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DEFICIENCY INDICES OF ODD-ORDER DIFFERENTIAL OPERATORS

DON B. HINTON[1]

1. **Introduction.** Considerable literature now exists for finding the deficiency indices of symmetric even-order differential expressions as is evident in the recent article by Atkinson [1]. For odd-order expressions much less is known and much of it has been obtained by asymptotic methods as in [6, 8, 9]. We apply here the methods of [4, 5] to odd order differential operators to compute the deficiency indices.

We assume that

(1.1) *$q_0, q_1, \cdots, q_n, p_0, p_1, \cdots, p_n$, and w are continuous, real functions defined on an open interval (a, b) with q_0 and w positive.*

Associated with the weight function w is the Hilbert space $\mathscr{L}^2(w; a, b)$ of all complex-valued, measurable functions f satisfying $\int_a^b w|f|^2 < \infty$. For $w = 1$ the usual notation $\mathscr{L}^2(a, b)$ is used. The differential operator $L[\cdot]$ is defined by

$$L[y] = w^{-1}(-1)^n[i(q_0(q_0y^{(n)})')^{(n)} + (p_0y^{(n)})^{(n)}]$$
$$+ w^{-1} \sum_{k=0}^{n-1} (-1)^k\{i[(q_{n-k}y^{(k)})^{(k+1)} \tag{1.2}$$
$$+ (q_{n-k}y^{(k+1)})^{(k)}] + (p_{n-k}y^{(k)})^{(k)}\}$$

where $a < x < b$ and $m = 2n + 1$ is odd. We assume $m \geqq 3$ since the first-order differential equation can be solved by quadratures.

The term $(q_0(q_0y^{(n)})')^{(n)}$ of (1.2) is usually written as

$$[(u_0y^{(n)})^{(n+1)} + (u_0y^{(n+1)})^{(n)}].$$

When q_0 is continuously differentiable, the two forms are seen to be equivalent by setting $q_0{}^2 = 2u_0$. We are using the form of symmetric differential expressions given by Walker [11].

We define the quasi-derivatives $y^{[k]}(k = 0, \cdots, 2n)$ by:

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$$
\begin{aligned}
y^{[k]} &= y^{(k)},\ k = 0, \cdots, n-1 \\
y^{[n]} &= -\theta q_0 (y^{[n-1]})',\ \theta = (1+i)/2^{1/2} \\
y^{[n+1]} &= -(\theta q_0)(y^{[n]})' + (i\theta p_0/q_0) y^{[n]} - i q_1 y^{[n-1]}, \qquad (1.3) \\
y^{[n+2]} &= -(y^{[n+1]})' - (\theta q_1/q_0) y^{[n]} + p_1 y^{[n-1]} \\
&\quad - i q_2 y^{[n-2]} \text{ if } n+2 \leqq m-1, \\
y^{[n+k+1]} &= -(y^{[n+k]})' + p_k y^{[n-k]} + i(q_k y^{[n-k+1]} \\
&\quad - q_{k+1} y^{[n-k-1]}) \text{ if } 2 \leqq k \leqq n-1.
\end{aligned}
$$

A function y is said to be *admissible* for $L[\cdot]$ provided the quasi-derivatives $y^{[k]}$, $k = 0, \cdots, 2n$, exist and are locally absolutely continuous on (a, b) in which case $L[\cdot]$ is given by [11]

$$L[y] = w^{-1}\{-(y^{[2n]})' + i q_n y' + p_n y\}. \qquad (1.4)$$

The linear manifold Δ is defined as the set of all admissible $y \in \mathcal{L}^2(w; a, b)$ such that $L[y] \in \mathcal{L}^2(w; a, b)$. A maximal operator L_1 in $\mathcal{L}^2(w; a, b)$ is defined by

$$L_1[y] = L[y],\ y \in \Delta.$$

Associated with L_1 is its minimal operator defined by letting $L_0' = L \mid \Delta_0'$ where Δ_0' consists of all $y \in \Delta$ with compact support. Then L_0' is a densely defined symmetric operator; hence L_0' has a closure L_0. From considerations similar to those in [7, § 17], it follows that $L_0^* = L_1$ and $L_1^* = L_0$.

The deficiency indices $N_+(b)$, $N_-(b)$ of L_0 at b are defined respectively as the number of linearly independent solutions of $L[y] = \lambda y$, $\operatorname{Im} \lambda > 0$, $\operatorname{Im} \lambda < 0$, which are square summable at b, i.e., satisfy $\int_c^b w|y|^2 < \infty$ for some (and hence every) c in (a, b). It is well known that the deficiency index is independent of λ if λ is restricted to either the half-plane $\operatorname{Im} \lambda > 0$ or to the half-plane $\operatorname{Im} \lambda < 0$. The deficiency indices $N_+(a)$, $N_-(a)$ are defined similarly by considering $L[y] = \lambda y$ on $(a, c]$. By N_+, N_- we will mean respectively, the number of linearly independent solutions of $L[y] = \lambda y$, $\operatorname{Im} \lambda > 0$, $\operatorname{Im} \lambda < 0$, which satisfy $\int_a^b w|y|^2 < \infty$.

