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AN ELEMENTARY DERIVATION OF THE DISTRIBUTION OF THE MAXIMA OF BROWNIAN MEANDER AND BROWNIAN EXCURSION

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ABSTRACT. An elementary proof is given to obtain the distributions of the maxima for the Brownian meander and Brownian excursion stochastic processes. The method involves application of conditional functional central limit theorems to simple random walk.

1. **Introduction.** The well-known theorem of Donsker identifies Brownian motion on [0, 1] as the weak limit of normalized random walk. Recently, the stochastic processes Brownian meander and Brownian excursion have been obtained through weak convergence of certain conditioned random walks by Belkin [1], Bolthausen [3], Iglehart [8], and Kaigh [9]. Such results are classified as functional central limit theorems or invariance principles and may be employed in conjunction wih the continuous mapping theorem to derive the asymptotic distributions of functionals of the limit process. To perform such calculations two techniques are generally available. One method involves a direct attack on the limit stochastic process itself while the other entails passage to the limit of expressions obtained for a corresponding random walk (usually, simple random walk).

This note is intended primarily to illustrate the use of the second method above to provide the distributions of the maxima for the Brownian meander and Brownian excursion. Recently, Chung [4], [5] and Durrett and Iglehart [6] also have obtained these distributions independently through direct analyses of the processes themselves instead of employing the functional central limit theorems mentioned previously. We remark that although not included here, similar results concerning first passages, expected occupation times, and other functionals also may be produced through our consideration of simple random walk.

2. **Notation and preliminaries.** Let $\{X_n : n \geqq 1\}$ be a sequence of i.i.d. random variables integer-valued with span 1 satisfying $EX_i = 0$, $EX_i^2 = \sigma^2 < \infty$. Form the random walk $\{S_n : n \geqq 0\}$ with $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geqq 1$, and define the random time T to be $\min\{n \geqq 1 : S_n = 0\}$ ($+\infty$ if no such n exists).

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On $C[0, 1]$, the space of real-valued continuous functions on the unit interval with the sigma field of Borel subsets generated by the supremum norm, define the process X_n by $X_n(k/n) = |S_k|/\sigma n^{1/2}$ and linearly interpolated elsewhere. Let P_n^+ and P_n° respectively be the probability measures on $C[0, 1]$ corresponding to the X_n process conditioned by the events $[T > n]$ and $[T = n]$. Belkin [1] has demonstrated the weak convergence of the P_n^+ to Brownian meander P^+ and in a similar manner Kaigh [9] has identified Brownian excursion P° as the weak limit of P_n°. In the following section we apply these results to obtain the distributions of the maxima of the two processes.

3. Distributions of the maxima of Brownian meander and Brownian excursion.

THEOREM. *For* $y > 0$

(i)
$$\lim_{n\to\infty} P[\max_{1\leqq k\leqq n} |S_k|/\sigma n^{1/2} \leqq y \mid T > n]$$
$$= P^+[\max_{0\leqq t\leqq 1} X(t) \leqq y]$$
$$= 1 + 2\sum_{k=1}^{\infty} (-1)^k \exp[-(ky)^2/2],$$

(ii)
$$\lim_{n\to\infty} P[\max_{1\leqq k\leqq n} |S_k|/\sigma n^{1/2} \leqq y \mid T = n]$$
$$= P^\circ[\max_{0<t<1} X(t) \leqq y]$$
$$= 1 + 2\sum_{k=1}^{\infty} [1 - (2ky)^2] \exp[-(2ky)^2/2].$$

PROOF. Consider the simple random walk $\{S_n : n \geqq 0\}$ defined by $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geqq 1$, where $\{X_n : n \geqq 1\}$ are i.i.d. Bernoulli r.v.'s with $P[X_i = \pm 1] = 1/2$ and let

$$m_n = \min_{1\leqq k\leqq n} S_k,\quad M_n = \max_{1\leqq k\leqq n} S_k,$$

$$T = \min\{n \geqq 1 : S_n = 0\} \qquad (\inf \phi = +\infty).$$

From pages 77–78 of Feller [7]

$$P[T > 2n] = P[S_{2n} = 0]$$
$$P[T = 2n] = P[S_{2n} = 0]/(2n - 1) \tag{1}$$

where

$$P[S_{2n} = \pm 2x] = \binom{2n}{n+x} 2^{-2n} \text{ for } 0 \leqq x \leqq n.$$

To prove (i) we have from the definition of the stopping time T

$$P\,[\max_{1 \leqq k \leqq 2n} |S_k| < x \mid T > 2n]$$

$$= 2P[0 < M_{2n} < x;\ T > 2n]/P[T > 2n]$$

(2) $$= 2P[X_1 = 1;\ 0 < M_{2n} < x;\ T > 2n]/P[T > 2n]$$

$$= P[-1 < m_{2n-1} \leqq M_{2n-1} < x - 1;$$
$$-1 < S_{2n-1} < x - 1]/P[T > 2n].$$

