ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Number 4, Fall 1978

AN APPROACH TO COMPLETELY ERGODIC TRANSFORMATIONS AND A STACKING METHOD

RICHARD J. TUREK

1. **Introduction.** A stacking method was introduced by Christiansen in [3]. The purpose of this paper is to generalize the application of this stacking procedure to a wider class of examples, and to develop a necessary and sufficient condition for the ensuing transformation to be completely ergodic. A very useful corollary for sufficiency is developed, from which several of the theorems in [3] can easily be strengthened and proven. The techniques used here are also quite different from those used by Christiansen.

2. **Preliminaries.** Let $(X, \mathscr{A}, m)$ denote the unit interval $X = [0, 1)$ with Lebesgue measure m and let T be an invertible measure preserving map from X onto X. T is *ergodic* if $m(A) > 0$ and $TA \subset A$ implies $m(A) = 1$, $A \in \mathscr{A}$. T is *completely ergodic* if T^k is ergodic for each positive integer k. Let U_T be the induced unitary operator on $L^2(X)$ defined by $U_Tf(x) = f(T(x))$. A well-known fact pertaining to completely ergodic transformations is that if T is ergodic but not completely ergodic, then there exists an eigenvalue λ of U_T such that $\lambda \neq 1$ and $\lambda^k = 1$ for some integer k.

Stacking methods, discussed by Friedman [4], were developed by Von Neumann and Kakutani as a source of examples of ergodic measure preserving transformations and were later modified by Chacon [2]. The stacking method discussed here is due to Christiansen [3] and is a generalization of Chacon's method. It will be clear from the construction that if T is produced by this procedure, then T is in the class of transformations studied by Baxter [1] and is therefore ergodic.

3. **Construction.** The following stacking method is described in [3], but it is included here for convenience and self-containment.

T is defined in stages, increasing the domain of definition at each stage, so that, in the limit, T is defined on X. At the n^{th} stage we have h_n intervals $I_n(1), I_n(2), \cdots, I_n(h_n)$ each of length w_n and a residual interval R_n. For $i = 1, 2, \cdots, h_n - 1$, T maps $I_n(i)$ to $I_n(i + 1)$ by translation. Thus we have $T(x)$ is the point directly above x. T is not yet defined on $I_n(h_n)$ or R_n. See Figure 1.

To create the $(n + 1)^{\text{th}}$ stack from the n^{th} stack, cut the n^{th} stack using vertical cuts into $k_n \geqq 2$ equal width columns. Then j_n intervals are

Received by the editors on October 20, 1976.