From [11] the indices $N_+(b)$ and $N_-(b)$ satisfy

$$n \leqq N_+(b) \leqq m,\ n+1 \leqq N_-(b) \leqq m,$$

and moreover, if either is m, then the other is also. Similarly, the indices $N_+(a)$ and $N_-(a)$ (see the transformation in section 3 below) satisfy

$$n+1 \leqq N_+(a) \leqq m,\ n \leqq N_-(a) \leqq m.$$

Thus for $m = 3$, the only possibilities for the pair $(N_+(b), N_-(b))$ are (1,2), (2,2), or (3,3); the only possibilities for the pair $(N_+(a), N_-(a))$ with $m = 3$ are (2,1), (2,2), or (3,3).

For the indices N_+, N_- the splitting method [2, p. 1302; 7, p. 72] ($w = 1$ in these references; similar arguments apply to the general case) yields that

$$N_+ = N_+(b) + N_+(a) - m,$$
$$N_- = N_-(b) + N_-(a) - m.$$

The classical result of Von Neumann gives that L_0 has self-adjoint extensions in $\mathcal{L}^2(w;\, a, b)$ if and only if $N_+ = N_-$; L_0 is self-adjoint if and only if $N_+ = N_- = 0$. Theorem 3.1 below yields a large class of operators for which $N_+ = N_- = 0$.

The results of this paper have application to criteria for the essential spectrum of L_0 [2, p. 1393] (and thus all self-adjoint extensions of L_0, if any) to consist of the entire real axis $(-\infty, \infty)$. The application uses a theorem from operator theory which states that unequal deficiency indices at either endpoint implies the essential spectrum is $(-\infty, \infty)$, [2, pp. 1396, 1438]. Thus a criterion which gives that $N_+(b) = n$, $N_-(b) = n + 1$ or that $N_+(a) = n + 1$, $N_-(a) = n$ is also a criterion for the essential spectrum of L_0 to be $(-\infty, \infty)$.

In section 2 some auxilary results are proved which are needed for the principal theorems in section 3. In section 4 we show how deficiency index criteria at a finite singularity may be deduced from one at infinity.

2. **Preliminary Results.** Using the quasi-derivatives (1.3), Walker [11] puts the scalar equation

$$L[y] = \lambda y + f \tag{2.1}$$

in the vector form

$$JY' = [\lambda A + B]\, Y + F \tag{2.2}$$

where

$$J_{kj} = \begin{cases} 0, & k + j \neq m + 1 \\ 1, & k + j = m + 1,\ 1 \leqq k \leqq n \\ i, & k = n + 1,\ j = n + 1 \\ -1, & k + j = m + 1,\ n + 1 < k \leqq m, \end{cases}$$

$$Y = [y^{[0]}, \cdots, y^{[m-1]}]^T,$$
$$F = JG,\ G = [0, \cdots, 0, -wf]^T,$$
$$A = JC,\ C_{m1} = -w,\ C_{kj} = 0 \text{ if } (k, j) \neq (m, 1),$$
$$B = JD;$$

and the non-zero elements of D are

$$D_{kj} = 1,\ j = k+1,\ k = 1, \cdots, n-1\ (n > 1)$$
$$D_{kj} = -1,\ j = k+1,\ k = n+2, \cdots, m\ (n > 1)$$

$$\begin{bmatrix} D_{n,n} & D_{n,n+1} & D_{n,n+2} \\ D_{n+1,n} & D_{n+1,n+1} & D_{n+1,n+2} \\ D_{n+2,n} & D_{n+2,n+1} & D_{n+2,n+2} \end{bmatrix} = \begin{bmatrix} 0 & -1/\theta q_0 & 0 \\ -\theta q_1/q_0 & ip_0/{q_0}^2 & -1/\theta q_0 \\ p_1 & -\theta q_1/q_0 & 0 \end{bmatrix}$$

$$D_{kj} = -iq_{k-n},\ k + j = m,\ k = n+2, \cdots, m-1\ (n > 1)$$
$$D_{kj} = iq_{k-n-1},\ k + j = m+2,\ k = n+3, \cdots, m\ (n > 1)$$
$$D_{kj} = p_{k-n-1},\ k + j = m+1,\ k = n+3, \cdots, m\ (n > 1).$$

Note that $A = A^*$, $B = B^*$, and $J^{-1} = J^* = -J$.

