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TORSION THEORY FOR NOT NECESSARILY ASSOCIATIVE RINGS

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1. **Introduction.** The purpose of this note is to investigate torsion theories in the category of not necessarily associative rings. Torsion theories for abelian groups and abelian categories were studied in Dickson's papers [4] and [5] and it is found that for associative and alternative rings radical and semisimple classes (in the sense of Kurosh and Amitsur) correspond to torsion and torsionfree classes, respectively.

We will be considering subclasses of some universal class of not necessarily associative rings, where a class is *universal* if it is homomorphically closed and hereditary (closed under taking ideals). Calling a class co-radical when its properties are dual to those of a radical class, it is well-known that a semisimple class need not be a co-radical class or vice versa. However, starting from Dickson's definition of a torsion theory, we nevertheless obtain a complete duality between torsion and torsionfree classes. Torsion classes turn out to be particular radical classes and torsionfree classes are special kinds of semisimple and co-radical classes. In Section 2 torsion and torsionfree classes will be characterized. In Sections 3 and 4 classes and constructions related to torsion theories will be investigated and further characterizations of torsion theories will be obtained. For fundamental definitions and properties of radical and semisimple classes we refer to [8] and [16].

2. **Characterizations of torsion theories.** All rings considered will be members of some fixed universal class of not necessarily associative rings. It is assumed that every class $\mathbf{X}$ considered contains the ring 0 and is an abstract class (that is, if $A \in \mathbf{X}$ and $A \cong B$ then $B \in \mathbf{X}$). Also remark that whenever we give an example we are tacitly assuming that our universal class is such that it contains the example. As usual, define the following functions $\mathcal{U}$ and $\mathcal{S}$ acting on classes of rings by $\mathcal{U}\mathbf{X} = \{A \mid A$ has no nonzero homomorphic image in $\mathbf{X}\}$. $\mathcal{S}\mathbf{X} = \{A \mid \mathbf{A}$ has no nonzero ideal in $\mathbf{X}\}$. Further, let us associate for any class $\mathbf{X}$ and to any ring A the ideals

$$\mathbf{X}(A) = \sum_{\alpha} (I_\alpha \lhd A \mid I_\alpha \in \mathbf{X})$$

and

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$$(A)\mathbf{X} = \bigcap_{\beta} (K_\beta \lhd A \mid A/K_\beta \in \mathbf{X}).$$

The fact that B is a homomorphic image of A will be denoted by $A \twoheadrightarrow B$. We recall that B is an *accessible subring* of a ring A if there are finitely many subrings $B_1, \cdots, B_n$ of A such that $B = B_n \lhd \cdots \lhd B_2 \lhd B_1 = A$. If B is an accessible subring of A we shall write $B \succ\!\!\!-\, A$.

DEFINITION. A pair $(\mathbf{T}, \mathbf{F})$ of classes of rings defines a *torsion theory*, if

(1) $\mathbf{T} \cap \mathbf{F} = 0$;

(2) $\mathbf{T}$ is homomorphically closed;

(3) $\mathbf{F}$ is hereditary, that is $I \lhd A \in \mathbf{F}$ implies $I \in \mathbf{F}$;

(4) for every ring A there exists an ideal A_T of A such that $A_T \in \mathbf{T}$ and $A/A_T \in \mathbf{F}$.

The classes $\mathbf{T}$ and $\mathbf{F}$ of a torsion theory $(\mathbf{T}, \mathbf{F})$ are called a *torsion class* and a *torsionfree class*, respectively.

This concept was introduced by Dickson [4], [5] for abelian groups and categories.

THEOREM 1. *The following four statements are equivalent*:

(A) *The pair* $(\mathbf{T}, \mathbf{F})$ *satisfies* $\mathbf{T} = \mathscr{U}\mathbf{F}$ *where* $\mathbf{F}$ *has the following properties*:

(i) $=(3)\mathbf{F}$ *is hereditary*;

(ii) $\mathbf{F}$ *has the co-inductive property, that is, if* $I_1 \supset \cdots \supset I_\alpha \supset \cdots$ *is a descending chain of ideals of a ring A such that every* A/I_α *is in* $\mathbf{F}$, *then also* $A/\bigcap_\alpha I_\alpha \in \mathbf{F}$;

(iii) $\mathbf{F}$ *has the extension property, that is, if* $I \lhd A$ *such that* $I \in \mathbf{F}$ *and* $A/I \in \mathbf{F}$, *then also A belongs to* $\mathbf{F}$;

(iv) $((A)\mathbf{F})\mathbf{F} \lhd A$ *for every ring A.*

(B) *The pair* $(\mathbf{T}, \mathbf{F})$ *satisfies*

(I) *if* $A \in \mathbf{T}$, $B \in \mathbf{F}$ *and* $A \twoheadrightarrow C \succ\!\!\!-\, B$, *then* $C = 0$;

(II) $=(4)$ *for every ring A there exists an ideal* A_T *of A such that* $A_T \in \mathbf{T}$ *and* $A/A_T \in \mathbf{F}$.

(C) $(\mathbf{T}, \mathbf{F})$ *is a torsion theory*;

(D) *The pair* $(\mathbf{T}, \mathbf{F})$ *satisfies* $\mathbf{F} = \mathscr{S}\mathbf{T}$ *where* $\mathbf{T}$ *has the properties*:

(a) $=(2)\mathbf{T}$ *is homomorphically closed*;

(b) $\mathbf{T}$ *has the inductive property, that is, if* $I_1 \subset \cdots \subset I_\alpha \subset \cdots$ *is an ascending chain of* $\mathbf{T}$*-ideals of a ring A, then also* $\bigcup_\alpha I_\alpha \in \mathbf{T}$.

