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SEMICOMPACT COZERO-FIELDS AND UNIFORM SPACES

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ABSTRACT. A cozero-field $\mathscr{A}$ is semicompact if each countable $\mathscr{A}$-cover has a finite subcover. This paper examines those uniform spaces X for which coz X is semicompact, and shows that each of the following conditions (among others) characterizes such spaces: Each completely additive coz X-cover has a finite subcover; X is the unique member of its cozero class; X is the unique member of its proximity class and each finite coz X-cover is uniform; X is precompact, and either cozero-fine or metric-fine; X is G_δ-dense in its Samuel compactification; Each metric uniformly continuous image of X is compact.

1. **Alexandroff Spaces.** We use the terminology of §1 of [8d]. Briefly: A pair $\langle X, \mathscr{A} \rangle$, where X is a set and $\mathscr{A}$ is a separated *cozero-field* of subsets, is called an Alexandroff space or A-space, the members of $\mathscr{A}$ are called cozero-sets (and the complements zero-sets), and an A-morphism between A-spaces is a function inversely preserving cozero-sets. When possible, we just write X for $\langle X, \mathscr{A} \rangle$. If X and Y are A-spaces, $A(X, Y)$ stands for the set of A-morphisms from X to Y. For $A(X, R)$ we just write $A(X)$ (R being the reals, whose topology is a cozero-field); $A^*(X)$ denotes the subset of bounded functions. For any A-space $\langle X, \mathscr{A} \rangle$, we have from [1] that $\mathscr{A} = \{\text{coz } f \mid f \in A(X)\}$, where $\text{coz } f = \{x \mid f(x) \neq 0\}$. A topology τ and a cozero-field $\mathscr{A}$ are coz-compatible if $\mathscr{A}$ is a base for τ; a compact Hausdorff space has a unique coz-compatible cozero-field [1]. (See also §9 of [8a].) There is an analogue of the Stone-Čech compactification [1], which we denote $\beta_A X$: an essentially unique compact A-space containing X as a dense A-subspace, such that if K is compact, then $A(X, K) = A(\beta_A X, K) \mid X$ (or just $A^*(X) = A(\beta_A X) \mid X$). A uniformity μ on X is coz-compatible with $\mathscr{A}$ if $\mathscr{A} = \{\text{coz } f \mid f \in U(\mu X)\}$, where $U(\mu X)$ denotes the real-valued uniformly continuous functions. (Similarly, $U^*(\mu X)$ denotes the subset of

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bounded functions.) Note that a topological (resp., uniform) space X can be equipped with the cozero-field $\{\text{coz } f \mid f \in C(X)\}$ (resp., $\{\text{coz } f \mid f \in U(X)\}$.) So it makes sense to speak of A-maps from an A-space into a topological (resp., uniform) space.

The objects we are calling Alexandroff spaces were introduced in [1] (called there *completely normal Hausdorff spaces*) and re-invented in [7] (defined dual to the above, called *Hausdorff zero-set spaces*). Other recent studies include [3], [6], and [8a, d].

2. **Semicompact Alexandroff spaces.** $\langle X, \mathscr{A} \rangle$ will be called *semicompact if each countable $\mathscr{A}$-cover has a finite subcover.* These spaces have been studied in [1] (called "countably compact") and [7] (called "pseudocompact"). We use the term "semicompact" (consistent with [12], at least) to avoid confusion.

Recall (say, from [5]) that a Tychonoff space X is called pseudocompact if $C(X) = C^*(X)$. Equivalently, if each countable cozero-cover (i.e., by sets coz f, $f \in C(X)$) has a finite subcover, that is , if the associated A-space is semicompact.

Let S be an A-subspace of the A-space X: S is called G_δ-dense if each non-empty zero-set of X meets S (equivalently, if each non-empty G_δ-set of X meets S, referring to the topology with the cozero-sets as base).

THEOREM 2.1. *The following conditions on the A-space X are equivalent.*

- (a) *X is semicompact.*
- (b) $A(X) = A^*(X)$.
- (c) *X is G_δ-dense (c_1) in $\beta_A X$; or (c_2) in every A-compactification; or (c_3) in some A-compactification.*
- (d) *X has a unique A-compactification.*
- (e) *X admits a unique* coz-*compatible uniformity.*
- (f_A) *Each A-image of X in an A-space is semicompact.*
- (f_J) *Each A-image of X in a uniform space is precompact.*
- (f_T) *Each A-image of X in a topological space is pseudocompact.*
- (f_M) *Each A-image of X in a metric space is compact.*
- (g) *Each A-morphism of X into a metric space extends over $\beta_A X$ (with values in the metric space).*

Much of this is known: the equivalence of (a), (b), (c_1), (c_2), and the first part of (f) is in both [1] and [7]. These are probably familiar as analogues of pseudocompactness. There are more analogues as well. For example, each of the following is equivalent to (a): (g) $\upsilon_A X = \beta_A X$ (υ_A being Gordon's A-space analogue of the Hewitt realcompactification);

(h) Each $f \in A^*(X)$ assumes its sup and inf; (i) Each infinite family of cozero-sets has a cluster point.