LEMMA 2.1. *Let* (1.1) *hold and* A, B, *and* J *be as above, and suppose that* Y, Z *satisfy for some complex number* λ,

$$JY' = (\lambda A + B)Y + F_1,\ JZ' = (\bar{\lambda} A + B)Z + F_2.$$

Then

$$(2.3)\qquad (Z^*JY)' = -F_2^*Y + Z^*F_1.$$

The proof is immediate from computing the derivative in (2.3). As an immediate corollary it follows from (2.2) for $\lambda = 0$ that if y, z are admissible for $L[\cdot]$, then

$$(2.4)\qquad \left[\sum_{k=o}^{n-1} \{\overline{z^{[k]}}y^{[m-1-k]} - \overline{z^{[m-1-k]}}y^{[k]}\} + \overline{iz^{[n]}}y^{[n]}\right]'$$
$$= -w\{L[y]\bar{z} - y\overline{L[z]}\},$$

which is the Lagrange identity for $L[\cdot]$.

LEMMA 2.2. *Suppose* (1.1) *holds,* $c \in (a, b)$, $y_j (j = 1, \cdots, m)$ *is the solution on* (a, b) *of* $L[y] = 0$ *satisfying the initial conditions*

$$y_j^{[k-1]}(c) = \delta_{kj}\ (k, j = 1, \cdots, m)$$

where δ_{kj} *is the Kronecker delta. Then*

$$(2.5)\qquad -\sum_{k=1}^{n} y_k\, \overline{y_{m+1-k}^{[m-1]}} - i\; y_{n+1}\overline{y_{n+1}^{[m-1]}} + \sum_{k=n+2}^{m} y_k\, \overline{y_{m+1-k}^{[m-1]}} \equiv -1.$$

PROOF. Define $\tilde{Y}$ by $\tilde{Y}_{kj} = y_j^{[k-1]}$ $(k, j = 1, \cdots, m)$. Then

$$J\tilde{Y}' = B\tilde{Y} \text{ and } \tilde{Y}(c) = I.$$

By Lemma 2.1, $\tilde{Y}^*J\tilde{Y}$ is constant; hence

$$\tilde{Y}^*(x)J\tilde{Y}(x) = J,\ a < x < b.$$

Since $J^{-1} = J^*$, this equation implies $\tilde{Y}^{-1} = J^*\tilde{Y}^*J$ from which we conclude that $\tilde{Y}J^*\tilde{Y}^*J = I$ and thus $\tilde{Y}J^*\tilde{Y} = J^*$. Equation (2.5) now follows by multiplying the first row of $\tilde{Y}$ times the mth column of $J^*\tilde{Y}^*$ and setting the product equal to $(J^*)_{1m}$.

LEMMA 2.3. *Suppose* (1.1) *holds,* $b = \infty$, $a < c < \infty$, q_0 *and* w *are continuously differentiable, and there is a continuously differentiable positive function* ρ *on* $[c, \infty)$ *such that*

$$\text{(i)}\qquad \frac{\rho^2}{w^{1/m}}\left[\frac{|w'|}{w} + \frac{|q_0'|}{q_0} + \frac{|\rho'|}{\rho}\right] = O(1) \text{ as } x \to \infty,$$

$$\text{(ii)}\qquad \frac{|p_i|\rho^{4i+2}}{q_0^{\,2}w^{(2i+1)/m}} = O(1) \text{ as } x \to \infty \text{ for } i = 0, \cdots, n,$$

$$\text{(iii)}\qquad \frac{|q_i|\rho^{4i}}{q_0^{\,2}w^{2i/m}} = O(1) \text{ as } x \to \infty \text{ for } i = 1, \cdots, n,$$

$$\text{(iv)}\qquad \rho^m/q_0 = O(1) \text{ as } x \to \infty.$$

Then if y *is admissible and* $y, L[y] \in \mathcal{L}^2(w; c, \infty)$, *it follows that*

$$\text{(v)}\qquad \int_c^\infty w^{(m-2i+2)/m}\rho^{4i-4}|y^{[i-1]}|^2 < \infty \text{ for } i = 1, \cdots, n,$$

$$\text{(vi)}\qquad \int_c^\infty w^{1/m}\rho^{4n}q_0^{-2}|y^{[n]}|^2 < \infty,$$

$$\text{(vii)}\qquad \int_c^\infty w^{(m-2i+2)/m}\rho^{4i-4}q_0^{-4}|y^{[i-1]}|^2 < \infty \text{ for } i = n+2, \cdots, m,$$

and

(viii) *The functions* $w^{(m-2i-1)/m}\rho^{2i-1}|y^{[i-1]}|$ $(i = 1, \cdots, n)$, $w^{(m-n-1)/m}\rho^{2n+1}|y^{[n]}|q_0^{-1}$, *and* $w^{(m-2i+1)/m}\rho^{2i-1}q_0^{-2}|y^{[i-1]}|$, $(i = n+2, \cdots, m)$ *are uniformly bounded on* $[c, \infty)$.

