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ON HYPERBOLIC POLYNOMIALS WITH CONSTANT COEFFICIENTS

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ABSTRACT. We give several criteria of hyperbolicity for polynomials P with constant coefficients. Our conditions include those of Svensson [5] (a necessary and sufficient condition), A. Lax [3] (a necessary condition) and Hörmander [2] (a sufficient condition).

Our criteria seem to be practical to handle because they are based either on the comparison of homogeneous functions of the same degree, or on the behaviour of the different homogeneous parts of P at the roots of its principal part.

On the other hand, our proof is quite elementary: in particular, it does not rely on Puiseux series or Seidenberg's lemma.

1. **Introduction.** The symbols $j, k, \ell, m,$ with or without indices, always denote integers $\geqq 0$.

Let $P_k(k = 0, \cdots, m)$ be a polynomials in n variables, homogeneous of degree k and with constant coefficients.

The polynomial

$$P = \sum_{k=o}^{m} P_{m-k}$$

is said to be *hyperbolic* with respect to $N \in R^n \setminus \{0\}$ if $P_m(N) \neq 0$ and if there exists a constant c such that

$$\left.\begin{array}{l} P(ix + \tau N) = 0 \\ x \in R^n,\ \tau \in \mathbf{C} \end{array}\right\} \Rightarrow |\mathscr{R}\tau| \leqq c.$$

One also says in this case, that P is *c-hyperbolic* with respect to N.

For any polynomial P, we denote by $P^{(j)}$ the polynomial defined by

$$P^{(j)}(ix + \tau N) = D_\tau^{\,j}[P(ix + \tau N)].$$

Let us recall some properties of hyperbolic polynomials. We give the proofs for the reader's convenience.

PROPERTY 1. *If P is c-hyperbolic with respect to N, then P_m is 0-hyperbolic with respect to N.*

PROOF. We have, for all $t > 0$,

$$\left.\begin{array}{l} \dfrac{P(itx + \tau tN)}{t^m} = 0 \\ x \in R^n,\ \tau \in \mathbf{C} \end{array}\right\} \Rightarrow |\mathscr{R}\tau| \leqq \frac{c}{t}.$$

Letting $t \to \infty$, we get the result, by Hurwitz's theorem (see e.g., [**6**, p. 119]).

PROPERTY 2. *If P is c-hyperbolic with respect to N, then $P^{(j)}$ is also c-hyperbolic with respect to N for all $j \in [0, m]$.*

PROOF. It is sufficient to treat the case $j = 1$. *Since*

$$P^{(1)} = \sum_{k=1}^{n} N_k D_{x_k} P,$$

its principal part takes, at N, the value $mP_m(N) \neq 0$ by Euler's identity.

In addition, for any x_0 in R^n, we have

$$\frac{P^{(1)}(ix_0 + \tau N)}{P(ix_0 + \tau N)} = \sum_{k=1}^{p} \frac{\alpha_k}{\tau - \tau_k} \text{ for } |\mathscr{R}\tau| > c,$$

where $\tau_k (k = 1, \cdots, p)$ are the distinct roots of $P(ix_0 + \tau N)$ and α_k their multiplicity.

Therefore

$$\mathscr{R}\left[\frac{P^{(1)}(ix_0 + \tau N)}{P(ix_0 + \tau N)}\right] = \sum_{k=1}^{p} \frac{\alpha_k \mathscr{R}(\tau - \tau_k)}{|\tau - \tau_k|^2} \neq 0 \text{ for } |\mathscr{R}\tau| > c$$

and $P^{(1)}$ differs from 0 for $|\mathscr{R}\tau| > c$.

PROPERTY 3. *If P is c-hyperbolic with respect to N, there exists a constant K such that*

$$\left| \frac{P^{(j)}(ix + \tau N)}{P(ix + \tau N)} \right| \leqq \frac{K}{(|\mathscr{R}\tau| - c)^j}$$

for all $x \in R^n$, $|\mathscr{R}\tau| > c$, $j \in [0, m]$.

PROOF. It is sufficient to prove it for $j = 1$.

For x_0 fixed in R^n, we have, with the same notations as in the preceding proof,

$$\left| \frac{P^{(1)}(ix_0 + \tau N)}{P(ix_0 + \tau N)} \right| = \left| \sum_{k=1}^{p} \frac{\alpha_k}{\tau - \tau_k} \right|$$

$$\leqq \sum_{k=1}^{p} \frac{\alpha_k}{|\mathscr{R}(\tau - \tau_k)|}$$

$$\leqq \sum_{k=1}^{p} \frac{\alpha_k}{|\mathscr{R}\tau| - c}$$

for $|\mathscr{R}\tau| > c$.

2. **Criteria of Hyperbolicity.**

THEOREM. *If P_m is hyperbolic with respect to N, the following are equivalent*:

(a) $P = \sum_{k=0}^{m} P_{m-k}$ *is hyperbolic with respect to N*;

(b) *there exists a constant K such that*

$$|P_{m-k}| \leqq K \sup[|P_m^{(j)}|^{(j'-k)/(j'-j)}|P_m^{(j')}|^{(k-j)/(j'-j)}: 0 \leqq j \leqq k < j' \leqq m]$$

for all $x \in R^n$, $\tau \in \mathbf{C}$, $k \in [0, m[$;

(*From now on, when the argument is omitted, it has the form $ix + \tau N$, with $x \in R^n$ and $\tau \in \mathbf{C}$, unless otherwise stated. Moreover, the same symbol K is used for possibly different constants.*)

(c) *for $\left\{\begin{smallmatrix} any \\ some \end{smallmatrix}\right\}$ $M > 0$, there exists a constant K_M such that*

$$|P_{m-k}| \leqq K_M \sup\{|P_m^{(j)}|^{(j'-k)/(j'-j)}|P_m^{(j')}|^{(k-j)/(j'-j)}: \\ 0 \leqq j \leqq k < j' \leqq m\}$$

for all $x \in R^n$, $|\mathscr{R}\tau| = M$, $k \in [0, m[$;

(d) *there exists a constant K_0 such that*

$$|P_{m-k}| \leqq K_0 \sup\{|P_m^{(j)}|^{(j'-k)/(j'-j)}|P_m^{(j')}|^{(k-j)/(j'-j)}: \\ 0 \leqq j \leqq k < j' \leqq m\}$$

for all $x \in R^n$, $\mathscr{R}\tau = 0$, $k \in [0, m[$.

