.20) admits a Z-periodic solution u_4 iff

$$\frac{1}{|Z|}\int_Z (A_2 u_3 + A_3 u_2)\, dz = f,$$

i.e.,

(5.21) $$\mathscr{A}\tilde{u}_2 = f, \quad x \in \Omega, \quad y \in Y - \mathcal{O},$$

with $\tilde{u}_2$ subject to

(5.22) $$\tilde{u}_2 \text{ is } Y\text{-periodic}, \ \tilde{u}_2 = 0 \text{ on } S.$$

We introduce $w(y)$ by

(5.23) $$\begin{aligned} &Aw = 1 \text{ in } Y - \mathcal{O}, \\ &w \text{ is } Y\text{-periodic}, \ w = 0 \text{ on } S. \end{aligned}$$

Then

(5.24) $$u_2 = \tilde{u}_2(x, y) = w(y)f(x).$$

Then (5.19) gives

(5.25) $$u_3 = -\chi^j(z)\frac{\partial w}{\partial y_j}(y)f(x) + \tilde{u}_3(x, y),$$

where $\tilde{u}_3$ is to be defined. For that one solves (5.20) in u_4 and one considers the term in ε^1

(5.26) $$A_1 u_5 + A_2 u_4 + A_3 u_3 + A_4 u_2 = 0;$$

equation (5.26) admits a Z-periodic solution u_5 iff

$$\int_Z (A_2 u_4 + A_3 u_3 + A_4 u_2)\, dz = 0,$$

which gives an elliptic equation

(5.27) $$\mathscr{A}\tilde{u}_3 = \text{given function of } x \text{ and } y \text{ in } Y - \mathcal{O}.$$

The boundary condition $u_3(x, y, z) = 0$ on S gives

$$\tilde{u}_3(x, y) = \chi^j(z) \frac{\partial w}{\partial y_j}(y) f(x) \text{ on } S$$

which is in general impossible to satisfy. Therefore in order to obtain higher order expansions, boundary layer terms "near" S_ε are necessary; but the construction of these terms is an open question.

In order to define $\tilde{u}_3$ one can take for instance

$$\tilde{u}_3 = 0 \text{ for } y \in S$$

and one continues the computation in this way.

We can prove the following

THEOREM 5.1. *We suppose that*

(5.28) $$f \in C^\sigma(\bar{\Omega}),\ D^p f = 0 \text{ on } \Gamma \text{ for every } |p| \leqq 4.$$

Then, assuming S smooth,

(5.29) $$\|u_\varepsilon - \varepsilon^2 w(y) f(x)\|_{L^\infty(\Omega_\varepsilon)} \leqq C\varepsilon^3,$$

where w and $\mathscr{A}$ are defined by (5.23) *and* (5.16).

PROOF. We introduce

(5.30) $$\varphi_\varepsilon = u_\varepsilon - (\varepsilon^2 u_2 + \cdots + \varepsilon^k u_k)$$

and we shall choose below $k \leqq 6$ (so that (5.28) is sufficient to have all terms well defined). We obtain

(5.31) $$A^\varepsilon \varphi_\varepsilon = \varepsilon^{k-3} g_\varepsilon,$$

(5.32)
$$\begin{aligned} g_\varepsilon = {} & A_2 u_k + A_3 u_{k-1} + A_4 u_{k-2} + A_5 u_{k-3} \\ & + \varepsilon(A_3 u_k + A_4 u_{k-1} + A_5 u_{k-2}) + \varepsilon^2(A_4 u_k + A_5 u_{k-1}) \\ & + \varepsilon^3 A_5 u_k, \end{aligned}$$

(5.33) $$\varphi_\varepsilon = 0 \text{ on } \Gamma_\varepsilon,$$

and

(5.34) $$\varphi_\varepsilon = -(\varepsilon^3 u_3 + \cdots + \varepsilon^k u_k) \text{ on } S_\varepsilon.$$

In (5.33) we have used the structure of u_k (as a combination of $D^p f$). Since we assume S smooth and since $\mathscr{A}$ is elliptic with constant coefficients, all functions $w(y), \ldots, w^{(p)}(y)$ are smooth so that, if we choose $k = 3$,

$$\|A^\varepsilon \varphi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon^3),$$
$$\|\varphi_\varepsilon\|_{L^\infty(\partial\Omega_\varepsilon)} = 0(\varepsilon^3).$$

so (5.29) follows.

6. **Various remarks.**

REMARK 6.1. The above methods are quite general for elliptic problems. For the case of higher order equations with singular perturbations, we refer to B. Desgraupes [**7**]. The case of elliptic systems can lead to some new difficulties. For the case of Stokes system, we refer to J. L. Lions [**10**].

REMARK 6.2. Some of the results of this paper can be obtained by probabilistic arguments; cf. A. Bensoussan [**4**].

REMARK 6.3. Similar methods apply to problems of evolution. Cf. J.L. Lions [**11**].

REMARK 6.4. Spectral problems for domains with holes (or obstacles) are studied in Kesavan and Vaninathan [**8**] and in the thesis of Vaninathan [15].

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