PROOF. Since $J^*J = I$, a multiplication of (2.2) by J^{-1} yields

$$Y' = (\lambda C + D)Y + G \tag{2.6}$$

where C, D, and G are as before. We will now make a transformation of (2.6) for $\lambda = 0$ and then apply Theorem A of [5]. In (2.6), let $X = MY$ where M is the diagonal matrix

$$M = \text{dia}\,\{\theta g^{\alpha}\rho,\ \theta g^{\alpha-1}\rho^3,\ \cdots,\ \theta g^{\alpha-n+1}\rho^{2n-1},\ g^{\alpha-n}\rho^{2n+1}/q_0,$$
$$g^{\alpha-n-1}\rho^{2n+3}/\theta q_0^{\,2},\ \cdots,\ g^{\alpha-m+1}\rho^{2m-1}/\theta q_0^{\,2}\},$$
$$\alpha = (m-1)/2,\ g = w^{1/m},\ \text{and}$$
$$X = (x_1, \cdots, x_m)^T.$$

Hence X satisfies

$$X' = [M'M^{-1} + MDM^{-1}]X + MG \tag{2.7}$$

with $G = (0, \cdots, 0, -wf)^T$ and $f = L[y]$.

A somewhat tedious calculation gives that

$$(MDM^{-1})_{kj} = \begin{cases} g/\rho^2,\ j = k+1,\ k = 1, \cdots, n-1, \\ -g/\rho^2,\ j = k+1,\ k = n, \cdots, 2n, \\ -(g/\rho^2)q_{k-n}\rho^{4k-4n}/q_0^{\,2}g^{2k-2n},\ k+j = 2n+1, \\ \qquad k = n+1, \cdots, 2n, \\ -(g/\rho^2)q_{k-n-1}\rho^{4k-4n-4}/q_0^{\,2}g^{2k-2n-2}, \\ k+j = 2n+3,\ k = n+2, \cdots, 2n+1, \\ (g/\rho^2)\, i\, p_0\rho^2/q_0^{\,2}g,\ k = j = n+1, \\ (g/\rho^2)p_{k-n-1}\rho^{4k-4n-2}/i\, q_0^{\,2}g^{2k-2n-1}, k+j = 2n+2, \\ \qquad k = n+2, \cdots, 2n+1, \\ 0 \text{ otherwise;} \end{cases}$$

hence conditions (ii) and (iii) above imply that the elements $\rho^2/g\,MDM^{-1}$ are uniformly bounded. Similarly condition (i) implies that the elements of $\rho^2/g\,M'M^{-1}$ are uniformly bounded. Thus we may write (2.7) as

$$X' = g/\rho^2[\rho^2/gMDM^{-1} + \rho^2/gM'M^{-1}]X + G_1 \tag{2.8}$$

where $G_1 = (0, 0, \cdots, 0, h)^T$ with

$$h = -wg^{\alpha-m+1}\rho^{2m-1}L[y]/\theta q_0{}^2$$
$$= -w^{(m+1)/2m}\rho^{2m-1}L[y]/\theta q_0{}^2.$$

From $L[y] \in \mathcal{L}^2(w;\ c, \infty)$ and condition (iv), we have that

$$\int_c^\infty (\rho^2/g)|h|^2\,dx = \int_c^\infty w(\rho^{4m}/q_o{}^4)|L[y]|^2\,dx < \infty.$$

Now $x_1 = \theta g^\alpha \rho y$ and $y \in \mathcal{L}^2(w;\ c, \infty)$ yield that

$$\int_c^\infty (g/\rho^2)|x_1|^2\,dx = \int_c^\infty w|y|^2\,dx < \infty;$$

hence Theorem A of [5] is applicable and results (v)–(viii) are an immediate consequence of Theorem A and the transformation $X = MY$.

Lemma 2.3 establishes certain *a priori* bounds on $[c, \infty)$ for members of Δ. For example, if $w = q_0 = 1$ and we take $\rho = 1$, then from Lemma 2.3 we conclude that if the other coefficients of $L[\cdot]$ are bounded, then for $y \in \Delta$ we have $y^{[i]} \in \mathcal{L}^2(c, \infty)$ for $i = 1, \cdots, 2n$. As a further example consider the Euler operator,

$$L_e[y] = -i(x^{3/2}(x^{3/2}y')')' + ky,\ 1 \leqq x < \infty$$

with $w = 1$. Then a choice of $\rho(x) = x^{1/2}$ in Lemma 2.3 implies that

$$xy',\ x^2y'' \in \mathcal{L}^2(1, \infty) \text{ for } y \in \Delta.$$

3. **Deficiency index theorems for $b = \infty$ or $a = -\infty$.** For the proof of our first theorem we define $(a < c < b)$,

$$V(\lambda) = \{y \in \mathcal{L}^2(w;\ c, b);\ y \text{ is admissible and } L[y] = \lambda y\}.$$

THEOREM 3.1. *Suppose the hypotheses of Lemma* 2.3 *hold and that also* $\int_c^\infty w^{1/m}\rho^{4n}/q_0{}^2 = \infty$. *Then* $N_+(\infty) = n$ *and* $N_-(\infty) = n + 1$.