(e) *there exists a constant K such that*

$$|P_{m-k}| \leqq K \sup\{|P_m^{(j)}|^{(j'-k)/(j'-j)}|P_m^{(j')}|^{(k-j)/(j'-j)}: \\ k - 1 \leqq j \leqq k < j' \leqq m\}$$

for all $x \in R^n$, $\mathscr{R}\tau = 0$, $k \in]0, m[$;

(f) *there exists a constant K such that*

$$\left.\begin{matrix} P_m^{(k+\ell-1)}(ix_0 + \tau_0 N) = 0 \\ x_0 \in R^n, \tau_0 \in \mathbf{C} \end{matrix}\right\} \Rightarrow$$

$$|P_{m-k}^{(\ell)}(ix_0 + \tau_0 N)| \leqq K|P_m^{(k+\ell)}(ix_0 + \tau_0 N)|$$

for all $k \in]0, m[$ and $\ell \in [0, m - k[$;

(g) *there exists a constant K such that*

$$x_0 \in R^n, \tau_0 \in \mathbf{C}, P_m^{(k-1)}(ix_0 + \tau_0 N) \\ = \cdots = P_m^{(k+\ell-1)}(ix_0 + \tau_0 N) = 0 \Rightarrow \\ |P_{m-k}^{(\ell)}(ix_0 + \tau_0 N)| \leqq K|P_m^{(k+1)}(ix_0 + \tau_0 N)|$$

for all $k \in]0, m[$ *and* $\ell \in [0, m-k[$;

(h) *for* $k \in]0, m[$, *there exists a constant* K *such that, for any* $x_0 \in R^n$ *and* $\tau_0 \in \mathbf{C}$ *such that* $P_m^{(k-1)}(ix_0 + \tau_0 N) = 0$, *there exists a neighbourhood* V *of* τ_0 *in* $\{\tau: \mathscr{R}\tau = 0\}$ *such that*

$$|P_{m-k}(ix_0 + \tau N)| \leqq K|P_m^{(k)}(ix_0 + \tau N)|, \quad \textit{for all } \tau \in V;$$

(i) (*Svennson's first condition*) *for* $\left\{\begin{smallmatrix} any \\ some \end{smallmatrix}\right\}$ $M > 0$ *there exists a constant* K_M *such that*

$$|P_{m-k}| \leqq K_M |P_m|$$

for all $x \in R^n$, $|\mathscr{R}\tau| \geqq M$, $k \in [0, m]$;

(j) (*Svensson's second condition*) *for* $\left\{\begin{smallmatrix} any \\ some \end{smallmatrix}\right\}$ $M > 0$ *there exists a constant* K_M *such that*

$$|P_{m-k}| \leqq K_M |P_m^{(k-1)}|$$

for all $x \in R^n$, $|\mathscr{R}\tau| \geqq M$, $k \in]0, m]$.

Before giving the proof, let us make some observations about this statement.

Conditions (b), (c), (d) and (e) are very similar: note that (b) is valid for any τ, (c) for $|\mathscr{R}\tau| = M > 0$, (d) for $\mathscr{R}\tau = 0$; (e) is also valid for $\mathscr{R}\tau = 0$ but many terms in the right member of (d) have disappeared: the index j takes only 2 values, k and $k - 1$.

Condition (b) (or even (c)) gives immediately Svensson's first condition (i). To obtain Svensson's second condition (j), which is more precise, we need condition (g), which comes from (d), (e) and (f). This is the first reason why we have distinguished the two cases $|\mathscr{R}\tau| = M > 0$ and $\mathscr{R}\tau = 0$. This last case is also useful in obtaining A. Lax's condition (see section 3).

Let us also note that the criteria (b), (d), (e), (f) and (g) relate functions which are homogeneous of the same degree in a cone. So, as necessary conditions, they are, in a certain sense, more precise than Svensson's conditions, (where the right members have a degree of homogeneity strictly greater than the left members.)

Furthermore, they also have some advantages as sufficient conditions, because of the following simple remark: if f and g are homegeneous of the same degree in a cone Γ, the following are equivalent:

(α) $\exists K > 0$: $|f| \leqq K|g|$ in Γ,

(β) $x_m \in \Gamma$, $g(x_m) \to 0 \Rightarrow f(x_m) \to 0$,

(γ) $x_m \in \Gamma$, $|f(x_m)| = 1 \Rightarrow g(x_m) \not\to 0$,

(δ) $x_m \in \Gamma$, $g(x_m) \to 0 \Rightarrow f(x_m) \not\to \infty$.

The proof is quite elementary: $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$ and (δ). Conversely, $(\gamma) \Rightarrow (\alpha)$ and $(\delta) \Rightarrow (\alpha)$ are easily obtained by contradiction.

PROOF. Let us now give the proof of our theorem. It will be divided in several steps.

I. (a) $\Rightarrow$ (b). To clarify the idea of the proof, we shall first give it for a polynomial of degree 2.

Let $P = P_2 + P_1 + P_0$ be c-hyperbolic with respect to N.

For any $\alpha > 0$, $t > 0$, $x \in R^n$ and $\tau \in \mathbf{C}$, we have

$$\begin{aligned} \alpha^2 P\left[\frac{itx + (z + t\tau)N}{\alpha}\right] &\equiv t^2 P_2(ix + \tau N) + tP_2^{(1)}(ix + \tau N)z \\ &\quad + \frac{P_2^{(2)}}{2} z^2 + \alpha t P_1(ix + \tau N) + \alpha P_1^{(1)} z \\ &\quad + \alpha^2 P_0 = 0 \Rightarrow \left|\mathscr{R}\left(\frac{z + t\tau}{\alpha}\right)\right| \leqq c \\ &\Rightarrow |\mathscr{R}z| \leqq c\alpha + t\,|\mathscr{R}\tau|. \end{aligned} \tag{1}$$

From this, it is impossible to find sequences $\alpha_p > 0$, $t_p > 0$, $x_p \in R^n$ and $\tau_p \in \mathbf{C}$ such that $c\alpha_p + t_p|\mathscr{R}\tau_p| \to 0$, $|\alpha_p t_p P_1(ix_p + \tau_p N)| = 1$ and all the other coefficients in (1) tend to 0 (except $P_2^{(2)}/2$, which is a constant $\neq 0$) i.e., such that

$$\begin{cases} \alpha_p \to 0,\ t_p\mathscr{R}\tau_p \to 0,\ |\alpha_p t_p P_1(ix_p + \tau_p N)| = 1, \\ t_p^2 P_2(ix_p + \tau_p N) \to 0,\ t_p P_2^{(1)}(ix_p + \tau_p N) \to 0. \end{cases}$$

Indeed, if it were possible, taking subsequences, we could also suppose that $\alpha_p t_p P_1(ix_p + \tau_p N)$ converges to a number r with $|r| = 1$. Passing then to the limit as $p \to \infty$ and using Hurwitz's theorem, we should obtain

$$\frac{P_2^{(2)}}{2} z^2 + r = o \Rightarrow \mathscr{R}z = 0$$

which implies $r/P_2^{(2)} \geqq 0$. But, choosing now the sequences α_p, t_p, $-x_p$ and $-\tau_p$, we should also obtain $-r/P_2^{(2)} \geqq 0$ (because P_1 is homogeneous of degree 1), and so $r = 0$, an absurdity.

Furthermore, note that $t_p|\mathscr{R}\tau_p| \to 0$ is a consequence of $t_p^2 P_2(ix_p + \tau_p N) \to 0$, because

$$\frac{|P_2^{(2)}|}{|P_2|} \leqq \frac{K}{|\mathscr{R}\tau|^2} \quad \text{for } \mathscr{R}\tau \neq 0$$

$$\Rightarrow t^2|\mathscr{R}\tau|^2 \leqq \frac{K}{|P_2^{(2)}|} t^2 |P_2| \equiv K' t^2 |P_2|.$$

Therefore, eliminating α_p, we find that it is impossible to find sequences $t_p > 0$, $x_p \in R^n$, $\tau_p \in \mathbf{C}$ such that

$$P_1(ix_p + \tau_p N) \neq 0$$

$$\frac{1}{t_p} \frac{1}{|P_1(ix_p + \tau_p N)|} \to 0,$$

$$t_p \sup\{|P_2(ix_p + \tau_p N)|^{1/2}, |P_2^{(1)}(ix_p + \tau_p N)|\} \to 0.$$

But given two sequences $a_p \geqq 0$, $b_p \geqq 0$, the following are equivalent:

(α) $a_p b_p \to 0$,

(β) $\exists\, t_p > 0$: $t_p a_p \to 0$ and $b_p / t_p \to 0$.