(c) $=(\mathrm{iii})\mathbf{T}$ *has the extension property*;

(d) $\mathbf{T}(L) \subseteq \mathbf{T}(A)$ *for every ideal L of any ring A.*

PROOF. (A) $\Rightarrow$ (B) Suppose that the class $\mathbf{F}$ satisfies conditions (i), (ii), (iii) and (iv) and consider the class $\mathscr{U}\mathbf{F}$. We claim that the pair $(\mathscr{U}\mathbf{F}, \mathbf{F})$ fulfills (I) and (II). The class $\mathscr{U}\mathbf{F}$ is clearly homomorphically closed, moreover $\mathscr{U}\mathbf{F} \cap \mathbf{F} = 0$. Since by (i) $\mathbf{F}$ is hereditary, in each relation $A \twoheadrightarrow C \rightarrowtail B$ it follows from $A \in \mathscr{U}\mathbf{F}$ and $B \in \mathbf{F}$ that $C \in \mathscr{U}\mathbf{F} \cap \mathbf{F} = 0$. Hence, (I) is valid.

By the co-inductive property (ii) the dual of the Zorn lemma is applicable, so there exists a minimal ideal $I \lhd A$ such that $A/I \in \mathbf{F}$. If K is an ideal of A with $A/K \in \mathbf{F}$, then we have

$$\frac{A/(I \cap K)}{K/(I \cap K)} \cong A/K \in \mathbf{F}$$

and

$$K/(I \cap K) \cong (K + I)/I \lhd A/I \in \mathbf{F}.$$

Hence, (i) and (iii) imply $A/(I \cap K) \in \mathbf{F}$ and so the minimality of I yields $I \subseteq K$. Thus I is unique and by definition $I = (A)\mathbf{F}$. Now condition (iv) yields $((A)\mathbf{F})\mathbf{F} \lhd A$. Further, since A has been chosen arbitrarily, also $(A)\mathbf{F}/((A)\mathbf{F})\mathbf{F} \in \mathbf{F}$ holds. Hence, taking into consideration

$$\frac{A/((A)\mathbf{F})\mathbf{F}}{(A)\mathbf{F}/((A)\mathbf{F})\mathbf{F}} \cong A/(A)\mathbf{F} \in \mathbf{F},$$

condition (iii) implies $A/((A)\mathbf{F})\mathbf{F} \in \mathbf{F}$. Consequently $(A)\mathbf{F} \subseteq ((A)\mathbf{F})\mathbf{F} \subseteq (A)\mathbf{F}$ holds and so $(A)\mathbf{F} \in \mathscr{U}\mathbf{F}$. Thus (II) has been established.

(B) $\Rightarrow$ (C) Suppose that the pair $(\mathbf{T}, \mathbf{F})$ satisfies (I) and (II). (I) implies (1) trivially. If $A \in \mathbf{T}$ and B is an arbitrary homomorphic image of A, then by (II) we have an ideal B_T of B such that $B_T \in \mathbf{T}$ and $B/B_T \in \mathbf{F}$. Since B/B_T is also a homomorphic image of A, we have the relation $A \twoheadrightarrow B/B_T \rightarrowtail B/B_T$ and condition (I) implies $B = B_T \in \mathbf{T}$. Thus (2) holds. If $A \in \mathbf{F}$ and $B \lhd A$, then again considering the ideal B_T of B, $B_T \rightarrowtail A$ is valid. Hence, by $B_T \in \mathbf{T}$ and by the relation $B_T \twoheadrightarrow B_T \rightarrowtail A$ condition (II) yields $B_T = 0$. Thus by (II) we have $B = B/B_T \in \mathbf{F}$ and (3) is valid. Finally, (4) is exactly (II).

(C) $\Rightarrow$ (D) Assume that $(\mathbf{T}, \mathbf{F})$ is a torsion theory. We claim that $\mathbf{T}$ fulfills the requirements of (a), (b), (c) and (d) and that $\mathbf{F} = \mathscr{S}\mathbf{T}$. (a) equals (2). Next, consider an ascending chain $I_1 \subset \cdots \subset I_\alpha \subset \cdots$ of $\mathbf{T}$-ideals of a ring A. (4) is applicable to the ring $K = \bigcup_\alpha I_\alpha$, and so there exists an ideal J of K such that $J \in \mathbf{T}$ and $K/J \in \mathbf{F}$. Since by (2) $\mathbf{T}$ is homomorphically closed and by (3) $\mathbf{F}$ is hereditary, from

$$I_\alpha/(I_\alpha \cap J) \cong (I_\alpha + J)/J \lhd K/J \in \mathbf{F}$$

follows $I_\alpha/(I_\alpha \cap J) \in \mathbf{T} \cap \mathbf{F} = 0$. Hence, $I_\alpha \subseteq J$ for every α which implies $K = J \in \mathbf{T}$. Thus (b) holds. To see (c), suppose that L is an ideal of A such that $L \in \mathbf{T}$ and $A/L \in \mathbf{T}$. In view of (4) there is an ideal A_T of A with $A_T \in \mathbf{T}$ and $A/A_T \in \mathbf{F}$. Hence, we have $L/(L \cap A_T) \cong (L + A_T)/A_T \lhd A/A_T \in \mathbf{F}$, and by (1), (2) and (3) it follows that $L/(L \cap A_T) = 0$. Hence, $L \subseteq A_T$. From

$$(A/L)/(A_T/L) \cong A/A_T \in \mathbf{F}$$

(1) and (2) yield $A/A_T \in \mathbf{T} \cap \mathbf{F} = 0$ that is $A = A_T \in \mathbf{T}$, so (c) has been established. For any ring A and ideal $L \lhd A$ consider the ideals $A_T \lhd A$ and $L_T \lhd L$ such that A_T, $L_T \in \mathbf{T}$ and A/A_T, $L/L_T \in \mathbf{F}$. We have

$$\begin{aligned} L_T/(L_T \cap A_T) &\cong (L_T + A_T)/A_T \\ &\lhd (L + A_T)/A_T \lhd A/A_T \in \mathbf{F} \end{aligned}$$

and conditions (1), (2) and (3) imply

$$L_T/(L \cap A_T) \in \mathbf{T} \cap \mathbf{F} = 0,$$

that is $L_T \subseteq A_T$. We still have to see that $A_T = \mathbf{T}(A)$ for every ring A. Since for any $\mathbf{T}$-ideal I of A

$$I/(I \cap A_T) \cong (I + A_T)/A_T \lhd A/A_T \in \mathbf{F}$$

holds, conditions (2) and (3) imply $I/(I \cap A_T) \in \mathbf{T} \cap \mathbf{F}$. Thus $I \subseteq A_T$ and so $\mathbf{T}(A) \subseteq A_T$. On the other hand $A_T \in \mathbf{T}$ implies trivially $A_T \subseteq \mathbf{T}(A)$. Hence, (d) holds. If $A \in \mathbf{F}$ then by (1) and (3) A has no nonzero $\mathbf{T}$-ideals. Conversely, if a ring A has no nonzero $\mathbf{T}$-ideals, then by (4) $A_T = 0$ and $A \cong A/A_T \in \mathbf{F}$ follow. Hence, $\mathbf{F} = \mathscr{S}\mathbf{T}$.