PROOF. For $y \in V(i)$, $z \in V(-i)$, the Lagrange identity (2.4) gives that

$$\sum_{k=0}^{n-1} \{\overline{z^{[k]}y^{[m-1-k]}} - \overline{z^{[m-1-k]}y^{[k]}}\} + i\,\overline{z^{[n]}y^{[n]}} = K \tag{3.1}$$

where K is a constant.

We now follow the linear algebra argument of Lemma 2.1 of [4]. If $N_+(\infty) = \dim V(i) > n$, we regard $V(i)$ and $V(-i)$ as subspaces of m-dimensional complex space (by the correspondence

$$y \to (y(c),\ y^{[1]}(c),\ \cdots,\ y^{[m-1]}(c))$$

and use the non-singularity of the transformation J to find $t \in V(-i) \cap JV(i)$, $t \neq 0$. If $t = z = Jy$, then the constant K in (3.1) is the norm of t squared; hence $K \neq 0$. It is sufficient that $K = 1$. Multiplication of (3.1) by $w^{1/m}\rho^{4n}/q_0^2$ yields

$$\text{(3.2)} \qquad \begin{aligned} \frac{w^{1/m}\rho^{4n}}{q_0^2} = i \Bigg[& \frac{w^{1/2m}\rho^{2n}}{q_0} \overline{z^{[n]}} \cdot \frac{w^{1/2m}\rho^{2n}}{q_0} y^{[n]} \\ & + \sum_{k=0}^{n-1} \Bigg[w^{(m-2k)/2m}\rho^{2k}\overline{z^{[k]}} \\ & \cdot \frac{w^{(2k+2-m)/2m}\rho^{2(m-1-k)}}{q_0^2} y^{[m-1-k]} \\ & - \frac{w^{(2k+2-m)/2m}\rho^{2(m-1-k)}}{q_0^2} \overline{z^{[m-1-k]}} \\ & \cdot w^{(m-2k)/2m}\rho^{2k} y^{[k]} \Bigg]. \end{aligned}$$

By conditions (v), (vi), and (vii) of Lemma 2.3, the right-hand side of (3.2) is Lebesgue integrable on (c, ∞); by hypothesis the left-hand side is not. This contradiction proves the theorem.

For Theorem 3.1, an optimal choice of ρ is given by

$$\text{(3.3)} \qquad w(x)^{1/m}\rho(x)^{4n}q_0(x)^{-2} = x^{-1}.$$

We now apply our results to the $m = 3$ case. The function ρ given by (3.3) is $\rho(x) = q_0^{1/2}(x)/w^{1/12}(x)x^{1/4}$. The operator $L[\cdot]$ is given by

$$\text{(3.4)} \qquad \begin{aligned} L[y] = \frac{1}{w}\{ & -i(q_0(q_0y')')' - (p_0y')' \\ & + i[q_1y' + (q_1y)'] + p_1y\}, \end{aligned}$$

and by Theorem 3.1, $N_+(\infty) = 1$ and $N_-(\infty) = 2$ if

$$\text{(3.5)} \qquad \frac{q_0}{(xw)^{1/2}} \left[\frac{|w'|}{w} + \frac{|q_0'|}{q_0} + \frac{1}{x} \right] = O(1) \text{ as } x \to \infty,$$

$$\text{(3.6)} \qquad \frac{|q_1|}{xw}, \quad \frac{|p_0|}{q_0(xw)^{1/2}}, \quad \frac{|p_1|q_0}{(xw)^{3/2}}, \quad \frac{q_0}{x^{3/2}w^{1/2}}, \quad \text{are } O(1) \text{ as } x \to \infty.$$

Consider now (3.4) with $(a, b) = (1, \infty)$ and

$$q_1(x) = ax^\alpha,\ q_0(x) = bx^\beta (b > 0),\ p_1(x) = cx^\gamma,$$
$$p_0(x) = dx^\delta,\ w(x) = x^\xi. \tag{3.7}$$

Then conditions (3.5) and (3.6) are equivalent to

$$\alpha \leqq 1 + \xi,\ 2\delta \leqq 2\beta + (1 + \xi),\ \beta + \gamma \leqq 3(1 + \xi)/2,\ \beta \leqq 3/2 + \xi/2. \tag{3.8}$$

For $\xi = 0$, these conditions are simply

$$\alpha \leqq 1,\ \delta \leqq \beta + 1/2,\ \gamma \leqq -\beta + 3/2,\ \beta \leqq 3/2. \tag{3.9}$$

Recently, Pfeiffer [8] has used asymptotic theorems to compute the deficiency indices of a class of third order operators with $w = 1$, $q_0 = 1$, and $p_0 = 0$. Applying his theorems to the case

$$q_1(x) = ax^\alpha \text{ and } p_1(x) = cx^\gamma,$$

Pfeiffer has shown that

$N_+(\infty) = 1$, $N_-(\infty) = 2$ if $\alpha \leqq 1$, $\gamma \leqq 3/2$ and certain additional restrictions hold on α, γ,

$N_+(\infty) = N_-(\infty) = 2$ if $c \neq 0$, $\gamma > 3/2$, and $\gamma > 2\alpha + 1$.