We need just 2.1 for application to uniform spaces so we give the proof.

PROOF OF 2.1. $(a) \Rightarrow (b)$. If $f \in A(X)$, then $\{\{x \mid |f(x)| < r\}\}$ is a countable cozero-cover (2.1 of [8d]); with a finite subcover, f is bounded.

$(b) \Rightarrow (c_1)$. If $\Phi \neq Z(f) \subset \beta_A X - X$, then for $x \in X$, $g(x) = 1/f(x)$ defines unbounded $g \in A(X)$ (1.2 of [8d]).

$(c_1) \Rightarrow (a)$. If $\{C_n\}$ is a countable cozero-cover with no finite subcover: for each n, choose a cozero-set C_n' of $\beta_A X$ with $C_n' \cap X = C_n$. Then $Z = \beta_A X - \cup_n C_n'$ works.

$(c_1) \Rightarrow (c_2) \Rightarrow (c_3)$. Obvious.

Now, 4.3 B of [8d] implies immediately: If S is G_δ-dense in Y, then $A(S) = A(Y) \mid S$.

Thus $(c_3) \Rightarrow (c_1)$, because the A-compactification in (c_3) has to be $\beta_A X$; and $(c_2) \Rightarrow (d)$, because every A-compactification has to be $\beta_A X$.

$(a) \Rightarrow (e)$. Assume (a), and let μ be coz-compatible. Since any uniformity has a base of some of its own cozero-covers, each μ-uniform cover has a finite subcover. So μ is precompact, and μX is a uniform subspace of its Samuel compactification $s\mu X$. But $s\mu X$ is an A-compactification of X (because $U^*(\mu X) = U(s\mu X) \mid X$). Since (d) holds too, μ is determined uniquely.

$(e) \Rightarrow (a)$. §2 of [8a] shows that, if $\mathscr{A}$ is a cozero-field, then the family of countable $\mathscr{A}$-covers is the base for a uniformity, say $\mu_1(\mathscr{A})$, which is coz-compatible with $\mathscr{A}$. The obvious variation on that construction shows the family of finite $\mathscr{A}$-covers is the base for another coz-compatible uniformity, $\mu_0(\mathscr{A})$. (These are provable as well by combinatorial means. For $\mu_0(\mathscr{A})$ one uses the "normality" of $\mathscr{A}$ and for $\mu_1(\mathscr{A})$, the "perfect normality".) Now, obviously, $(a) \Leftrightarrow [\mu_0 = \mu_1]$, which is implied by (e).

Certainly $(a) \Rightarrow (f_A)$. $(f_A) \Rightarrow f_U$ as in $(a) \Rightarrow (e)$. And $(f_U) \Rightarrow (a)$ by considering the identity $X \rightarrow \mu_1 X$ (as in $(e) \Rightarrow (a)$). Next: $(f_A) \Rightarrow (f_T)$, clearly. $f_T \Rightarrow (f_M)$ because a pseudocompact metric space is compact [5],

$(f_M) \Rightarrow (g)$. Obvious.

$(g) \Rightarrow (c_1)$. If $f \in A^*(\beta_A X)$ and $\Phi \neq Z(f) \subset \beta_A X - X$, then with $M = f(X)$, the restriction $f \mid X$ violates (g).

3. **Semicompact uniform spaces.** A separated uniform space X (we can usually suppress indicating the uniformity) will be called semicompact if the associated A-space is semicompact, that is, if each countable cover by sets in coz $X \equiv \{\text{coz } f \mid f \in U(X)\}$ has a finite

shall give some characterizations.

Since each uniform space "is" an A-space the notation $A(X, Y)$ (for $X, Y \in$ Unif) is clear. Evidently, $U(X, Y) \subset A(X, Y)$ always. But hardly conversely: for X, Y metric, $A(X, Y) = C(X, Y)$, but only rarely is $U(X, Y) = C(X, Y)$. In case $X \in$ Unif has the property that $U(X, Y) = A(X, Y)$ for each $Y \in$ Unif, (or equivalently, for each metric Y), then X is called *coz-fine*. It is a small theorem that X is coz-fine iff X is finest in its cozero-class (which by definition consists of all uniform spaces X' with coz X' = coz X); see [8c].

Given $X \in$ Unif, let $\overline{X}$ be the uniform space weakly generated by all functions in all $A(X, Y)$, $Y \in$ Unif. That is, $\overline{X}$ carries the coarsest uniformity making all these functions uniformly continuous; this uniformity is at least as fine as X's (since $U(X, Y) \subset A(X, Y)$ always) and it is easily seen that $\overline{X} = X$ iff X is coz-fine.