(D) $\Rightarrow$ (A) Let us suppose that a class $\mathbf{T}$ has (a), (b), (c) and (d). We show that $\mathbf{F} = \mathscr{S}\mathbf{T}$ has (i), (ii), (iii) and (iv), and that $\mathscr{U}\mathscr{S}\mathbf{T} = \mathbf{T}$. If $I \lhd A \in \mathbf{F}$, then by (d) it follows that $\mathbf{T}(I) \subseteq \mathbf{T}(A) = 0$. Thus (i) is satisfied. Consider a descending chain $I_1 \supset \cdots \supset I_\alpha \supset \cdots$ of ideals of a ring A such that each A/I_α is in $\mathbf{F}$. Putting $K = \bigcap_\alpha I_\alpha$, take an arbitrary $\mathbf{T}$-ideal L/K of A/K. Then for each α we have

$$L/(L \cap I_\alpha) \cong (L + I_\alpha)/I_\alpha \lhd A/I_\alpha \in \mathbf{F}.$$

Hence (a) and (i) yield $L/(L \cap I_\alpha) \in \mathbf{T} \cap \mathbf{F} = 0$. Thus $L \subseteq I_\alpha$ for each α, that is $L = K$, and therefore $A/K \in \mathbf{F}$. Hence, (ii) has been established. Let us suppose that $I \in \mathbf{F}$ and $A/I \in \mathbf{F}$ for an ideal I of a ring A. We have

$$\mathbf{T}(A)/(\mathbf{T}(A) \cap I) \cong (\mathbf{T}(A) + I)/I \lhd A/I \in \mathbf{F}.$$

Applying (i) and (a) it follows that

$$\mathbf{T}(A)/(T(A) \cap I) \in \mathbf{T} \cap \mathbf{F} = 0.$$

Consequently, $\mathbf{T}(A) \subseteq I \in \mathbf{F}$ holds, so again by (i) we obtain $\mathbf{T}(A) \in \mathbf{F}$. But as is well-known, (a), (b), (c) imply $\mathbf{T}$ is radical so $\mathbf{T}(A) \in \mathbf{T}$ and we have $\mathbf{T}(A) \in \mathbf{T} \cap \mathbf{F} = 0$. Thus $A \in \mathbf{F}$ and (iii) is satisfied. To see (iv), take an ideal I of a ring A and consider

$$(I)\mathbf{F} = \bigcap_{\alpha} (K_\alpha \lhd I \mid I/K_\alpha \in \mathbf{F}).$$

We claim that $I/\mathbf{T}(I) \in \mathbf{F}$. Otherwise there would be an ideal L of I such that $0 \neq L/\mathbf{T}(I) \lhd I/\mathbf{T}(I)$ and $L/\mathbf{T}(I) \in \mathbf{T}$. But then by (c) we would get $L \in \mathbf{T}$ and so also $L \subseteq \mathbf{T}(I)$, a contradiction. Thus $I/\mathbf{T}(I) \in \mathbf{F}$ holds, and the definition of $(I)\mathbf{F}$ gives $(I)\mathbf{F} \subseteq \mathbf{T}(I)$. By (ii) and the dual of Zorns lemma there exists a minimal $L \lhd I$ such that $I/L \in \mathbf{F}$. Then $(I)\mathbf{F} \subseteq L$ and if they were unequal there would exist some $J \lhd I$ such that $I/J \in \mathbf{F}$ but $L \nsubseteq J$. But then (i) would imply $0 \neq (L + J)/J \cong L/(L \cap J) \in \mathbf{F}$ and by (iii) from $I/L \in \mathbf{F}$ would follow $I/L \cap J \in \mathbf{F}$ contradicting the minimality of L. Thus $I/(I)\mathbf{F} \in \mathbf{F}$ so from $\mathbf{T}(I)/(I)\mathbf{F} \lhd I/(I)\mathbf{F}$ it follows by (i) that $\mathbf{T}(I)/(I)\mathbf{F} \in \mathbf{T} \cap \mathbf{F} = 0$. Hence $\mathbf{T}(I) = (I)\mathbf{F}$ and so

$$((A)\mathbf{F})\mathbf{F} = \mathbf{T}(\mathbf{T}(A)) = \mathbf{T}(A) \lhd A$$

and thus (iv) has been proved. If $A \in \mathbf{T}$, then A has clearly no nonzero homomorphic image in $\mathbf{F}$. Therefore, $A \in \mathcal{U}\mathbf{F}$ holds. If $A \in \mathcal{U}\mathbf{F}$, then A has no nonzero homomorphic image in $\mathbf{F}$, so $A = (A)\mathbf{F} = \mathbf{T}(A) \in \mathbf{T}$ holds. Thus, Theorem 1 has been proved.

We say that the class $\mathcal{U}\mathbf{X}$ has the *intersection property relative to* $\mathbf{X}$, if $\mathcal{U}\mathbf{X}(A) = (A)\mathbf{X}$ for every ring A.

COROLLARY 1. *In* (A) *conditions* (ii) *and* (iv) *can be replaced by*

(ii′) $\mathbf{F}$ *is closed under subdirect sums, that is if A is a subdirect sum of* $\mathbf{F}$*-rings then also* $A \in \mathbf{F}$;

(iv′) $\mathcal{U}\mathbf{F}$ *has the intersection property relative to* $\mathbf{F}$.