Indeed, (β) obviously implies (α). Conversely, if (α) is true, we can take, for instance,

$$t_p = \begin{cases} \left(\dfrac{b_p}{a_p}\right)^{1/2} & \text{if } a_p \neq 0,\ b_p \neq 0, \\ p b_p & \text{if } a_p = 0,\ b_p \neq 0, \\ \dfrac{1}{p a_p} & \text{if } a_p \neq 0,\ b_p = 0, \\ 1 & \text{if } a_p = 0,\ b_p = 0, \end{cases}$$

to obtain (β).

From this remark, we see that there exist no sequences $x_p \in R^n$, $\tau_p \in \mathbf{C}$ such that

$$P_1(ix_p + t_p N) \neq 0,$$

$$\frac{1}{|P_1(ix_p + \tau_p N)|} \sup\{|P_2(ix_p + \tau_p N)|^{1/2}, |P_2^{(1)}(ix_p + \tau_p N)|\} \to 0.$$

In other words, there exists $\epsilon > 0$ such that

$$P_1 \neq 0 \Rightarrow \frac{1}{|P_1|} \sup\{|P_2|^{1/2}, |P_2^{(1)}|\} \geqq \epsilon$$

or, with $K = 1/\epsilon$,

$$|P_1| \leqq K \sup\{|P_2|^{1/2}, |P_2^{(1)}|\},$$

which is equivalent to condition (b) of the theorem for $k = 1$. Since (b) is obvious for $k = 0$, the proof is complete for polynomials of degree 2.

Let us now treat the general case.

We prove it by induction on m.

For $m = 1$, it is obvious because $P_1^{(1)} = P_1(N) \neq 0$.

From now on, we suppose that (a)$\Rightarrow$(b) for polynomials of degree $< m\,(m > 1)$ and we prove it for polynomials of degree m in several steps.

Let us fix k^* in $]0, m[$ and $j^* \in\,]k^*, m]$.

(Note that b) is obvious for $k = 0$).

(1) *It is impossible to find sequences* $\alpha_p > 0$, $t_p > 0$, $x_p \in R^n$ *and* $\tau_p \in \mathbf{C}$ *such that*

$$\begin{cases} |\alpha_p{}^{k^*} t_p{}^{m-k^*} P_{m-k^*}(ix_p + \tau_p N)| \\ \quad = |t_p{}^{m-j^*} P_m{}^{(j^*)}(ix_p + \tau_p N)| \neq 0, \\ \dfrac{\alpha_p{}^k t_p{}^{m-k-j} P^{(j)}_{m-k}(ix_p + \tau_p N)}{t_p{}^{m-j^*} P_m{}^{(j^*)}(ix_p + \tau_p N)} \to 0, & \textit{for all } k \in [0, m], \\ & j \in [0, m-k] \textit{ except} \\ & \textit{for } (k, j) = (0, j^*) \textit{ and} \\ \alpha_p \to 0. & (k^*, 0), \end{cases}$$

Note first that these conditions imply $t\mathscr{R}\tau_p \to 0$: take $j = k = 0$ and recall the existence of a constant K such that

$$\frac{|P_m{}^{(j^*)}|}{|P_m|} \leqq \frac{K}{|\mathscr{R}\tau|^{j^*}}$$

for all $x \in R^n$ and $\mathscr{R}\tau \neq 0$.

Since P is c-hyperbolic with respect to N, for any $t > 0$, $\alpha > 0$, $\tau \in \mathbf{C}$, we have

$$\alpha^m P\left[\frac{itx + (z + t\tau)N}{\alpha}\right]$$
$$\equiv \sum_{k=0}^{m} \alpha^k \sum_{j=0}^{m-k} t^{m-k-j} P^{(j)}_{m-k}(ix + \tau N)\frac{z^j}{j!} = 0$$
$$\Rightarrow |\mathscr{R}z| \leqq c\alpha + t|\mathscr{R}\tau| \quad (x \in R^n,\ z \in \mathbf{C}).$$

Replacing (α, t, x, τ) by $(\alpha_p, t_p, x_p, \tau_p)$ in the equation, dividing this equation by $t_p{}^{m-j^*} P_m{}^{(j^*)}(ix_p + \tau_p N)$, taking subsequences so that

$$\frac{\alpha_p{}^{k^*} t_p{}^{m-k^*} P_{m-k^*}(ix_p + \tau_p N)}{t_p{}^{m-j^*} P_m{}^{(j^*)}(ix_p + \tau_p N)}$$

converges to $r(|r| = 1)$ and letting $p \to +\infty$, we would obtain, by Hurwitz's theorem,

$$\frac{z^{j*}}{j^*!} + r = 0 \Rightarrow \mathscr{R}z = 0,$$

which is obviously impossible if $j^* > 2$. If $j^* = 2$ (thus $k^* = 1$), replacing (x_p, τ_p) by $(-x_p, -\tau_p)$ and keeping the same sequences t_p and α_p would simply change r into $-r$ and we would still obtain a contradiction:

$$|r| = 1 \text{ and } \frac{z^2}{2} \pm r = 0 \Rightarrow |\mathscr{R}z| = 0.$$

(2) *It is impossible to find sequences* $\alpha_p > 0$, $t_p > 0$, $x_p \in R^n$ *and* $\tau_p \in \mathrm{C}$ *such that*

$$\begin{cases} |\alpha_p^{k*}\, t_p^{m-k*}\, P_{m-k*}(ix_p + \tau_p N)| = |t_p^{m-j*}\, P_m^{(j*)}\,(ix_p + \tau_p N)| \neq 0, \\ \dfrac{\alpha_p^{k}\, t_p^{m-k}\, P_{m-k}\,(ix_p + \tau_p N)}{t_p^{m-j*}\, P_m^{(j*)}\,(ix_p + \tau_p N)} \longrightarrow 0, \text{ for } k \in\,]0, m[\,\setminus\{k^*\}, \\ \dfrac{t_p^{m-j}\, P_m^{(j)}\,(ix_p + \tau_p N)}{t_p^{m-j*}\, P_m^{(j*)}\,(ix_p + \tau_p N)} \longrightarrow 0, \text{ for } j \in [0, m]\setminus\{j^*\}, \\ \alpha_p \longrightarrow 0. \end{cases}$$

If it were possible, we should also have, for the same sequences,

$$\frac{\alpha_p^{k}\, t_p^{m-k-j}\, P_{m-k}^{(j)}\,(ix_p + \tau_p N)}{t_p^{m-j*}\, P_m^{(j*)}\,(ix_p + \tau_p N)} \longrightarrow 0, \text{ for all } k \in [0, m],\ j \in [0, m-k] \text{ except for } (k, j) = (0, j^*) \text{ and } (k^*, 0)$$

which is absurd, by (1).