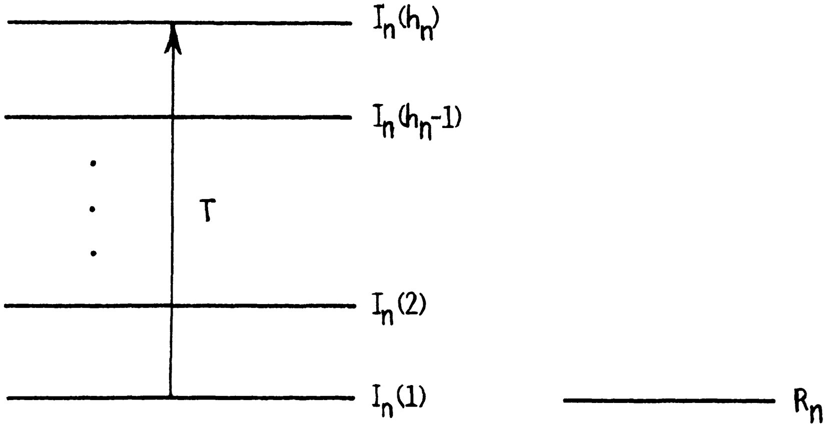

Figure 1

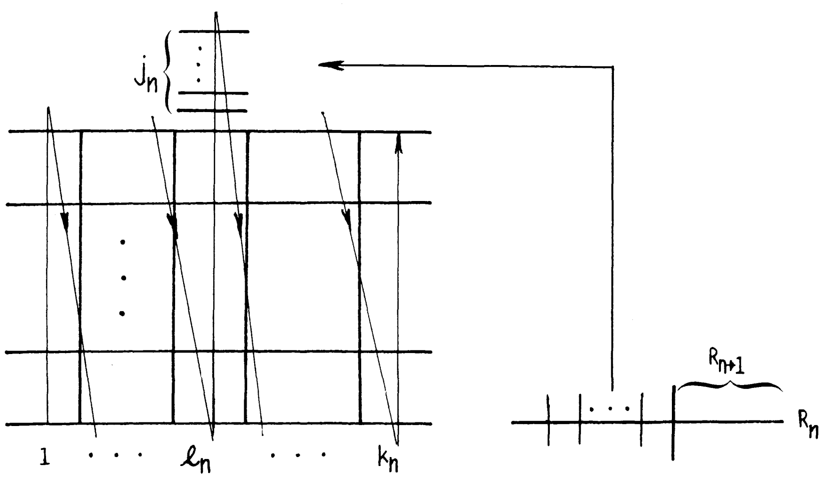

Figure 2

cut from R_n of an appropriate length as to maintain the measure preserving property of T, and are added to column ℓ_n. Now T is extended as shown in Figure 2. Then the $h_{n+1} = k_n h_n + j_n$ subintervals are restacked one above the other as indicated in Figure 2 to form the $(n + 1)^{\text{th}}$ stack with residual R_{n+1}. The only restriction on this technique used here is at least one subinterval of the top is mapped into the bottom at each stage.

In [3], a restriction on the size of k_n and j_n is also employed. Although implicit, this restriction is used in the proofs of theorems 3.2 and 4.2 in [3]. Indeed, with some ingenuity, one can produce interesting examples with $\lim_{n\to\infty} k_n = \infty$ and $\lim_{n\to\infty} j_n = \infty$. In this paper, no such restrictions are required, and thus the results are more general.

4. Main Results.

THEOREM. *Let T be constructed by the above stacking procedure. A necessary and sufficient condition for T to be completely ergodic is that if g_n is the greatest common divisor of $h_n, h_{n+1}, \cdots$ for $n = 1, 2, \cdots$, then $g_n = 1$ for all n.*

PROOF. To show necessity, assume there exists an N such that $g_N = (h_N, h_{N+1}, \cdots) > 1$. Let $\lambda = \exp(2\pi i/g_N)$. The proof of necessity will be complete if we show λ is an eigenvalue of U_T.

Define a sequence of approximate eigenfunctions f_m by $f_m(x) = 0$ if $m < N$ and for $m \geqq N$, define $f_m(x) = \lambda^{i-1}$ for $x \in I_m(i)$, $1 \leqq i \leqq h_m$ and $f_m(x) = 0$ if $x \in R_m$. For integers m and n with $N \leqq n \leqq m$ and $x \in \bigcup_{i=1}^{h_n} I_n(i)$, it happens that $f_m(x) = f_n(x)$. To see why this is so, we will show it is true for the case $m = n + 1$ and the statement would follow by induction. Let $x \in I_n(i)$ for some fixed i, $1 \leqq i \leqq h_n$. $I_n(i)$ splits into k_n intervals of the form

$$I_n(i) = \bigcup_{j=0}^{\ell_n - 1} I_{n+1}(i + jh_n) \cup \bigcup_{j=\ell_n}^{k_n - 1} I_{n+1}(i + jh_n + j_n)$$

where we will take the union on the right to be the empty set if $\ell_n = k_n$. Thus $x \in I_n(i)$ implies $x \in I_{n+1}(i + ah_n + bj_n)$ for some integers a and b with $0 \leqq a \leqq k_n - 1$ and $b = 0$ or $b = 1$ and we have:

$$f_{n+1}(x) = \lambda^{i+ah_n+bj_n-1} = \lambda^{i-1}(\lambda^{h_n})^a(\lambda^{j_n})^b.$$

By the choice of $g_N = (h_N, h_{N+1}, \cdots)$ and λ, $\lambda^{h_n} = 1$ for $n \geqq N$. Since $h_{n+1} = k_n h_n + j_n$, $1 = \lambda^{h_{n+1}} = (\lambda^{h_n})^{k_n} \lambda^{j_n} = \lambda^{j_n}$, and so $\lambda^{j_n} = 1$ for $n \geqq N$. Using this in the above equality, $f_{n+1}(x) = \lambda^{i-1} = f_n(x)$. Define f by $f(x) = \lim_{m\to\infty} f_m(x)$. The convergence of f_m to f is as follows. If $x \in \bigcup_{i=1}^{h_N} I_N(i)$, then $f_m(x) = f_N(x) = f(x)$ for all $m \geqq N$. If $x \in R_N$, let $r > N$ be the least positive integer such that $x \in \bigcup_{i=1}^{h_r} I_r(i)$. Then $f_m(x) = f_r(x) = f(x)$ for $m \geqq r$. Clearly $\lambda f(x) = f(T(x))$ for $x \in [0, 1)$ and λ is an eigenvalue.