An application of the repeated reflection formula on page 78 of Billingsley [2] provides

$$P[-1 < m_{2n-1} \leqq M_{2n-1} < x - 1;\ -1 < S_{2n-1} < x - 1]$$

$$= \sum_{k=-\infty}^{\infty} P[2kx - 1 < S_{2n-1} < (2k+1)x - 1]$$

$$- \sum_{k=-\infty}^{\infty} P[(2k+1)x - 1 < S_{2n-1} < 2(k+1)x - 1]$$

where all but a finite number of terms vanish. Employing the symmetry of the distribution of S_{2n-1}, termwise manipulation of the series above gives

$$P[-1 < m_{2n-1} \leqq M_{2n-1} < x - 1;\ -1 < S_{2n-1} < x - 1]$$

$$= \sum_{k=0}^{\infty} \{P[2kx - 1 < S_{2n-1} < (2k+1)x - 1]$$

$$- P[2kx + 1 < S_{2n-1} < (2k+1)x + 1]$$

$$+ P[(2k+1)x + 1 < S_{2n-1} < 2(k+1)x + 1]$$

$$- P[(2k+1)x - 1 < S_{2n-1} < 2(k+1)x - 1]\}$$

$$= \sum_{k=0}^{\infty} \{P[S_{2n-1} = 2kx + 1] - P[S_{2n-1} = (2k+1)x]$$

$$+ P[S_{2n-1} = 2(k+1)x - 1] - P[S_{2n-1} = (2k+1)x]\}$$

where the second equality is valid for odd x and a similar expression holds for the case x even.

Combination of (1), (2), and (3) followed by an application of the local central limit theorem for simple random walk given on page 81 of Billingsley [2] will show that if $x/(2n)^{1/2} \to y > 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} P[\max_{1\leqq k\leqq 2n} |S_k|/(2n)^{1/2} \leqq y \mid T > 2n]$$

$$= \sum_{k=0}^{\infty} [\exp\{-(2ky)^2/2\} - \exp\{-[(2k+1)y]^2/2\}$$

$$+ \exp\{-[2(k+1)y]^2/2\} - \exp\{-[(2k+1)y]^2/2\}]$$

$$= 1 + 2 \sum_{k=1}^{\infty} (-1)^k \exp[-(ky)^2/2].$$

The justification for interchange of limit with summation follows from the fact that $\overline{\lim}_n E\,|\,S_n/n^{1/2}| < +\infty$.

Thus, (i) follows from the continuous mapping theorem (see [2], Theorem 5.1) and the weak convergence of P_n^+ to P^+.

To prove (ii) we have

$$P[\max_{1\leqq k\leqq 2n} |S_k| < x \mid T = 2n]$$

$$= 2P[0 < M_{2n} < x;\ T > 2n]/P[T = 2n]$$

(4) $$= 2P[X_1 = 1;\ 0 < M_{2n} < x;\ X_{2n} = -1;\ T = 2n]/P[T = 2n]$$

$$= (1/2)P[-1 < m_{2n-2} \leqq M_{2n-2} < x - 1;\ S_{2n-2} = 0]/P[T = 2n].$$

Again from the repeated reflection formula and the symmetry of the distribution of S_{2n-2}, after manipulation we obtain

$$P[-1 < m_{2n-2} \leqq M_{2n-2} < x - 1;\ S_{2n-2} = 0]$$

$$= \sum_{k=-\infty}^{\infty} P[S_{2n-2} = 2kx] - \sum_{k=-\infty}^{\infty} P[S_{2n-2} = 2(k+1)x - 2]$$

$$= P[S_{2n-2} = 0] - P[S_{2n-2} = 2]$$

$$+ \sum_{k=1}^{\infty} \{2P[S_{2n-2} = 2kx] - P[S_{2n-2} = 2kx - 2]$$

(5) $$- P[S_{2n-2} = 2kx + 2]\}$$

$$= (1/n)P[S_{2n-2} = 0]$$

$$+ (2/n) \sum_{k=1}^{\infty} \frac{[1 - 2k^2x^2/n]}{[1 - k^2x^2/n^2]} P[S_{2n-2} = 2kx]$$

where the last equality follows from the expression for $P[S_{2n} = 2x]$ given in (1) and simplification.

A combination of (1), (4), and (5) with the local central limit theorem produces for $x/(2n)^{1/2} \to y > 0$ as $n \to \infty$,

$$\lim_{n\to\infty} P[\max_{1<k<2n} |S_k|/(2n)^{1/2} < y \mid T = 2n]$$

$$= 1 + 2 \sum_{k=1}^{\infty} [1 - (2ky)^2] \exp[-(2ky)^2/2].$$

The justification for interchange of limit with summation follows from dominated convergence and the fact that $\lim_n E \mid S_n/n^{1/2}|^5 < +\infty$. Hence, (ii) follows from the continuous mapping theorem and the fact that $P_n{}^\circ$ converges weakly to P°.

The proof of the theorem is complete. We conclude with the remark that Chung [5] has verified analytically that the expressions given in (i) and (ii), in act, are cumulative distribution functions.

References

1. B. Belkin, *An invariance principle for conditioned random walk attracted to a stable law,* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **21** (1972), 45–64.

2. P. Billingsley, *Convergence of Probability Measures,* Wiley, New York: 1968.

3. E. Bolthausen, *On a functional central limit theorem for random walks conditioned to stay positive,* Ann. Probability **4** (1976), 480–485.

4. K. L. Chung, *Maxima in Brownian excursions,* Bull. Amer. Math. Soc. **81** (1975), 742–745.

5. ———, *Excursions in Brownian motion* (1976), (to appear).

6. R. T. Durrett and D. L. Iglehart, *Functionals of Brownian meander and Brownian excursion* (1976), (to appear).

7. W. Feller, *An Introduction to Probability Theory and Its Applications,* 3rd ed., Wiley, New York: 1968.

8. D. L. Iglehart, *Functional central limit theorems for random walks conditioned to stay positive,* Ann. Probability **2** (1974), 608–619.

9. W. D. Kaigh, *An invariance principle for random walk conditioned by a late return to zero,* Ann. Probability **4** (1976), 115–121.

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