We have only to verify this for $k \in [1, m[$, $j \in [1, m-k]$, for $k = 0$, $j = m$, and for $k = m$, $j = 0$. For $k \in [0, m[$ and $j = m - k$, $p_{m-\kappa}^{(j)}$ is a constant and

$$\frac{\alpha_p^{k}}{t_p^{m-k*}\, P_m^{(j*)}\,(ix_p + \tau_p N)} \longrightarrow 0$$

because $\alpha_p \longrightarrow 0$, and

$$\begin{cases} \dfrac{P_m^{(m)}\,(ix_p + \tau_p N)}{t_p^{m-j*}\, P_m^{(j*)}\,(ix_p + \tau_p N)} \longrightarrow 0, \text{ if } j^* \neq m, \\ t_p^{m-j*}\, P_m^{(j*)}(ix_p + \tau_p N) \text{ is a constant, if } j^* = m. \end{cases}$$

The same argument may be used for $k = m$ and $j = 0$. For $k \in [1, m[$ and $j \in [1, m-k[$, we shall apply the hypothesis of induction (a) $\Rightarrow$ (b) for polynomials of degree $< m$. Since $P^{(j)} \equiv \Sigma_{k=0}^{(j)}$

$P_{m-k}^{(j)}$ is hyperbolic with respect to N, there exists a constant K such that

$$|P_{m-k}^{(j)}| \leqq K \sup\{|P_m^{(\ell)}|^{\frac{\ell'-k-j}{\ell'-\ell}} |P_m^{(\ell')}|^{\frac{k+j-\ell}{\ell'-\ell}} : \\ j \leqq \ell \leqq k+j \leqq \ell' \leqq m\}$$

for all $x \in R^n$, $\tau \in \mathrm{C}$. Then, omitting here the arguments $ix_p + \tau_p N$, we have, for all p,

$$\left| \frac{\alpha_p{}^k t_p{}^{m-k-j} P_{m-k}^{(j)}}{t_p{}^{m-j^*} P_m{}^{(j^*)}} \right|$$

$$\leqq K \sup\left\{ \frac{\alpha_p{}^k t_p{}^{m-k-j} |P_m{}^{(\ell)}|^{\frac{\ell'-k-j}{\ell'-\ell}} |P_m{}^{(\ell')}|^{\frac{k+j-\ell}{\ell'-\ell}}}{t_p{}^{m-j^*} P_m{}^{(j^*)}} \right. \\ \left. : j \leqq \ell \leqq k+j < \ell' \leqq m \right\}$$

$$\leqq K \sup\left\{ \alpha_p{}^k \left[\frac{t_p{}^{m-\ell}|P_m{}^{(\ell)}|}{t_p{}^{m-j^*}|P_m{}^{(j^*)}|} \right]^{\frac{\ell'-k-j}{\ell'-\ell}} \left[\frac{t_p{}^{m-\ell'}|P_m{}^{(\ell')}|}{t_p{}^{m-j^*}|P_p{}^{(j^*)}|} \right]^{\frac{k+j-\ell}{\ell'-\ell}} \right. \\ \left. : j \leqq \ell \leqq k+j < \ell' \leqq m \right\}$$

and each term in the braces tends to 0 as p tends to infinity.

(3) *There exists a constant K such that, for any $x \in R^n$ and $\tau \in \mathrm{C}$, we have either*

$$|P_m{}^{(j^*)}| \leqq K \sup\{|P_m{}^{(j)}|^{\frac{j'-j^*}{j'-j}} |P_m{}^{(j')}|^{\frac{j^*-j}{j'-j}} : \\ 0 \leqq j < j^* < j' \leqq m\}$$

or

$$|P_{m-k^*}| \leqq k \sup\{|P_{m-k}|^{\frac{k'-k^*}{k'-k}} |P_{m-k'}|^{\frac{k^*-k}{k'-k}}, \\ \frac{|P_m{}^{(j)}|^{\frac{j^*(k'-k^*)}{k'(j^*-j)}} |P_{m-k'}|^{\frac{k^*}{k'}}}{|P_m{}^{(j^*)}|^{\frac{j(k'-k^*)}{k'(j^*-j)}}},$$

$$\frac{|P_{m-k}|^{\frac{k^*}{k}}\ |P_m^{(j')}|^{\frac{j^*(k^*-k)}{k(j'-j^*)}}}{|P_m^{(j^*)}|^{\frac{j'(k^*-k)}{k(j'-j^*)}}},$$

$$|P_{m-k}|^{\frac{j^*-k^*}{j^*-k}}\ |P_m^{(j^*)}|^{\frac{k^*-k}{j^*-k}},$$

$$\frac{|P_m^{(j)}|^{\frac{j^*-k^*}{j^*-j}}}{|P_m^{(j^*)}|^{\frac{j-k^*}{j^*-j}}} : 0 \leqq j < j^* < j' \leqq m,\ 0 < k < k^* < k' < m\}.$$

Here, the second inequality is valid everywhere for $j^* = m$. In this case, we shall conventionally say that the first inequality is "false" everywhere; in fact, it has no meaning! In the same way, any term which has no meaning has to be omitted: it is the case, for instance, of terms including P_{m-k} if $k^* = 1$.

At a point where $P_m^{(j^*)} = 0$ (resp. $P_{m-k^*} = 0$), the first (resp. second) inequality is valid for any K.

So, we may suppose $P_m^{(j^*)}$ and $P_{m-k^*} \neq 0$.

By (2), replacing α_p by its value and then eliminating t_p, we find that it is impossible to construct sequences $x_p \in R^n$ and $\tau_p \in \mathrm{C}$ such that the quotients of each expression in the braces by the first members, calculated at $ix_p + \tau_p N$, all tend to zero as $p \to \infty$. This is equivalent to assertion (3).

We shall now proceed to eliminate in (3) the terms containing P_{m-k} and $P_{m-k'}$.

For the reader's convenience, we shall first study the case of a polynomial of degree $3 : P = P_3 + P_2 + P_1 + P_0$.

In this case, the inequalities in (3) take the form:

(a) $|P_1| \leqq K \sup\{|P_2|^{1/2}, |P_3|^{1/3}, |P_3^{(1)}|^{1/2}, |P_3^{(2)}|\}$;

(b) $|P_2| \leqq K \sup\{|P_3|^{1/2}|P_1|^{1/2}, |P_3^{(1)}|^{3/4}|P_1|^{1/2}, |P_3^{(2)}|^{3/2}|P_1|^{1/2},$
$|P_3|^{2/3}, |P_3^{(1)}|, |P_3^{(2)}|^2\}$;

either

(c) $$|P_3^{(2)}| \leqq K \sup\{|P_3|^{1/3}, |P_3^{(1)}|^{1/2}\},$$

or

(d) $$|P_2| \leqq K \sup\left\{ |P_3|^{1/2}|P_1|^{1/2}, \frac{|P_3^{(1)}|\ |P_1|^{1/2}}{|P_3^{(2)}|^{1/2}}, |P_3|^{1/2}|P_3^{(2)}|^{1/2}, |P_3^{(1)}| \right\}.$$

To eliminate P_2 in the bound for P_1, we only need (a) and (b) (i.e., the inequalities obtained for $j^* = 3$). Indeed, if we replace $|P_2|$ by the right member of (b) in (a), we find

$$\begin{aligned}|P_1| &\leqq K \sup\{K^{1/2}|P_3|^{1/4}|P_1|^{1/4},\ K^{1/2}|P_3{}^{(1)}|^{3/8}|P_1|^{1/4},\\ &\qquad K^{1/2}|P_3{}^{(2)}|^{3/4}|P_1|^{1/4},\ K^{1/2}|P_3|^{1/3},\\ &\qquad K^{1/2}|P_3{}^{(1)}|^{1/2},\ K^{1/2}|P_3{}^{(2)}|,\ |P_3|^{1/3},\ |P_3{}^{(1)}|^{1/2},\ |P_3{}^{(2)}|\}\\ &\leqq K^{3/2} \sup\{|P_3|^{1/4}|P_1|^{1/4},\ |P_3{}^{(1)}|^{3/8}|P_1|^{1/4},\\ &\qquad |P_3{}^{(2)}|^{3/4}|P_1|^{1/4},\ |P_3|^{1/3},\ |P_3{}^{(1)}|^{1/2},\ |P_3{}^{(2)}|\}\end{aligned}$$

if K is greater than 1, which is no restriction.