PROOF. Clearly (i), (ii), and (iii) is equivalent to (i), (ii′), and (iii). Now (ii′) implies $A/(A)\mathbf{F} \in \mathbf{F}$ so $\mathcal{U}\mathbf{F}(A) \subseteq (A)\mathbf{F}$. Also $(A)\mathbf{F}/((A)\mathbf{F})\mathbf{F} \in \mathbf{F}$ so assuming (iv) by (iii) we have $A/((A)\mathbf{F})\mathbf{F} \in \mathbf{F}$. Thus $((A)\mathbf{F})\mathbf{F} = (A)\mathbf{F}$, that is $(A)\mathbf{F}$ has no non-zero images in $\mathbf{F}$ so is in $\mathcal{U}\mathbf{F}$ and we have $\mathcal{U}\mathbf{F}(A) = (A)\mathbf{F}$. On the other hand $\mathcal{U}\mathbf{F}(A) = (A)\mathbf{F}$ clearly implies $((A)\mathbf{F})\mathbf{F} = (A)\mathbf{F} \lhd A$.

The proofs of the following two corollaries are straightforward so we omit them.

COROLLARY 2. *Conditions* (2) *and* (3) *can be replaced in the definition of torsion theories by*

(2′) *if* $A \in \mathbf{T}$ *and* $A \twoheadrightarrow B \neq 0$, *then* $B \notin \mathbf{F}$;

(3′) *if* $A \in \mathbf{F}$ *and* $0 \neq B \rightarrowtail A$, *then* $B \notin \mathbf{T}$.

COROLLARY 3. *In* (D) *for condition* (b) *can be substituted* (b′) $\mathbf{T}(A) \in \mathbf{T}$ *for every ring* A.

Remark that in the proof of Theorem 1 we have used only the isomorphism theorems, and did not use the operations of rings. Thus, Theorem 1 could have been proved in any category satisfying some additional requirements (see for instance, Rjabuhin [**12**], Suliński [**14**], and Wiegandt [**15**]). In particular, *Theorem* 1 *is valid also for multioperator groups.*

As is well-known, a class $\mathbf{R}$ of rings having properties (a), (b) and (c) is said to be a *radical class in the sense of Kurosh and Amitsur,* (cf. Amitsur [**1**]), and the class $\mathscr{S}\mathbf{R}$ is the *semisimple class* for the radical class $\mathbf{R}$ (cf. Amitsur [**1**] and Leavitt [**8**]). In view of Theorem 1 we may say that *a torsion class is a radical class with the additional property* (d), *and a torsionfree class is a hereditary semisimple class.* For associative and alternative rings condition (d) is a consequence of (a), (b) and (c) (cf. Anderson, Divinsky, and Suliński [**2**]) and every semisimple class is hereditary. Hence, every radical class is a torsion class and every semisimple class is a torsionfree class. On the other hand, in the case of associative and alternative rings condition (iv) follows from conditions (i), (ii) and (iii), so the semisimple classes are characterized by conditions (i), (ii) and (iii) (cf. Sands [**13**] and van Leeuwen, Roos, and Wiegandt [**11**]).

A radical class $\mathbf{R}$ is called a strict radical class, if $\mathbf{R}(A)$ contains every $\mathbf{R}$-subring of A. Clearly any strict radical class fulfills condition (d) and so is a torsion class. Examples of strict radicals are Gardner's A-radical classes (cf. [**6**]): a radical class $\mathbf{R}$ is said to be an A-radical class, if $A \in \mathbf{R}$ and $A^+ \cong B^+$ imply $B \in \mathbf{R}$ where X^+ denotes the additive group of the ring X. This means that belonging to an A-radical class depends only on the additive structure of the ring. Thus we can easily get *examples of torsion classes.* For instance

$$T_0 = \{A \mid \text{every } a \in A \text{ is torsion (that is, has finite additive order)}\},$$

or

$$T_p = \{A \mid A \text{ has } p\text{-primary additive group for some fixed prime } p\}.$$

To construct another *example of a torsion theory,* let $\mathbf{V}$ be the class of all rings with characteristic 2 and claim that $(\mathscr{U}\mathbf{V}, \mathscr{S}\mathscr{U}\mathbf{V})$ is a torsion theory. Clearly $\mathbf{V} \subseteq \mathscr{S}\mathscr{U}\mathbf{V}$ and $\mathscr{U}\mathbf{V}$ contains all rings of characteristic $p \neq 2$. Since $\mathscr{U}\mathbf{V}$ is a radical class, we only need $\mathscr{S}\mathscr{U}\mathbf{V}$ hereditary. Let $A \in \mathscr{S}\mathscr{U}\mathbf{V}$ and define $A_1 = A$, $A_{\alpha+1} = 2A_\alpha$ for all ordinals α, and $A_\beta = \bigcap_{\alpha<\beta} A_\alpha$ for limit ordinals β. By induction we have a descending chain $A_1 \supset \cdots \supset A_\alpha \supset \cdots$ of ideals of A which must stabilize at some $A_\gamma = A_{\gamma+1}$. If $A_\gamma \neq 0$ then since $A_\gamma \triangleleft A$ we would have some $0 \neq A_\gamma/J \in \mathbf{V}$. But this leads to the contradiction $A_\gamma = 2A_\gamma \subseteq J$ and thus $A_\gamma = 0$. Now consider an arbitrary subring B of A. We claim that if $B = 2B$ then $B = 0$ for we have $B \subseteq A = A_1$ and if $B \subseteq A_\alpha$ for all $\alpha < \beta$ then either $B \subseteq \bigcap_{\alpha<\beta} A_\alpha = A_\beta$ where β is a limit ordinal or $B = 2B \subseteq A_{\alpha+1}$. Thus by induction $B \subseteq A_\gamma = 0$. Thus if $0 \neq B$ then $0 \neq B/2B \in \mathbf{V}$ and since this is true for all ideals of B it follows that $B \in \mathscr{S}\mathscr{U}\mathbf{V}$.