To show the condition is sufficient, assume that T is not completely ergodic. Therefore, there exists an eigenvalue $\lambda \neq 1$ such that $\lambda^k = 1$ for some positive integer k, and we may assume k is the least such positive integer. Let f be an eigenfunction corresponding to the eigenvalue λ. Since f has constant absolute value, we choose f with range in the unit circle. Let $h_n = a_n k + r_n$ with $0 \leqq r_n < k$. According to Lusin's theorem, f is uniformly continuous on a set A of Lebesgue measure arbitrarily close to 1. Choose an arbitrary $\epsilon > 0$ and choose n large enough such that whenever x and y are in $A \cap I_n(i)$ for some i, $|f(x) - f(y)| < \epsilon$. Let

$$E_n = \{x | x \in I_n(i) \text{ and } T^{h_n}(x) \in I_n(i) \text{ for some } i = 1, 2, \cdots, h_n\}.$$

By the construction of T, $m(E_n) \geqq (1/3)m(S_n) \geqq (1/3)m(S_1)$ for all n and, when the measure of A is large enough, there exists an $x_0 \in A \cap E_n$ for which $T^{h_n}(x_0) \in A$. Since $x_0 \in E_n$, both x_0 and $T^{h_n}(x_0)$ are in the same $I_n(i)$. Therefore

$$|f(x_0) - f(T^{h_n}(x_0))| = |1 - \lambda^{h_n}| = |1 - \lambda^{a_n k + r_n}| = |1 - \lambda^{r_n}| < \epsilon.$$

Thus $\lim_{n\to\infty} \lambda^{r_n} = 1$.

Now assume there exists a subsequence $\{n_j\}$ such that $r_{n_j} > 0$ for all j. But for any exponent r_{n_j} such that $0 < r_{n_j} < k$, $|\lambda^{r_{n_j}} - 1| \geqq |e^{2\pi i/k} - 1| \neq 0$ which contradicts the convergence of this subsequence to 1. Therefore, no such subsequence exists. This implies $r_n = 0$ for all n sufficiently large, say $n \geqq N$. Hence $h_n = a_n k$ for all $n \geqq N$ and $g_N = (h_N, h_{N+1}, \cdots) \geqq k > 0$ and the theorem is proved.

The following corollary uses a slightly stronger condition than that in the theorem. From this corollary, theorems 3.3, 3.4 and 3.5 of [3] follow easily and can be strengthened.

COROLLARY. *If there exists a subsequence $\{h_{n_j}\}$ of heights such that $(h_{n_{2j-1}}, h_{n_{2j}}) = 1$ for all j, then T is completely ergodic.*

PROOF. Suppose $\{h_n\}$ has a subsequence $\{h_{n_j}\}$ as described above. For fixed n, if $n_{2j-1} > n$, then

$$g_n = (h_n, h_{n+1}, \cdots, h_{n_{j-1}}, \cdots, h_{n_j}, \cdots) = 1.$$

Hence $g_n = 1$ for all n and the conclusion follows from the theorem.

REFERENCES

1. J. R. Baxter, *A class of ergodic transformations having simple spectrum*, Proc. Amer. Math. Soc. **27** (1971), 275–279.

2. R. V. Chacon, *A geometric construction of measure preserving transformations*, Proc. Fifth Berkeley Sym. Math. Stat. Prob., 1967, **2**, part 2, 335–360.

3. S. L. M. Christiansen, *Complete ergodicity, weak mixing and stacking methods*, Rocky Mountain Journal of Math. **5** (1975), 79–86.

4. N. Friedman, *Introduction to ergodic theory*, Van Nostrand Reinhold Math. Studies, New York, 1970.

UNIVERSITY OF AKRON, AKRON, OHIO 44325