$N_+(\infty) = N_-(\infty) = 2$ if $a < 0$, $\alpha > 1$, and $\alpha > (4\gamma + 2)/5$,

$N_+(\infty) = N_-(\infty) = 3$ if $a > 0$, $\alpha > 1$, and $\alpha > (4\gamma + 2)/5$.

Thus for $w = q_0 = 1$, $p_0 = 0$, the conditions (3.9) are sharp when q_1 and p_1 are powers of x. In the example following Theorem 3.2 we show that the exponent of 3/2 for β in (3.9) is also sharp; this fact has also been shown by Unsworth [9].

The asymptotic behavior of solutions of (3.4) has been investigated also by Unsworth. In [9], Unsworth applied his methods to give the asymptotic solutions and deficiency indices of (3.4) with a singularity at infinity and the coefficients given by (3.7) with $w_0(x) = 1$ and $q_0(x) = 2$. His results for the deficiency indices are:

(i) $N_+(\infty) = N_-(\infty) = 3$ if $a > 0$, $\alpha > 1$, $\delta < \alpha/2$, $\gamma < 3\alpha/2$

(ii) $N_+(\infty) = N_-(\infty) = 2$ if $a < 0$, $\alpha > 1$, $\delta < \alpha/2$, $\gamma < 3\alpha/2$

(iii) $N_+(\infty) = 1$, $N_-(\infty) = 2$ if $0 < \alpha \leqq 1$, $\delta < \alpha/2$, $\gamma < 3\alpha/2$.

Furthermore, Unsworth states that with $w(x) = 1$ in (3.7), $N_+(\infty) = N_-(\infty) = 2$ if $\beta > 3/2$, $\alpha < 2\beta - 2$, $\gamma < 2\beta - 3$, and $b \neq 0$. For convenience we have stated Unsworth's results in our notation.

Deficiency index criteria for $a = -\infty$ may be deduced from those at $b = \infty$ by application of the transformation $s = -x$. This technique has been used by Eastham and Unsworth [3]. As applied to (3.4), this transformation transforms $L[\cdot]$ into $K[\cdot]$ where $K[\cdot]$ has the same form as $L[\cdot]$, $L[y] = -K[z]$, $z(s) = y(x)$, and the coefficients of $K[\cdot]$ are given by:

$$\tilde{w}(s) = w(x),\ \tilde{q}_0(s) = q_0(x),\ \tilde{q}_1(s) = q_1(x),\ \tilde{p}_0(s) = -p_0(x),$$
$$\tilde{p}_1(s) = -p_1(x).$$

Hence, if the conditions (3.5) and (3.6) hold as $x \to -\infty$ (with x replaced by $|x|$), then

$$N_+(-\infty) = 2,\ N_-(-\infty) = 1.$$

As mentioned in the introduction, the essential spectrum of L_0 is $(-\infty, \infty)$ if either $N_+(b) \neq N_-(b)$ or $N_+(a) \neq N_-(a)$. To apply the theory of [2], we assume in addition that the coefficients are of class $C^\infty(a, b)$. To illustrate Theorem 3.1, we consider only the case where L is given by (3.4) and compare the results with those of Unsworth [10] where the method of singular sequences is used (cf. [2, p. 1435]). As applied to (3.4) with $w = 1$ and the interval $(a, b) = (-\infty, \infty)$,

$$\begin{aligned} &q_1(x) = a(1 + x^2)^{\alpha/2} && q_0(x) = b(1 + x^2)^{\beta/2},\ b > 0, \\ &p_1(x) = c(1 + x^2)^{\gamma/2} && p_0(x) = d(1 + x^2)^{\delta/2}, \end{aligned} \tag{3.10}$$

Unsworth gives the following four criteria for the essential spectrum of L_0 to be $(-\infty, \infty)$:

(a) $\beta < 3/2,\ \alpha < 2\beta/3,\ c = d = 0,$

(b) $\beta < 3/2,\ \gamma < 0,\ \alpha < 2\beta/3,\ \delta < 4\beta/3,$

(c) $\alpha < 1,\ \gamma < 0,\ \beta < 3\alpha/2,\ \delta < 2\alpha,$

(d) $\gamma > 0,\ \beta + \gamma < 3/2,\ \alpha < (2\beta - \gamma)/3,\ \delta < 2(2\beta - \gamma)/3.$

Note that because of the different form of (3.4) used in [10] our β is half that given on page 303 of [10].