But, if

$$|P_1| \leqq K^{3/2} \sup\{|P_3|^{1/4}|P_1|^{1/4},\ |P_3{}^{(1)}|^{3/8}|P_1|^{1/4},\ |P_3{}^{(2)}|^{3/4}|P_1|^{1/4}\}$$

and if $P_1 \neq 0$, dividing by $|P_1|^{1/4}$, we find

$$|P_1|^{3/4} \leqq K^{3/2} \sup\{|P_3|^{1/4},\ |P_3{}^{(1)}|^{3/8},\ |P_3{}^{(2)}|^{3/4}\}$$

or

$$\text{(a}'\text{)} \qquad |P_1| \leqq K^2 \sup\{|P_3|^{1/3},\ |P_3{}^{(1)}|^{1/2},\ |P_3{}^{(2)}|\}.$$

So, in fact, this last inequality is valid everywhere and we have eliminated P_2 from the right member of (a) (and this gives the desired inequality for $|P_1|$).

From (a′), one can see that the terms including P_1 in (b) and (d) may be eliminated if we change the constant K. Indeed, (a′) shows, for instance, that

$$\left.\begin{aligned} t_p{}^3 P_3(ix_p + \tau_p N) &\to 0\\ t_p{}^2 P_3{}^{(1)}(ix_p + \tau_p N) &\to 0\\ t_p P_3{}^{(2)}(ix_p + \tau_p N) &\to 0\\ \alpha_p &\to 0\end{aligned}\right\} \Rightarrow t_p P_1(ix_p + \tau_p N) \to 0 \Rightarrow \alpha_p t_p P_1(ix_p + \tau_p N) \to 0.$$

So, in (2), adapted to the particular case treated here, we can drop the condition about $P_1(\alpha_p t_p P_1(ix_p + \tau_p N) \to 0)$; we can therefore replace P_1 by 0 in the rest of the proof.

We thus obtain the existence of a constant K such that

$$\text{(b}'\text{)} \qquad |P_2| \leqq K \sup\{|P_3|^{2/3},\ |P_3{}^{(1)}|,\ |P_3{}^{(2)}|^2\}$$

and either

(c′) $$|P_3^{(2)}| \leqq K \sup\{|P_3|^{1/3}, |P_3^{(1)}|^{1/2}\}$$

or

(d′) $$|P_2| \leqq K \sup\{|P_3|^{1/2}|P_3^{(2)}|^{1/2}, |P_3^{(1)}|\}.$$

At a given point $ix + \tau N$, if

$$|P_3^{(2)}| > K \sup\{|P_3|^{1/3}, |P_3^{(1)}|^{1/2}\},$$

then, at this point

$$\begin{aligned} |P_2| &\leqq K \sup\{|P_3|^{1/2}|P_3^{(2)}|^{1/2}, |P_3^{(1)}|\} \\ &\leqq K \sup\{|P_3|^{1/2}|P_3^{(2)}|^{1/2}, |P_3^{(1)}|, |P_3|^{2/3}\}. \end{aligned}$$

If, on the contrary,

$$|P_3^{(2)}| \leqq K \sup\{|P_3|^{1/3}, |P_3^{(1)}|^{1/2}\},$$

then we have also

$$\begin{aligned} |P_2| &\leqq K \sup\{|P_3|^{2/3}, |P_3^{(1)}|, K^2|P_3|^{2/3}, K^2|P_3^{(1)}|\} \\ &\leqq K^3 \sup\{|P_3|^{2/3}, |P_3^{(1)}|\} \\ &\leqq K^3 \sup\{|P_3|^{2/3}, |P_3^{(1)}|, |P_3|^{1/2}|P_3^{(2)}|^{1/2}\} \end{aligned}$$

(if $K \geqq 1$). So, this last inequality is valid everywhere and this is the desired inequality for $|P_2|$.

Let us now come to the general case.

Let k_0^* be fixed in $]0, m[$.

For x and τ fixed in R^n and C respectively, denote by $j_0^* = j_0^*(x, \tau, k_0^*)$ the smallest integer $j^* > k_0^*$ for which the first inequality in (3), with K replaced by $K' \geqq K$, is false at $ix + \tau N$. Here, K' is a sufficiently large constant independent of x and τ; its value will be specified in the sequel.

(4) *There exists a constant* K_1 *such that*

$$|P_{m-k_0^*}| \leqq K_1 \sup \left\{ \frac{|P_m^{(j)}|^{\frac{j_0^*(k'-k_0^*)}{k'(j_0^*-j)}} \; |P_{m-k'}|^{\frac{k_0^*}{k'}}}{|P_m^{(j_0^*)}|^{\frac{j(k'-k_0^*)}{k'(j_0^*-j)}}}, \right.$$

$$\left. \frac{|P_m^{(j)}|^{\frac{j_0^*-k_0^*}{j_0^*-j}}}{|P_m^{(j_0^*)}|^{\frac{j-k_0^*}{j_0^*-j}}} : 0 \leqq j < j_0^*, \; k_0^* < k' < m \right\}.$$

For every $j^* \in]k_0^*, j_0^*[$ (if any), we have, by the definition of j_0^*,

$$|P_m^{(j^*)}| \leqq K' \sup \left\{ |P_m^{(j)}|^{\frac{j'-j^*}{j'-j}} \; |P_m^{(j')}|^{\frac{j^*-j}{j'-j}} : \right.$$
$$\left. 0 \leqq j < j^* < j' \leqq m \right\}$$

for all $x \in R^n$, $\tau \in \mathbf{C}$.

It can be shown by induction on j_0^* that, with another constant K'', we have

$$|P_m^{(j^*)}| \leqq K'' \sup \left\{ |P_m^{(j)}|^{\frac{j'-j^*}{j'-j}} \; |P_m^{(j')}|^{\frac{j^*-j}{j'-j}} : \right.$$
$$\left. 0 \leqq j \leqq k_0^*, \, j_0^* \leqq j' \leqq m \right\}$$

for all $x \in R^n$, $\tau \in \mathbf{C}$. (Start with $j_0^* = k_0 + 2$.)

In the same way, from (3), one can prove, by induction on k_0^* (start with $k_0^* = 1$) the following result: there exists a constant $K_1 \geqq K$ such that, at every point of the form $ix + \tau N$, we have either

$$|P_m^{(j^*)}| \leqq K_1 \sup\{ |P_m^{(j)}|^{\frac{j'-j^*}{j'-j}} \; |P_m^{(j)}|^{\frac{j^*-j}{j'-j}} :$$
$$0 \leqq j < j^* < j' \leqq m\}$$

or

$$|P_{m-k^*}| \leqq K_1 \sup \left\{ \frac{|P_m^{(j)}|^{\frac{j^*(k'-k^*)}{k'(j^*-j)}} \; |P_{m-k'}|^{\frac{k^*}{k'}}}{|P_m^{(j^*)}|^{\frac{j(k'-k^*)}{k'(j^*-j)}}} , \right.$$

$$\left. \frac{|P_m^{(j)}|^{\frac{j^*-k^*}{j^*-j}}}{|P_m^{(j^*)}|^{\frac{j-k^*}{j^*-j}}} : 0 \leqq j < j^*, \, k_0^* < k' < m \right\}$$

for all $k^* \in]0, k_0^*]$.