Note that if our universal class is the class of all not necessarily associative rings torsionfree classes tend to be very large. For example, it follows from Proposition 1.2 of Andrunakievič and Rjabuhin [3, p. 25] that a torsionfree class containing any non-zero Φ-algebra (for an arbitrary field Φ) contains all Φ-algebras. This is, of course, not in general true for more restricted universal classes. Another consequence of this result is that while if $(\mathbf{T}, \mathbf{F})$ is a torsion theory in the class of all rings then $(\mathbf{T} \cap \mathbf{A}, \mathbf{F} \cap \mathbf{A})$ is a torsion theory in the class $\mathbf{A}$ of all associative rings, there are associative torsion theories not obtainable in this way since, for example, many associative radicals have semisimple classes containing some but not all Z_2-algebras.

3. **Classes related to torsion theories.** Conditions (i), (ii), and (iii) are dual to (a), (b) and (c) which define the radical classes. A class with properties (i), (ii) and (iii) is therefore called a *co-radical class* (cf. Rjabuhin [12]). Leavitt and Armendariz [9] have shown that in the category of not necessarily associative rings there are non-hereditary semisimple classes, hence, a semisimple class is not always a co-radical class (or a torsionfree class).

Next, we shall construct a co-radical class containing a given hereditary class. This construction yields another characterization of torsionfree classes and shows that a co-radical class need not be a torsionfree class.

LEMMA 1. *If* $\mathbf{M}$ *is a hereditary class of rings, then* $\mathscr{U}\mathbf{M} = \mathscr{U}\overline{\mathbf{M}}$ where

$$\overline{\mathbf{M}} = \{A \mid \textit{every accessible subring of } A \textit{ is in } \mathscr{S}\mathscr{U}\mathbf{M}\}$$

is the largest hereditary class continued in $\mathscr{S}\mathscr{U}\mathbf{M}$.

PROOF. Since $\mathbf{M} \subseteq \overline{\mathbf{M}} \subseteq \mathscr{S}\mathscr{U}\mathbf{M}$ we get $\mathscr{U}\mathbf{M} \supseteq \mathscr{U}\overline{\mathbf{M}} \supseteq \mathscr{U}\mathscr{S}\mathscr{U}M = \mathscr{U}M$.

THEOREM 2. *For* $\mathbf{M}$ *an arbitrary class, the largest hereditary subclass* $\overline{\mathbf{M}}$ *of* $\mathscr{S}\mathscr{U}\mathbf{M}$ *is a co-radical class.*

PROOF. Let $\overline{\mathbf{M}} \subseteq \mathscr{S}\mathscr{U}\mathbf{M}$ where $\overline{\mathbf{M}}$ is hereditary. Then $\overline{M} \subseteq \mathscr{S}\mathscr{U}\overline{M} \subseteq \mathscr{S}\mathscr{U}\mathscr{S}\mathscr{U}M \subseteq \mathscr{S}\mathscr{U}M$. Thus $\overline{M}$ is also the largest hereditary subclass of $\mathscr{S}\mathscr{U}\overline{\mathbf{M}}$ and we might as well assume $\mathbf{M} = \overline{\mathbf{M}}$ from the beginning. To see (iii) let I be an ideal of a ring A with $I \in \mathbf{M}$ and $A/I \in \mathbf{M}$. Certainly $A \in \mathscr{S}\mathscr{U}\mathbf{M}$ since if $0 \neq J \lhd A$ with $J \in \mathscr{U}\mathbf{M}$, then $J \nsubseteq I$. But this leads to the contradiction

$$0 \neq J/(I \cap J) \cong (J + I)/I \in \mathscr{U}\mathbf{M} \cap \mathbf{M} = 0.$$

If $A \notin \mathbf{M}$, then the maximality of $\mathbf{M}$ would require that there is some accessible subring $B \in \mathscr{U}\mathbf{M}$ with

$$0 \neq B = A_n \lhd A_{n-1} \lhd \cdots \lhd A_1 = A.$$

Then $B \nsubseteq I$, for otherwise $I \in \mathbf{M}$ implies $B \in \mathbf{M} \cap \mathscr{U}\mathbf{M} = 0$. But this leads to the contradiction

$$\begin{aligned} 0 \neq (B + I)/I = (A_n + I)/I \lhd \cdots &\lhd (A_1 + I)/I \\ &= A/I \in \mathbf{M}. \end{aligned}$$

Thus, the conclusion is $A \in \mathbf{M}$.

In order to prove (ii), without loss of generality let us consider a chain $I_1 \supset \cdots \supset I_\alpha \supset \cdots$ of ideals of a ring A where each $A/I_\alpha \in \mathbf{M}$ *and* $\bigcap_\alpha I_\alpha = 0$. Again $A \in \mathscr{S}\mathscr{U}\mathbf{M}$, for if $0 \neq J \lhd A$ with $J \in \mathscr{U}\mathbf{M}$ then for some α we would have $J \nsubseteq I_\alpha$ leading to the contradiction

$$0 \neq J/(J \cap I_\alpha) \cong (J + I_\alpha)/I_\alpha \in \mathscr{U}\mathbf{M} \cap \mathbf{M} = 0.$$

As before, if $A \notin \mathbf{M}$, then the maximality of $\mathbf{M}$ would require some accessible subring $0 \neq A_n \in \mathscr{U}\mathbf{M}$. But then $A_n \nsubseteq I_\alpha$ would lead to the contradiction

$$(A_n + I_\alpha)/I_\alpha \lhd \cdots \lhd A/I_\alpha \in \mathbf{M}.$$

Theorem 2 and Corollary 1 yield immediately the following characterization of torsionfree classes.

COROLLARY 4. *A class* $\mathbf{F}$ *is a torsionfree class if and only if* $\mathbf{F}$ *satisfies*
(α) $\mathbf{F}$ *is the largest hereditary class in* $\mathscr{S}\mathscr{U}\mathbf{F}$;
(β) $\mathscr{U}\mathbf{F}$ *has the intersection property relative to* $\mathbf{F}$.