For (3.10), Theorem 3.1 and the transformation above yield that $N_+(\infty) = 1 = N_-(-\infty)$ and $N_-(\infty) = 2 = N_+(-\infty)$ if (3.9) holds. We see that if (a) or (b) or (d) holds, then so does (3.9) while (c) and (3.9) are independent.

Additional criteria for the essential spectrum to be $(-\infty, \infty)$ or to contain a ray $[0, \infty)$ or $(-\infty, 0]$ are contained in [3].

In our next theorem we use an argument similar to that of [12].

THEOREM 3.2. *Suppose* (1.1) *holds,* q_n *is continuously differentiable,* $b = \infty$, *and for some* $c \in (a, b)$, $c > 0$,

$$\int_c^x p_n^2/w, \quad \int_c^x (q_n')^2/w, \text{ and} \tag{3.11}$$
$$\int_c^x w \text{ are } O(xw(x)) \text{ as } x \to \infty.$$

Then $N_+(\infty) \neq m$ *and* $N_-(\infty) \neq m$.

PROOF. From Schwarz's inequality we have that

$$|q_n(x)| = |q_n(c) + \int_c^x q_n'| \leqq |q_n(c)|$$
$$+ \left[\int_c^x q_n'^2/w \right]^{1/2} \left[\int_c^x w \right]^{1/2}$$

from which it follows that $|q_n(x)| = O(xw(x))$ as $x \to \infty$.

Suppose now y is admissible, $y \in \mathcal{L}^2(w;\ c, \infty)$ and $L[y] = 0$. From (1.4) we have $(y^{[2n]})' = iq_n y' + p_n y$; hence

$$y^{[2n]}(x) = y^{[2n]}(c) + \int_c^x [iq_n y' + p_n y] \tag{3.12}$$
$$= y^{[2n]}(c) + iq_n y|_c^x + \int_c^x [-iq_n' y + p_n y].$$

By (3.11) and Schwarz's inequality, each of $\int_c^x q_n' y$ and $\int_c^x p_n y$ is $O(x^{1/2} w^{1/2}(x))$ as $x \to \infty$. Applying these to (3.12) yields that for some constant M_y

$$|y^{[2n]}(x)| \leqq M_y[xw(x)|y(x)| + x^{1/2} w^{1/2}(x)], \quad c \leqq x < \infty. \tag{3.13}$$

Suppose now $N_+(\infty) = m$ or $N_-(\infty) = m$; then all solutions of $L[y] = 0$ are in $\mathcal{L}^2(w;\ c, \infty)$ (cf. [11]). Let $y_1, \cdots, y_m$ be as in Lemma 2.2. From (2.5) and (3.13) we see that there is a constant M such that

$$1 \leqq M[xw(x)f(x) + x^{1/2} w^{1/2}(x)]f(x), \quad c \leqq X < \infty \tag{3.14}$$

where

$$f(x) = |y_1(x)| + \cdots + |y_m(x)|.$$

Since $f \in \mathcal{L}^2(w;\ c, \infty)$, inequality (3.14) and Schwarz's inequality yield that

$$\ln(x/c) = \int_c^x dt/t \leqq M \left[\int_c^x wf^2\, dt + \int_c^x t^{-1/2} w^{1/2} f\, dt \right]$$

$$= O\left(\int_c^x dt/t \right)^{1/2} = O([\ln x/c]^{1/2})$$

as $x \to \infty$. This contradiction completes the proof.

As an example of this theorem set $w = 1$, $p_1 = p_0 = q_1 = 0$ in (3.4); then

$$L[y] = -i(q_0(q_0 y')')'.$$

Suppose also the integrals

$$(3.15) \qquad y_1(x) = \int_x^\infty 1/q_0,\ y_2(x) = \int_x^\infty y_1(t)/q_0(t)\, dt$$

are finite and that y_1 and y_2 are in $\mathcal{L}^2(c, \infty)$ for some c. Then $N_+(\infty) = N_-(\infty) = 2$ since $L[y_i] = 0$, $i = 1, 2$, and from [2, p. 1398] the number of linearly independent $\mathcal{L}^2(c, \infty)$ solutions of $L[y] = 0$ does not exceed the minimum of $N_+(\infty)$, $N_-(\infty)$. For $q_0(x) = x^\beta$, $1 \leqq x < \infty$, the integrals in (3.15) are finite for $\beta > 1$ and are in $\mathcal{L}^2(1, \infty)$ for $\beta > 3/2$.