This obviously implies (4) if K' (in the definition of j_0^*) is taken equal to K_1.

5. *End of the proof of* (a) $\Rightarrow$ (b). Let us first examine the last term in (4).

For $0 \leqq j \leqq k_0^*$, we have

$$\frac{|P_m^{(j)}|^{\frac{j_0^*-k_0^*}{j_0^*-j}}}{|P_m^{(j_0^*)}|^{\frac{j-k_0^*}{j_0^*-j}}} = |P_m^{(j')}|^{\frac{j_0^*-k_0^*}{j_0^*-j}} \; |P_m^{(j_0^*)}|^{\frac{k_0^*-j}{j_0^*-j}}$$

$$\leqq \sup\{|P_m^{(j)}|^{\frac{j-k_0^*}{j'-j}} \; |P_m^{(j')}|^{\frac{k_0^*-j}{j'-j}} :$$

$$0 \leqq j \leqq k_0^* < j' \leqq m\}.$$

For $k_0^* < j < j_0^*$, we have, with the same K'' as in the proof of (4),

$$\frac{|P_m^{(j)}|^{\frac{j_0^*-k_0^*}{j_0^*-j}}}{|P_m^{(j_0^*)}|^{\frac{j-k_0^*}{j_0^*-j}}}$$

$$\leqq (K'')^{\frac{j_0^*-k_0^*}{j_0^*-j}} \sup\left\{ \frac{|P_m^{(\ell)}|^{\frac{(\ell'-j)(j_0^*-k_0^*)}{(\ell'-\ell)(j_0^*-j)}} \; |P_v^{(\ell')}|^{\frac{(j-\ell)(j_0^*-k_0^*)}{(\ell'-\ell)(j_0^*-j)}}}{|P_m^{(j_0^*)}|^{\frac{j-k_0^*}{j_0^*-j}}}, \right.$$

$$\left. 0 \leqq \ell \leqq k_0^*,\, j_0^* \leqq \ell' \leqq m \right\}.$$

But, for each ℓ, ℓ', of the indicated type with $\ell' > j_0^*$, we have with the same K' as before,

$$\frac{|P_m^{(\ell)}|^{\frac{(\ell'-j)(j_0^*-k_0^*)}{(\ell'-\ell)(j_0^*-j)}} \; |P_m^{(\ell')}|^{\frac{(j-\ell)(j_0^*-k_0^*)}{(\ell'-\ell)(j_0^*-j)}}}{|P_m^{(j_0^*)}|^{\frac{j-k_0^*}{j_0^*-j}}}$$

$$\leqq \frac{1}{(K')^{\frac{j-k_0^*}{j_0^*-j}}} \; |P_m^{(\ell)}|^{\frac{\ell'-k_0}{\ell'-\ell}} \; |P_m^{(\ell')}|^{\frac{k_0^*-\ell}{\ell'-\ell}}$$

$$\leqq \frac{1}{(K')^{\frac{j-k_0^*}{j_0^*-j}}} \sup\{|P_m^{(j)}|^{\frac{j'-k_0^*}{j'-j}} \; |P_m^{(j')}|^{\frac{k_0^*-j}{j'-j}} :$$

$$0 \leqq j \leqq k_0^* < j' \leqq m\}.$$

The same inequality is valid for $\ell' = j_0^*$ with K' replaced by 1.

It remains to show that the first term in the braces in (4) may be suppressed.

This is obvious for $k_0^* = m - 1$, which proves the desired inequality for $|P_1|$.

Suppose now that the desired inequality is proved for $|P_{m-k^*}|$

$(k^* > k_0^*)$. Then, in the second condition in (2), with (k^*, j^*) replaced by (k_0^*, j_0^*), we may keep only those k which are strictly less than k_0^* (the proof is the same as that of (2) itself). In other words, we may replace here $P_{m-k'}$ by 0 and the proof of (b) is complete.

II. (b) $\Rightarrow$ (c) and (d). Obvious: take $K_M = K$ for $M \geqq 0$.

III. (c) $\Rightarrow$ (i). For $k = m$, note that P_0 is a constant and

$$|P_m^{(m)}| \leqq \frac{K|P_m|}{|\mathscr{R}\tau|^m}$$

for some K and all $\mathscr{R}\tau \neq 0$.

For $k \in [0, m[$, the inequality in (i) is obviously true for $|\mathscr{R}\tau| = M$, with the same constant K_M as in (c).

To prove it for $|\mathscr{R}\tau| \geqq M$, observe that

$$\mathscr{R}\tau \neq 0 \Rightarrow \left| \frac{P_{m-k}(ix + \tau N)}{P_m(ix + \tau N)} \right|$$

$$= \frac{M^k}{|\mathscr{R}\tau|^k} \left| \frac{P_{m-k}\left(\dfrac{ixM}{|\mathscr{R}\tau|} + \dfrac{\tau M}{|\mathscr{R}\tau|} N \right)}{P_m\left(\dfrac{ixM}{|\mathscr{R}\tau|} + \dfrac{\tau M}{|\mathscr{R}\tau|} N \right)} \right| \leqq \frac{K_M M^k}{|\mathscr{R}\tau|^k}.$$

IV. (d) $\Rightarrow$ (e). In fact, we shall prove the following:

Lemma 1. *If P_m is a polynomial homogeneous of degree m and hyperbolic with respect to N, there exists a constant K such that*

$$|P_m^{(j)}|^{\frac{j'-k}{j'-j}} |P_m^{(j')}|^{\frac{k-j}{j'-j}} \leqq k \sup\{|P_m^{(\ell)}|^{\frac{\ell'-k}{\ell'-\ell}} |P_m^{(\ell')}|^{\frac{k-\ell}{\ell'-\ell}}$$
$$: k - 1 \leqq \ell \leqq k < \ell' \leqq m\}$$

for all $x \in R^n$, $\mathscr{R}\tau = 0$, $k \in]0, m[$, $j \in [0, k[$, $j' \in]k, m]$.

Proof. For $m = 2$, this is obvious. Let us suppose the result is true for polynomials of degree $< m$. Since $P_m^{(1)}$ is homogeneous of degree $m - 1$ and hyperbolic with respect to N, the inequality is valid for $k \in]1, m[$ and $j \in]0, k[$, $j' \in]k, m]$.

It remains to prove the existence of a constant K' such that

$$\text{(1)} \quad |P_m|^{\frac{j'-k}{j'}} |P_m^{(j')}|^{\frac{k}{j'}} \leqq K' \sup\{|P_m^{(\ell)}|^{\frac{\ell'-k}{\ell'-\ell}} |P_m^{(\ell')}|^{\frac{k-\ell}{\ell'-\ell}} :$$
$$k - 1 \leqq \ell \leqq k < \ell' \leqq m\}$$

for all $x \in R^n$, $\mathscr{R}\tau = 0$, $k \in]0, m[$, $j' \in]k, m]$.

For $k = 1$, this is obvious. Suppose $k > 1$.