COROLLARY 5. *If* $\mathbf{M}$ *is a hereditary class such that* $\mathcal{SU}\mathbf{M}$ *is not hereditary, then the largest hereditary subclass* $\overline{\mathbf{M}}$ *of* $\mathcal{SU}\mathbf{M}$ *is not a torsionfree class (though a co-radical class).*

PROOF. Suppose that $\overline{\mathbf{M}}$ is a torsionfree class, then by Lemma 1 $\mathcal{U}\overline{\mathbf{M}} = \mathcal{U}\mathbf{M}$ and by Theorem 1 $\mathcal{U}\overline{\mathbf{M}}$ is a torsion class. Hence, $\mathcal{SU}\mathbf{M}$ is a torsionfree class and so hereditary, a contradiction.

Thus for $\mathbf{M} = \{Z_2{}^0, 0\}$ where $Z_2{}^0$ is the zero ring with two elements, $\overline{\mathbf{M}}$ is a co-radical class which (in the universal class of all rings) is not a torsionfree class since it is known that in this case $\mathcal{SU}\mathbf{M}$ is not hereditary. (cf. Leavitt and Armendariz [9].)

The following theorem characterizes the maximal hereditary subclass $\overline{\mathbf{M}}$ of $\mathcal{SU}\mathbf{M}$ and gives another characterization of a torsion theory. For this purpose we shall need condition

(v) *Let* $A = I_1 \supset \cdots \supset I_\alpha \supset \cdots$ *be a transfinite chain of subrings of a ring* A *such that* $I_{\alpha+1} \lhd I_\alpha$ *and* $I_\alpha/I_{\alpha+1} \in M$ *for all ordinals* α, *and* I_β *is the largest ideal of* A *contained in* $\bigcap_{\alpha<\beta} I_\alpha$ *when* β *is a limit ordinal. If* $\bigcap_\gamma I_\gamma = 0$ *then* $A \in \mathbf{M}$.

THEOREM 3. *If a class* $\mathbf{M}$ *is hereditary and has property* (v), *then* $\mathbf{M}$ *is the largest hereditary subclass* $\overline{\mathbf{M}}$ *in* $\mathcal{SU}\mathbf{M}$.

PROOF. By the hereditariness of $\mathbf{M}$ we have $\mathbf{M} \subseteq \mathcal{SU}\mathbf{M}$. Take an arbitrary ring $A \in \overline{\mathbf{M}}$ For induction assume that for all $\alpha < \beta$ distinct subrings I_α have been defined with the desired properties. If β is a limit ordinal then define

$$I_\beta = \Sigma(I_\lambda \mid I_\lambda \lhd A \text{ and } I_\lambda \subseteq \textstyle\bigcap_{\alpha<\beta} I_\alpha).$$

Otherwise since $A \in \overline{\mathbf{M}}$ it is easy to check that $I_\alpha \in \mathcal{SU}\mathbf{M}$ so there exists an $I_{\alpha+1} \lhd I_\alpha$ such that $0 \neq I_\alpha/I_{\alpha+1} \in \mathbf{M}$. Note that we could have $I_\beta = 0$ when β is a limit ordinal, or if $\beta = \alpha + 1$ we could have $I_\alpha = 0$. If so we simply let $I_\gamma = 0$ for all $\gamma \geqq \beta$.

Now the descending chain of distinct I_α must stabilize at some $I_\gamma = I_{\gamma+1}$ which implies $I_\gamma = 0$. We thus have $\bigcap_\gamma I_\gamma = 0$, so by (v) it follows that $A \in \mathbf{M}$.

Theorems 2 and 3 imply

COROLLARY 6. *A class* $\mathbf{F}$ *is a torsionfree class if and only if* $\mathbf{F}$ *has properties* (i), (iv) *and* (v).

PROOF. The sufficiency is clear from Theorem 3 and Theorem 2, and for necessity it is enough to prove that a torsionfree class has property (v). Thus let $\mathbf{M}$ be a torsionfree class and $A = I_1 \supset \cdots$ the ring of property (v). If $A \notin \mathbf{M}$ then A would have $0 \neq I \lhd A$ with $I \in \mathscr{U}\mathbf{M}$. Then $I \subseteq I_1$ and since $\bigcap_\gamma I_\gamma = 0$ there would be some ordinal γ such that $I \nsubseteq I_\gamma$. Let β be the least ordinal such that $I \nsubseteq I_\beta$, so that $I \subseteq I_\alpha$ for all $\alpha < \beta$. Clearly β cannot be a limit ordinal so $\beta = \alpha + 1$ for some α. But then I would have a non-zero image under $I_\alpha \to I_\alpha/I_{\alpha+1}$, so $I_\alpha/I_{\alpha+1}$ would have a non-zero ideal in $\mathscr{U}\mathbf{M}$ contrary to the hereditariness of $\mathbf{M}$.

COROLLARY 7. *A class* $\mathbf{S}$ *of associative or alternative rings is a semi-simple class if and only if it has properties* (i) *and* (v).

4. **Smallest and largest constructions.** First we shall construct the smallest co-radical class containing a given hereditary class. For a class $\mathbf{X}$ define

$$\mathscr{F}\mathbf{X} = \{A \mid A \supset I_1 \supset \cdots \supset I_\alpha \supset \cdots \text{ where all } I_\alpha \lhd A,\ A/I_\alpha \in \mathbf{X},$$

and
$$\bigcap_\alpha I_\alpha = 0\},$$

$$\mathscr{G}\mathbf{X} = \{A \mid I,\ A/I \in \mathbf{X} \text{ for some ideal } I \lhd A\}.$$

Clearly $\mathbf{X}$ has (ii) if and only if $\mathscr{F}\mathbf{X} = \mathbf{X}$, and has (iii) if and only if $\mathscr{G}\mathbf{X} = \mathbf{X}$.

LEMMA 2. *If* $\mathbf{M}$ *is hereditary, so is* $\mathscr{F}\mathbf{M}$.