There is a somewhat complicated cover-theoretic description of $\overline{X}$ which we shall need: Given $X \in$ Unif, a cover $\mathscr{U}$ (not necessarily uniform) is called a completely additive (*ca*) coz X-cover if (*a*) $\mathscr{U}' \subset \mathscr{U}$ implies $\cup \mathscr{U}' \in$ coz X, and (*b*) $\mathscr{U}$ initiates a normal sequence of covers with property (*a*). Then (§4 of [8c]), $\overline{X}$ has subbasis of *ca* coz X-covers.

We write $X \in$ coz! if X is the only member of its coz-class.

$\mathscr{P}$ denotes the class of precompact uniform spaces, p is the precompact reflection (see [10a]) and the p-(proximity, or precompactness) class of X consists of all X' with $pX = pX'$ (equivalently, $U^*(X) = U^*(X')$). We write $X \in \mathscr{P}!$ if X is the only member of its p-class. Isbell [10b] and Polyakov [13] have studied $\mathscr{P}!$. One sees easily that $X \in \mathscr{P}!$ iff $X \in \mathscr{P} \cap$ (proximally-fine). (See [2] on proximally-fine spaces.)

For $X \in$ Unif, we write $X \in$ coz-$\mathscr{P}$ if each A-image of X in a uniform space is in $\mathscr{P}$ (i.e., precompact). Cf. 2.1(*f*).

Finally, X is called metric-fine [8a] if $U(X, M) = U(X, \alpha M)$ for any metric M. (Here α is the fine coreflector in Unif [10a]: Given $Y \in$ Unif, αY carries the finest uniformity compatible with the underlying topology of Y.) We need only this (2.3 of [8a]): if X has a base of countable covers, then X is metric-fine iff each countable coz X-cover is uniform, i.e., $\mu_2 X = X$, where $\mu_1 = \mu_1(\text{coz } X)$ is the uniformity with base of countable coz X-covers mentioned in the proof of 2.1(*e*) $\Rightarrow$ (*a*).

THEOREM 3.1.

(*a*) *X is semicompact.*

(*b*) *The image of X in its Samuel compactification sX is G_δ-dense.*

(*c*) *X is a G_δ-dense uniform subspace (c_1) of its Samuel compactification; or equivalently, (c_2) of some compact space.*

(*d*) $X \in$ coz!

(*e*) $X \in$ coz-$\mathscr{P}$.

(*f*) $X \in \mathscr{P} \cap$ (coz-*fine*).
(g) *Each ca* coz *X-cover has a finite subcover.*
(*h*) $X \in \mathscr{P}!$ *and each finite* coz *X-cover is uniform.*
(*i*) $X \in \mathscr{P}! \cap$ (*metric-fine*).
(*j*) $X \in \mathscr{P} \cap$ (*metric-fine*).
(*k*) *Each metric uniformly continuous image of X is compact.*
(*l*) *Each uniformly continuous function from X into a metric space extends over sX (with values in the metric space).*

PROOF. (*a*) through (*e*) are equivalent: 2.1 shows the equivalence of (*a*), (*b*), (*d*), (*e*). Such an X is precompact, hence $X \subset sX$, hence (*b*) $\Rightarrow$ (c_1). Clearly, (c_1) $\Rightarrow$ (c_2). And (c_2) $\Rightarrow$ (*b*) because the compact space in (c_2) must be sX.

(*d*) $\Rightarrow$ (*h*). The p-class of any space is a subset of the coz-class; so coz! $\subset \mathscr{P}!$. The uniformity μ_0 defined by finite coz X-covers is in the coz-class of X (see 2.1(*e*) $\Rightarrow$ (*a*)). So (*d*) $\Rightarrow$ $[\mu_0 X = X]$.

(*h*) $\Rightarrow$ (*i*). Assume (*h*). Since each finite coz X-cover is uniform, and $X \in \mathscr{P}$, $\mu_0 X = X$. Evidently, $p\mu_1 X = \mu_0 X$ for any Y. Thus, $\mu_1 X$ is in the p-class of X. Since $X \in \mathscr{P}!$, $\mu_1 X = X$. Since X has a base of countable covers (even finite ones, since $X \in \mathscr{P}$), X is metric-fine.

(*i*) $\Rightarrow$ (*j*) is clear, since $\mathscr{P}! \subset \mathscr{P}$.

(*j*) $\Rightarrow$ (*k*). Let $f \in U(X, M)$ be onto, with M metric. If X is metric-fine, then $f \in U(X, \alpha M)$. If $X \in \mathscr{P}$, then $\alpha M \in \mathscr{P}$. Since the base for an αM is all open covers, each open cover has a finite subcover.