Let us first note that, for any $k' \in]0, m]$, there exist constants $c_j > 0$ ($j \in [0, k']$) such that

$$\left.\begin{array}{c} \sum_{j=0}^{k'} c_j t^{k'-j} P_m^{(j)}(ix + \tau N) z^j = 0 \\ P_m(ix + \tau N) \neq 0 \\ x \in R^n, \mathscr{R}\tau = 0, \ t > 0 \end{array}\right\} \Rightarrow \mathscr{R}z = 0.$$

Indeed,

$$P_m[itx + (z + t\tau)N] \equiv \sum_{j=0}^{m} \frac{z^j}{j!} P_m^{(j)} (ix + \tau N) t^{m-j} = 0 \Rightarrow \mathscr{R}z = 0.$$

Replacing z by $1/z$ and multiplying the new equation by z^m, we get

$$\sum_{j=0}^{m} \frac{z^{m-j}}{j!} P_m^{(j)} (ix + \tau N)\, t^{m-j} = 0 \Rightarrow \mathscr{R}z = 0.$$

Differentiating $m - k'$ times with respect to z, we obtain

$$\left.\begin{array}{c} \sum_{j=0}^{k'} \frac{z^{k'-j}}{(k'-j)!} \frac{(m-j)!}{j!} P_m^{(j)} (ix + \tau N) t^{m-j} = 0 \\ P_m(ix + N) \neq 0 \end{array}\right\} \Rightarrow \mathscr{R}z = 0,$$

(the condition $P_m(ix + \tau N) \neq 0$ is sufficient to assert that the equation does not take the form $0 = 0$).

Replacing now z by $1/z$ and multiplying the new equation by $z^{k'}$, we have

$$\left.\begin{array}{c} \sum_{j=0}^{k'} \frac{z^j}{(k'-j)!} \frac{(m-j)!}{j!} P_m^{(j)} (ix + \tau N) t^{m-j} = 0 \\ P_m\, (ix + \tau N) \neq 0 \end{array}\right\} \Rightarrow \mathscr{R}z = 0$$

(the condition $P_m(ix + \tau N) \neq 0$ implies that 0 is not a solution of the equation). The result follows after division by $t^{m-k'}$.

We shall now use this result with $k' = k + 1$ to prove inequality (1) for $j' = k + 1$.

It is easy to verify as previously that it is impossible to find sequences $x_p \in R^n$, $\tau_p \in \mathbf{C}$ (with $\mathscr{R}\tau_p = 0$) and $t_p > 0$ such that

$$t_p^{k+1}|P_m(ix_p + \tau_p N)| = |P_m^{(k+1)}(ix_p + \tau_p N)| \neq 0,$$

$$\frac{t_p^{k+1-j} P_m^{(j)}(ix_p + \tau_p N)}{P_m^{(k+1)}(ix_p + \tau_p N)} \to 0, \text{ for all } j \in \,]0, k].$$

This implies the existence of a constant $K^* > 0$ such that

$$\sup\left\{ \frac{|P_m^{(j)}|}{|P_m^{(k+1)}|^{\frac{j}{k+1}} \quad |P_m|^{\frac{k+1-j}{k+1}}} \; : 0 < j < k+1 \right\} \geqq \frac{1}{K^*}$$

for all $x \in R^n$, $\mathscr{R}\tau = 0$ such that $P_m \neq 0$ and $P_m^{(k+1)} \neq 0$. In other words, for any $x \in R^n$, $\mathscr{R}\tau = 0$, there exists $j = j(x, \tau) \in \,]0, k+1[$ such that

$$|P_m|^{\frac{k+1-j}{k+1}} |P_m^{(k+1)}|^{\frac{j}{k+1}} \leqq K^* |P_m^{(j)}|.$$

Then,

$$\begin{aligned} &|P_m|^{\frac{1}{k+1}} \quad |P_m^{(k+1)}|^{\frac{k}{k+1}} \\ (2) \qquad &= \; |P_m|^{\frac{1}{k+1}} \, |P_m^{(k+1)}|^{\frac{j}{(k+1)(k+1-j)}} \; |P_m^{(k+1)}|^{\frac{k-j}{k+1-j}} \\ &\leqq (K^*)^{\frac{1}{k+1-j}} \; |P_m^{(j)}|^{\frac{1}{k+1-j}} \, |P_m^{(k+1)}|^{\frac{k-j}{k+1-j}} \end{aligned}$$

If $j \in \,]0, k[$, we get, from the hypothesis of induction,

$$\begin{aligned} &|P_m|^{\frac{1}{k+1}} \, |P_m^{(k+1)}|^{\frac{k}{k+1}} \leqq \\ &\leqq (K^*)^{\frac{1}{k+1-j}} K \sup \{ |P_m^{(\ell)}|^{\frac{\ell'-k}{\ell'-\ell}} \quad |P_m^{(\ell')}|^{\frac{k-\ell}{\ell'-\ell}} \\ &\qquad : k-1 \leqq \ell \leqq k < \ell' \leqq m \}. \end{aligned}$$

For $j = k$, (2) becomes

$$|P_m|^{\frac{1}{k+1}} \quad |P_m^{(k+1)}|^{\frac{k}{k+1}} \leqq K^* |P_m^{(k)}|$$

and the result is proved for $j' = k + 1$.

The general case follows in the same way by induction on j'.

V. $a \Rightarrow f$. We know that $a \Rightarrow b \Rightarrow d \Rightarrow e$. Applying this to $P^{(\ell)}$, we obtain that, for some constant K,

$$|P^{(\ell)}_{m-k}| \leqq \sup\{|P_m^{(j)}|^{\frac{j'-k-\ell}{j'-j}} \; |P_m^{(j')}|^{\frac{k+\ell-j}{j'-j}} \\ : k+\ell-1 \leqq j \leqq k+\ell < j' \leqq m\}$$

for all $x \in R^n$, $\mathscr{R}\tau = o$, $k \in]0, m[$ and $\ell \in [0, m-k[$. Property (f) follows immediately with the same constant K.

VI. $(f) \Rightarrow (g)$. Obvious with the same constant K.

VII. $(g) \Rightarrow (j)$. For $k = m$, note that P_0 is a constant (see III). We shall thus suppose $k \in]0, m[$.

For x fixed in R^n, let τ_j $(j = 1, \cdots, p)$ denote the distinct roots of $P_m^{(k-1)}(ix + \tau N)$ and α_j their multiplicity.

From (g), $\alpha_j > 1 \Rightarrow \tau_j$ is a root of $P_{m-k}(ix + \tau N)$ with multiplicity $\geqq \alpha_j - 1$.

So, $P_{m-k}/P_m^{(k-1)}$ has only simple poles and it may be written as

$$\frac{P_{m-k}}{P_m^{(k-1)}} = \sum_{j=1}^{q} \frac{A_j}{\tau - \tau_j} \qquad (q \leqq p),$$

where

$$|A_j| = \left| \lim_{\tau \to \tau_j} \frac{(\tau - \tau_j) P_{m-k}}{P_m^{(k-1)}} \right| = \left| \alpha_j \frac{P^{(\alpha_j - 1)}_{m-k}(ix + \tau_j N)}{P_m^{(k-1+\alpha_j)}(ix + \tau_j N)} \right| \leqq K\alpha_j$$

by Taylor's formula; the result follows because

$$\mathscr{R}\tau_j = 0 \Rightarrow \left| \frac{1}{\tau - \tau_j} \right| \leqq \frac{1}{|\mathscr{R}\tau|} \leqq \frac{1}{M} \text{ for } |\mathscr{R}\tau| \geqq M.$$

VIII. $(e) \Rightarrow (h)$. In fact, there exists a neighborhood V' of τ_0 in $\mathbf{C}$ such that

$$|P_m^{(k-1)}(ix_0 + \tau N)|^{\frac{j'-k}{j'-k+1}} \; |P_m^{(j')}(ix_0 + \tau N)|^{\frac{1}{j'-k+1}} \leqq |P_m^{(k)}(ix_0 + \tau N)|$$

for all $\tau \in V'$, $j' \in]k, m]$.