4. **Deficiency index theorems for a finite singularity.** We consider here a finite singular endpoint of the interval (a, b). For convenience we assume it to be at $a = 0$. In the equation $L[y] = \lambda y$ with $L[\cdot]$ given by (1.4), make the transformation (assume $b < \infty$)

$$Z(t) = R(t)Y(1/t),\ t \geqq 1/b,$$

where Y is as in (2.2) and $R(t)$ is the diagonal matrix,

$$R(t) = \operatorname{diag}\{t^{m-1}, t^{m-3}, \cdots, t^{-m+1}\}.$$

Then from (2.6) we have $(\cdot = d/dt)$

$$\begin{aligned} \dot{Z}(t) &= (-1/t^2)R(t)Y'(1/t) + \dot{R}(t)T(1/t) \\ &= [(-1/t^2)R(t)(\lambda C(1/t) + D(1/t))R^{-1}(t) \\ &\quad + \dot{R}(t)R^{-1}(t)]Z(t) + (-1/t^2)R(t)G(1/t) \qquad (4.1) \\ &= [\lambda\tilde{C}(t) + \tilde{D}(t) + \dot{R}(t)R^{-1}(t)]Z(t) + (-1/t^2)R(t)G(1/t) \end{aligned}$$

where $\tilde{C}(t) = (-1/t^2)R(t)C(1/t)R^{-1}(t)$ and $\tilde{D}(t) = (-1/t^2)R(t)D(1/t)R^{-1}(t)$. Note that D and $\tilde{D}$ and C and $\tilde{C}$ are related by:

$$\tilde{D}_{kj}(t) = -D_{kj}(1/t)t^{2(j-k)-2}(k, j = 1, \cdots, m),$$
$$\tilde{C}_{m1}(t) = -C_{m1}(1/t)t^{-4n-2} = w(1/t)t^{-4n-2} \equiv \tilde{w}(t).$$

The argument of Lemma 2.3 may be applied to the system (4.1). It differs from the system of Lemma 2.3 only in the additional diagonal terms of $\dot{R}(t)R^{-1}(t)$. As long as $\rho^2(t)/g(t)\,\dot{R}(t)R^{-1}(t)$ is bounded, the argument is the same as before. This requirement is equivalent to $\rho^2(t)t/w(1/t)^{1/m}$ being bounded as $t \to \infty$. Conditions (v)–(vii) of Theorem 4.1 below are the analogues of conditions (v)–(vii) of Lemma 2.3. Condition (ix) is proved in the same manner as Theorem 3.1. Although we omit the details, the function σ below is related to ρ by $\rho(t) = \sigma(1/t)$ and we note that for $y \in \Delta$,

$$\int_0^b w(x)|y(x)|^2\,dx = \int_{1/b}^{\infty} \tilde{w}(t)|z_1(t)|^2\,dt,$$

with similar relations holding among the derivatives of y.

THEOREM 4.1. *Suppose* (1.1) *holds,* $a = 0$, $b < \infty$, q_0 *and* w *are continuously differentiable, and there is a continuously differentiable positive function* σ *on* $(0, b]$ *such that*

(i) $$\frac{\sigma^2}{w^{1/m}}\left[\frac{|w'|}{w} + \frac{|q_0'|}{q_0} + \frac{|\sigma'|}{\sigma} + \frac{1}{x}\right] = O(1) \text{ as } x \to 0,$$

(ii) $$\frac{|p_i|\sigma^{4i+2}}{q_0^{\,2}w^{(2i+1)/m}} = O(1) \text{ as } x \to 0\ (i = 0, \cdots, n),$$

(iii) $$\frac{|q_i|\sigma^{4i}}{q_0^{\,2}w^{2i/m}} = O(1) \text{ as } x \to 0\ (i = 1, \cdots, n),$$

(iv) $$\frac{\sigma^m}{q_0} = O(1) \text{ as } x \to 0.$$

Then for $y \in \Delta$,

(v) $$\int_0^b w^{(m-2i+2)/m}\sigma^{4i-4}|y^{[i-1]}|^2\,dx < \infty,\ i = 1, \cdots, n,$$

(vi) $$\int_0^b w^{1/m}\sigma^{4n}q_0^{-2}|y^{[n]}|^2\,dx < \infty,$$

(vii) $$\int_0^b w^{(m-2i+2)/m}\sigma^{4i-4}q_0^{-4}\,|y^{[i-1]}|\,dx < \infty,\ i = n+2, \cdots, m.$$

Moreover, if also

(viii) $$\int_0^b w^{1/m}\sigma^{4n}/q_0^{\,2} = \infty,$$

then

(ix) $$N_+(0) = n + 1 \text{ and } N_-(0) = n.$$

Considering again (3.4) with the coefficients given by (3.7), we choose

$$\sigma(x) = x^\eta,\ \eta = (2\beta - 1 - \xi/3)/4.$$

Then condition (viii) above holds and (i)–(iv) above are equivalent to

$$\alpha \geqq 1 + \xi,\ 2\delta \geqq 2\beta + (1 + \xi),\ \beta + \gamma \geqq 3(1 + \xi)/2,\ \beta \geqq 3/2 + \xi/2. \tag{4.2}$$

These inequalities are the reverse of those given by (3.8).

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UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916