(*k*) $\Rightarrow$ (*l*) is obvious.

(*l*) $\Rightarrow$ (*c*) is like 2.1 (*g*) $\Rightarrow$ (c_1).

(*f*) $\Leftrightarrow$ (g) by the description of $\overline{X}$ given above.

(*f*) $\Rightarrow$ (*e*). Each uniform image of X is precompact (since $X \in \mathscr{P}$), and each A-image is a uniform image (since $X \in$ coz-fine).

(*c*) $\Rightarrow$ (*f*). Assume (*c*), with $X \subset K$. Then $X \in \mathscr{P}$ and $K = sK$, the Samuel compactification. A uniform subspace is an A-subspace (by Katětov's extension theorem for bounded functions [11]). Thus, as in the proof of 2.1 (c_3) $\Rightarrow$ (c_1), it follows that $sX = \beta_A X$. Now let $f \in A(X, M)$, M metric and f onto. There is an extension $f' \in A(\beta_A X, \beta_A M)$. Since (*c*) holds, (*a*) holds also, and by 2.1(*f*), $M = \beta_A M$. Since $\beta_A X = sX$, then, $f' \in A(sX, M) = U(sX, M)$ (by compactness of sX). Thus, the restriction $f = f' \mid X \in U(X, M)$.

COROLLARY 3.2. *The class* coz! *of semicompact uniform spaces is closed under formation of: uniformly continuous images, G_δ-dense subspaces, and arbitrary products.*

PROOF. In each case, 3.1(*c*) can be used.

REMARKS 3.3. We compare coz! with $\mathscr{P}!$:

Isbell [10b] has shown (1) that the cone T in precompact spaces over the countably infinite free precompact space is a test space for spaces in $\mathscr{P}!$, i.e., $X \in \mathscr{P}!$ iff each $f \in U(X, T)$ extends over sX; this is to be compared with the much simpler condition 3.1(1) for coz!; and (2), that $\mathscr{P}!$ is closed under uniformly continuous images and products (the latter result using (1)). It can be shown that $\mathscr{P}!$ is closed under forming G_δ-subspaces, as well.

Polyakov [13] has shown that $\mathscr{P}!$ is closed under finite products, and Hušek [9], that a product is proximally-fine iff each finite subproduct is. These combine to give another proof of Isbell's product theorem.

REMARK. 3.4. In [4], A. Diabes has shown, independently, 3.1 $(a) \Leftrightarrow (c)$, and given a number of other equivalences, mostly involving uniform measures.

REFERENCES

1. A. D. Alexandroff, *Additive set-functions in abstract spaces*, Mat. Sbornik **50** (1940), 307–348, **51** (1941), 563–628; **55** (1943), 169–238.

2. E. M. Alfsen and O. Njastad, *Proximity and generalized uniformity*, Fund. Math. **5** (1963), 235–252.

3. M. J. Canfell, *Dimension theory in zero-set spaces*, Pac. J. Math. **47** (1973), 393–400.

4. A. Diabes, *Espace Uniformes et Espace de Mesures*, Thesis, Universite Claude-Bernard-Lyon, 1976.

5. L. Gillman and M. Jerison, *Rings of Continuous Functions*, D. van Nostrand Co., Princeton, 1960.

6. C. R. A. Gilmour, *Special morphisms for zero-set spaces*, Bull. Austral. Math. Soc. **13** (1975), 57–68.

7. H. Gordon, *Rings of functions determined by zero-sets*, Pac. J. Math., **36** (1971), 133–157.

8. A. W. Hager, *Some nearly fine uniform spaces*, Proc. London Math Soc., **28** (1974), 517–546.

9. ———, *Pseudocompact cozero fields and uniform spaces*, Notices Amer. Math. Soc. **21** (1974), A-562.

10. ———, *Uniformities induced by cozero-and Baire-sets*, Proc. Amer. Math. Soc. **63** (1977), 153–159.

11. ———, *Real-valued functions on Alexandroff (zero-set) spaces*, Comm. Math. Univ. Carolinae. **16** (1975), 755–769.

12. M. Hušek, *Factorization of mappings (products of proximally fine spaces)*, Seminar Uniform Spaces 1973-1974, ČSAV, Prague, 173–190.

13. J. R. Isbell, *Uniform Spaces*, Providence, 1964.

14. ———, *Spaces without large projective subspaces*, Math. Scand. **17** (1965), 89–105.

15. M. Katětov, *On real-valued functions in topological spaces*, Fund. Math. **38** (1951), 85–91.

16. P. Meyer, *Probabilities and Potentials*, Blaisdell Publ. Co., Waltham, 1966.

17. V. Polyakov, *Regularity, product and spectra of proximity spaces*, Soviet Math. Dokl. **5** (1964), 45–49.

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