This follows immediately from the fact that

$$\lim_{\tau \to \tau_0} \frac{[P_m^{(k-1)}(ix_0 + \tau N)]^{j'-k} P_m^{(j')}(ix_0 + \tau N)}{[P_m^{(k)}(ix_0 + \tau N)]^{j'-k+1}}$$

exists and is < 1: it is indeed easy to verify that, if ℓ is the multiplicity of τ_0 as a root of $P_m^{(k-1)}(ix_0 + \tau N)$, this limit is 0 if $\ell < j' - k + 1$ and it is equal to

$$\frac{(\ell - j' + k) \cdots (\ell - 1)}{\ell^{j'-k}} < 1$$

if $\ell \geqq j' - k + 1$.

IX. $(h) \Longrightarrow (g)$. Let ℓ_0 be the smallest integer such that $P_m^{(k+\ell_0-1)}(ix_0 + \tau_0 N) \neq 0$, so $\ell_0 \in]\ell, m - k + 1]$. It follows from (h) that

$$P_{m-k}(ix_0 + \tau_0 N) = \cdots = P_{m-k}^{(\ell_0-2)}(ix_0 + \tau_0 N) = 0.$$

Indeed, there exists a polynomial Q such that

$$P_m^{(k-1)}(ix_0 + \tau N) \equiv (\tau - \tau_0)^{\ell_0} Q(\tau).$$

Then, for some polynomial Q_1,

$$P_m^{(k)}(ix_0 + \tau N) \equiv (\tau - \tau_0)^{\ell_0 - 1} Q_1(\tau).$$

But, by (h),

$$|P_{m-k}(ix_0 + \tau N)| \leqq K\, |(\tau - \tau_0)^{\ell_0-1} Q_1(\tau)| \text{ for all } \tau \in V,$$

so $P_{m-k}(ix_0 + \tau_0 N) = 0$.

Suppose $P_{m-k}(ix_0 + \tau N)$ takes the form

$$P_{m-k}(ix_0 + \tau N) \equiv (\tau - \tau_0)^{\alpha} P'(\tau) \ (\alpha < \ell_0 - 1),$$

where P' is a polynomial. We can write (h) in the form

$$|P'(\tau)| \leqq K\, |(\tau - \tau_0)^{\ell_0 - 1 - \alpha} Q_1(\tau)| \text{ for all } \tau \in V,$$

and τ_0 is a root of P'. Therefore, we can take $\alpha = \ell_0 - 1$ and the result follows.

Then, by Taylor's formula,

$$\lim_{\substack{\tau \to \tau_0 \\ \mathscr{R}\tau = 0}} \frac{P_{m-k}(ix_0 + \tau N)}{P_m^{(k)}(ix_0 + \tau N)} = \lim_{\substack{\tau \to \tau_0 \\ \mathscr{R}\tau = 0}} \frac{P_{m-k}^{(\ell_0-1)}(ix_0 + \tau N)}{P_m^{(k+\ell_0-1)}(ix_0 + \tau N)}$$

$$= \lim_{\substack{\tau \to \tau_0 \\ \mathscr{R}\tau = 0}} \frac{P_{m-k}^{(\ell')}(ix_0 + \tau N)}{P_m^{(k+\ell')}(ix_0 + \tau N)}$$

for all $\ell' \in [0, \ell_0[$.

But the absolute value of the first limit is $\leqq K$ (by (h)) and this implies (g) with the same constant K.

REMARK. Property (g) is not implied by

$$\exists K : P_m^{(k-1)}(ix_0 + \tau_0 N) = 0 \Rightarrow |P_{m-k}(ix_0 + \tau_0 N)| \leqq K|P_m^{(k)}(ix_0 + \tau_0 N)|.$$

Take, for instance, with $n = 2$, $N = (0, 1)$, $m = 3$, $k = 1$,

$$P_3(ix + \tau N) = (\tau + ix_2 - ix_1)^3$$
$$P_2(ix + \tau N) = (\tau + ix_2 - ix_1)(\tau + ix_2).$$

X. $(j) \Rightarrow (i)$. Obvious, if we recall the existence of a constant K such that

$$\left| \frac{P_m^{(k-1)}}{P_m} \right| \leqq \frac{K}{|\mathscr{R}\tau|^{k-1}} \text{ for } \mathscr{R}\tau \neq 0,\ k > 0.$$

XI. $(i) \Rightarrow (a)$. If (i) is satisfied for some $M > 0$, we have, for $\mathscr{R}\tau \neq 0$ and $k \in]0, m]$,

$$\left| \frac{P_{m-k}(ix + \tau N)}{P_m(ix + \tau N)} \right| = \frac{M^k}{|\mathscr{R}\tau|^k} \left| \frac{P_{m-k}\left(\frac{ixM}{|\mathscr{R}\tau|} + \frac{\tau M}{|\mathscr{R}\tau|} N \right)}{P_m\left(\frac{ixM}{|\mathscr{R}\tau|} + \frac{\tau M}{|\mathscr{R}\tau|} N \right)} \right|$$
$$\leqq \frac{K_m M^k}{|\mathscr{R}\tau|^k} \equiv \frac{K_M'}{|\mathscr{R}\tau|^k}.$$

So, for $|\mathscr{R}\tau|$ large enough, we have

$$|P| = |P_m + \sum_{k=1}^{m} P_{m-k}| \geqq |P_m| - \sum_{k=1}^{m} |P_{m-k}|$$
$$> \frac{|P_m|}{2} \neq 0.$$

This completes the proof of the theorem.

3. Corollaries.

COROLLARY 1. (*A. Lax's theorem*) *If* $P \equiv \sum_{k=0}^{m} P_{m-k}$ *is hyperbolic with respect to* N, *then* $P_m^{(j)}(ix_0 + \tau_0 N) = 0$, *for all* $j \in [0, \ell] \Rightarrow P_{m-k}^{(j)}(ix_0 + \tau_0 N) = 0$ *for all* $j \in [0, \ell - k]$ *for* $k \in [0, \ell]$.

This is an immediate consequence of (f). See [4] for a simpler proof.

COROLLARY 2. (*Hörmander's theorem*) *If P_m is a homogeneous polynomial of degree m hyperbolic with respect to N and if, for any $x \in R^n$ not proportional to N, $P_m(ix + \tau N)$ possesses only roots of multiplicity $\leqq k_0$, where k_0 does not depend on x, then $P_m + \sum_{k=k_0}^m P_{m-k}$, is hyperbolic with respect to N, whatever be the polynomials P_{m-k} homogeneous of degree $m - k$.*

This follows from criterion (*d*): both members are homogeneous of degree $m - k$ and the second member never vanishes, by hypothesis, for x not parallel to N.

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