PROOF. Let $A \supset I_1 \supset \cdots \supset I_\alpha \supset \cdots$ be a descending chain of ideals of A such that all A/I_α are in $\mathbf{M}$ and $\bigcap_\alpha I_\alpha = 0$. If $J \lhd A$ then

$$J \supset (I_1 \cap J) \supset \cdots \supset (I_\alpha \cap J) \supset \cdots$$

where $\bigcap_\alpha (I_\alpha \cap J) \subseteq \bigcap_\alpha I_\alpha = 0$ and

$$J/(I_\alpha \cap J) \cong (J + I_\alpha)/I_\alpha \lhd A/I_\alpha \in \mathbf{M}.$$

By the hereditariness of $\mathbf{M}$ each $J/(I_\alpha \cap J)$ is in $\mathbf{M}$ and so $J \in \mathscr{F}\mathbf{M}$.

LEMMA 3. *If* $\mathbf{M}$ *is hereditary, so is* $\mathscr{G}\mathbf{M}$.

PROOF. Let I, A/I be in $\mathbf{M}$ and $J \lhd A$. The hereditariness of $\mathbf{M}$ implies $J \cap I \in \mathbf{M}$. Further, by $J/(J \cap I) \cong (J + I)/I \lhd A/I \in \mathbf{M}$ we have $J/(J \cap I) \in \mathbf{M}$. Hence, $J \in \mathscr{G}\mathbf{M}$.

Note that $M \subseteq \mathscr{F}\mathbf{M}$ and $M \subseteq \mathscr{G}\mathbf{M}$ for any class $\mathbf{M}$, and that $\mathbf{M} \subseteq \mathbf{N}$ implies $\mathscr{F}\mathbf{M} \subseteq \mathscr{F}\mathbf{N}$ and $\mathscr{G}\mathbf{M} \subseteq \mathscr{G}\mathbf{N}$.

We also have

LEMMA 4. *If* $\{\mathbf{M}_\gamma\}$ *is any ascending chain of classes defined for all ordinals* γ *then* $\mathscr{F} \cup \mathbf{M}_\gamma = \cup \mathscr{F}\mathbf{M}_\gamma$ *and* $\mathscr{G} \cup \mathbf{M}_\gamma = \cup \mathscr{G}\mathbf{M}_\gamma$.

PROOF. From $\mathbf{M}_\gamma \subseteq \cup \mathbf{M}_\gamma$ it is clear that $\cup \mathscr{F}\mathbf{M}_\gamma \subseteq \mathscr{F} \cup \mathbf{M}_\gamma$ so to establish the reverse inequality let $A \in \mathscr{F} \cup \mathbf{M}_\gamma$. Then $A \supset I_1 \supset \cdots \supset I_\alpha \supset \cdots$ where $A/I_\alpha \in \cup \mathbf{M}_\gamma$ for all I_α. Thus for some ordinal $\lambda(\alpha)$ we have $A/I_\alpha \in \mathbf{M}_{\lambda(\alpha)}$. Sine $\{I_\alpha\}$ is a set there must be some ordinal $\mu > \lambda(\alpha)$ for all α and so all $A/I_\alpha \in \mathbf{M}_\mu$. Thus $A \in \mathscr{F}\mathbf{M}_\mu \subseteq \cup \mathscr{F}\mathbf{M}_\gamma$. The proof that $\mathscr{G} \cup \mathbf{M}_\gamma = \cup \mathscr{G}\mathbf{M}_\gamma$ is similar.

For a class $\mathbf{M}$ define $\mathbf{M}_1 = \mathbf{M}$, $\mathbf{M}_{\alpha+1} = \mathscr{F}\mathscr{G}\mathbf{M}_\alpha$ and $\mathbf{M}_\beta = \cup_{\alpha<\beta} \mathbf{M}_\alpha$ if β is a limit ordinal, and consider the class $\mathbf{M}' = \cup M_\gamma$ where γ ranges over all ordinals.

THEOREM 4. *$\mathbf{M}'$ is the smallest class having properties* (ii) *and* (iii) *and containing $\mathbf{M}$. In particular, if $\mathbf{M}$ is hereditary, then $\mathbf{M}'$ is the smallest co-radical class containing $\mathbf{M}$. Even when $\mathbf{M}$ is hereditary, the class $\mathbf{M}'$ need not coincide with the largest hereditary class $\overline{\mathbf{M}}$ of $\mathscr{S}\mathscr{U}\mathbf{M}$.*

PROOF. From Lemma 4 we have $\mathbf{M}' \subseteq \mathscr{F}\mathbf{M}' \subseteq \mathscr{F}\mathscr{G}\mathbf{M}' = \in \mathbf{M}_{\gamma+1} \subseteq \mathbf{M}'$ and $\mathbf{M}' \subseteq \mathscr{G}\mathbf{M}' \subseteq \mathscr{F}\mathscr{G}\mathbf{M}' \subseteq \mathbf{M}'$. Thus $\mathbf{M}' = \mathscr{F}\mathbf{M}'$ and $\mathbf{M}' = \mathscr{G}\mathbf{M}'$, so $\mathbf{M}'$ has properties (ii) and (iii). If $\mathbf{M}$ is hereditary, then Lemmas 2 and 3 imply that $\mathbf{M}'$ is also hereditary and so is a co-radical class. Let $\mathbf{N}$ be a class such that $\mathbf{M} \subseteq \mathbf{N}$, $\mathscr{G}\mathbf{N} = \mathbf{N}$ and $\mathscr{F}\mathbf{N} = \mathbf{N}$. Since $\mathscr{F}\mathscr{G}\mathbf{M}_\alpha \subseteq \mathscr{F}\mathscr{G}\mathbf{N} = \mathbf{N}$ for every non-limit ordinal, it follows that $\mathbf{M}' \subseteq \mathbf{N}$, that is $\mathbf{M}'$ is the smallest class having properties (ii) and (iii) and containing $\mathbf{M}$.

Finally we shall give a hereditary class $\mathbf{M}$ such that $\mathbf{M}' \neq \overline{\mathbf{M}}$. Let $\mathbf{M} = \{Z_2^0, 0\}$. Let A be the ring generated over the two-element finite field Z_2 by the symbols $x_1, x_2, \cdots$ where $x_1^2 = x_2^2 = \cdots = 0$ and $x_ix_j = x_jx_i = x_{i+1}$ for all $i < j$. Note that A is isomorphic to the ring I_i generated over Z_2 by $x_i, x_{i+1}, \cdots$ for any i. We claim that I_2 is the only non-trivial ideal of A. Let $I \lhd A$ and notice that if $x_2 \in I$ then $x_2x_3 = x_3 \in I$ and so on, that is, $I_2 \subseteq I$. First suppose $y \in I$, $y \notin I_2$ so $y = x_1 + x_{i_1} + \cdots + x_{i_n}$. It is easy to check that in all cases $x_2 = (y - yx_2)x_2$ so $x_2 \in I$; then $x_1 \in I$ and therefore $I = A$. Thus if $I \neq A$ we have $I \subseteq I_2$, and if $0 \neq y \in I$ then $y = x_{i_1} + \cdots + x_{i_n}$ where $2 \leqq i_1 < \cdots < i_n$. Then $x_2 = yx_1$ if n is odd or $x_2 = (yx_{i_1})x_1$ if n is even, so in either case $x_2 \in I$. Therefore, $I = I_2$ and the only image of A is $A/I_2 \cong Z_2^0$. By the above remark on $A \cong I_i$ we have $A = I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ where each I_{i+1} is the sole ideal in I_i and each $I_i/I_{i+1} \cong Z_2^0$. Hence, it follows that the hereditary closure of A is in $\mathscr{S}\mathscr{U}\mathbf{M}$ so $A \in \overline{\mathbf{M}}$. But $A \notin \mathbf{M} = \mathbf{M}_1$ and suppose $A \notin \mathbf{M}_\alpha$ for all $\alpha < \beta$ then certainly when β is a limit ordinal $A \in \mathbf{M}_\beta$. Also if $A \in \mathscr{G}\mathbf{M}_\alpha$ then we would have I_2, $A/I_2 \in \mathbf{M}_\alpha$ contradicting $I_2 \cong A \notin \mathbf{M}_\alpha$. Thus $A \notin \mathscr{G}\mathbf{M}_\alpha$, and since the sole descending chain of ideals in A is $A \supset I_2 \supset 0$ and $I_2 \cong A/0 \cong A \notin \mathscr{G}\mathbf{M}_\alpha$, so $A \notin \mathscr{F}\mathscr{G}\mathbf{M}_\alpha$.

Thus $A \notin \mathbf{M}'$ follows and so in this case $\mathbf{M}'$ is properly contained in $\overline{\mathbf{M}}$.

A variation of the above example shows that the sufficient condition of Theorem 3 is not necessary; that is, for the class $\mathbf{M} = \{Z_2^0, 0\}$, $\overline{\mathbf{M}}$ does not satisfy (v). This will follow when we show the existence of a ring B satisfying the hypothesis of (v) relative to $\overline{\mathbf{M}}$, with $B \notin \overline{\mathbf{M}}$ Let B be generated over Z_2 by $x_1, \cdots, x_n, \cdots, x_\omega$ with relations $x_i^2 = 0$ for all $i < \omega$; $x_1 x_\omega = x_\omega x_1 = x_2$; $x_i x_\omega = x_\omega x_i = x_\omega$ for all $2 \leqq i \leqq \omega$ and $x_i x_j = x_j x_i = x_{i+1}$ for all $i < j < \omega$. Let I_i, $2 \leqq i < \omega$, be the ring generated by $x_i, x_{i+1}, \cdots, x_\omega$. We claim as before that I_2 is the only ideal of B. This follows as before from the fact that if $y = x_1 + x_{i_1} + \cdots + x_{i_n}$ then $x_2 = (y - yx_2)x_2$ and if $y = x_{i_1} + \cdots + x_{i_n}$ (with $2 \leqq i_1 < \cdots < i_n \leqq {}_\omega$), then $x_2 = yx_1$ when n is odd or $x_2 = (yx_{i_1})x_1$ when n is even. We have $B = I_1 \supset I_2 \supset \cdots$ where each $I_n/I_{n+1} \cong Z_2^0 \in \overline{\mathbf{M}}$. Also $\cap I_n = \{0, x_\omega\}$ is not an ideal of B so $I_\omega = 0$. Thus, B satisfies the hypothesis of (v) relative to $\overline{\mathbf{M}}$, but since $\{0, x_\omega\} \in \mathscr{U}\mathbf{M}$ is an ideal of I_2 it follows that $B \notin \overline{\mathbf{M}}$.

The smallest torsionfree (i.e., hereditary semisimple class) containing a given class, was given in Leavitt [7] Theorem 2. Let $\mathscr{I}\mathbf{X}$ denote the hereditary closure of any class $\mathbf{X}$.

Proposition 1. *Every class* $\mathbf{M}$ *is contained in a smallest torsionfree class* $\mathbf{F}$. $\mathbf{F}$ *can be constructed as follows: Define* $\mathbf{M}_1 = \mathscr{I}\mathbf{M}$, $\mathbf{M}_{\alpha+1} = \mathscr{I}\mathscr{S}\mathscr{U}\mathbf{M}_\alpha$ *and* $\mathbf{M}_\beta = \cup_{\alpha<\beta} \mathbf{M}_\alpha$ *if* β *is a limit ordinal. Then* $\mathbf{F} = \cup \mathbf{M}_\gamma$ *where* γ *runs over all ordinals.*

In view of Theorem 1 it is obvious that $\mathbf{T} = \mathscr{U}\mathbf{F} = \mathscr{U}(\cup \mathbf{M}_\gamma)$ *is the largest torsion class such that* $\mathbf{T} \cap \mathbf{F} = 0$.

In Leavitt and Watters [**10**, p. 103] an example was given showing that in general there does not exist a smallest torsion class containing a given class. However, it was shown that every class $\mathbf{M}$ is contained in a smallest radical class $\mathbf{P}$ with the somewhat stronger property $\mathbf{P}(I) \lhd A$ for every $I \lhd A$. It is also true that every class M is contained in a smallest hereditary torsion class (called strongly hereditary radical in [**10**]).

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