ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Number 4, Fall 1978

ENVELOPING W^*-ALGEBRAS

JOHN DAUNS

ABSTRACT. Starting witth a **C** (complex)-algebra R having a conjugate linear (idempotent) involution on R, and a set P of positive **C**-linear functionals on R (with no topology on R), the $\mathbf{C}^*$ and W^*-enveloping algebras of R are constructed. They are uniquely determined by universal mapping properties (Theorem IV).

Let S be an involutive semi-topological semigroup (not necessarily locally compact), H any Hilbert space, and $L(H)$ the W^*-algebra of all bounded operators in the σ (ultraweak) topology. As a special case with $R = \mathbf{C}S$ the semigroup algebra, and with P arising from a set of bounded continuous positive definite functions on S, the $\mathbf{C}^*$ and W^*-enveloping algebras **A**S and **W**S of S are obtained. There is a map $S \to S/\Omega$, and $S \to S/\Omega \subset \mathbf{A}S \subset \mathbf{W}S$, where S/Ω is the image of S in **A**S. Then **W**S is uniquely determined by the universal property that any σ-continuous representation $S \to L(H)$ factors uniquely through a (σ-continuous homomorphism $S \to \mathbf{W}S \to L(H)$ of the W^*-algebras **W**S and $L(H)$.

0. **Introduction.** There is such a large number of duality theories for various classes of semigroups and groups, some of them overlapping, that old ones are as quickly forgotten as new ones are invented. Thus any new development that would in some way tend to simplify, unify, and generalize these would be welcome. A duality can perhaps be regarded as a functor from some category of semigroups to some well defined category of $\mathbf{C}^*$ or W^*-algebras. For example, in [8] this functor is an equivalence of categories.

Such a functor should preserve as much of the available structure on the given category of semigroups or groups as possible. Although the direct product of semigroups in itself is of minor interest, its importance lies in the fact that in subsequent generalizations of duality it should single out the appropriate subcategory for the equivalence among all the W^*-algebras, namely those that carry an additional coalgebra structure. Thus the direct product of semigroups should map into a tensor product of W^*-algebras.

Frequently, duality theories ([6] and [9]) use C^* and W^*-tensor products that do not have the usual universal mapping property. Here we will be forced to use the categorical W^*-tensor product ([7]); others simply would not work because they lack the above multiplicative property. Also, here the W^*-tensor product will have to be defined in terms of a (tensor) product on the preduals of W^*-algebras.

Received by the editors on April 19, 1976, and in revised form on January 13, 1977.

The present paper is motivated by what seems presently an unattainable objective—to generalize [19] to some bigger class of not necessarily locally compact groups by use of the more categorical concepts and techniques of [8]. In [19] each locally compact group G is recovered from its W^*-enveloping algebra $W^*(G)$ with the aid of the predual of $W^*(G)$ by use of a long, technical and difficult argument using $L^1(G)$ and Haar measure. As the latter two are no longer available in the non-locally compact case, also, all the proofs in [19] would fail.

In the locally compact case, the indispensable algebra W^*G is the von-Neumann or W^*-enveloping algebra of $\mathbf{C}^*(G)$, the $\mathbf{C}^*$-enveloping algebra of G. The algebra $\mathbf{C}^*(G)$ is constructed from two ingredients: an involutive complex algebra $L^1(G)$ and a set $P(G)$ of positive functionals on $L^1(G)$, where $P(G)$ are the continuous, positive definite functions on the group G. The usual presently accepted and used $\mathbf{C}^*$-enveloping algebra $\mathbf{C}^*(G)$ of G has three drawbacks limiting its usefulness. It does not always contain G. Secondly, it contains an identity if and only if G is discrete. Lastly, arbitrary continuous homomorphisms of groups $G \to H$, do not induce a map $L^1(G) \to L^1(H)$, and consequently there is no map $\mathbf{C}^*(G) \to \mathbf{C}^*(H)$.

A general construction will be given that not only will include the above usual $\mathbf{C}^*(G)$ as a special case, but also simultaneously will be used to construct a new $\mathbf{C}^*$-enveloping algebra remedying all of the above three defects. Moreover, the group G will be replaced by a not necessarily locally compact involutive semigroup S, where, in addition, multiplication on S need only be separately continuous. The semigroup S may contain also a zero element.

However, the appropriate framework within which to develop the subject is to begin with an involutive complex algebra R with no topology, some given set of positive linear functionals P on R, and then manufacture a semi-norm on R. In the locally compact group case, take $R = L^1(G)$ and $P = P(G)$. Having chosen $P = P(G)$, the L^1-norm on $L^1(G)$ becomes superfluous for the purpose of constructing $\mathbf{C}^*(G)$.

For a semigroup S, to obtain the universal $\mathbf{C}^*$ and W^*-enveloping algebras AS and **W**S, R is specialized to R = CS, the ordinary (algebraic) semigroup algebra. Both AS and **W**S are uniquely determined by universal mapping properties. In the locally compact group case when S = G, this process will produce the replacement AG of $\mathbf{C}^*(G)$. Since $W^*(G)$ also has the universal property uniquely determining **W**G, it follows that $W^*(G) \cong \mathbf{W}G$. However, $\mathbf{C}^*(G) \cong \mathrm{A}G$ if and only if G is discrete. If G is a locally compact nondiscrete topological group, let G_d be G with the discrete topology, and $\mathbf{C}^*(G_d)$ the usual $\mathbf{C}^*$-enveloping algebra of G_d ([6; p. 188(1.18)]). Since $\mathbf{C}^*(G)$ does not contain an identity while AG does, AG $\neq$ $C^*(G)$. Furthermore $AG \neq C^*(G_d)$ (see 7.7, 7.8). With this purpose in mind, one of the aims here is to establish the existence on AS and WS all those objects, functions, involutions, and relations that are available in the case of $S = G$, a locally compact group as in [6]. However, also some new spaces and functions are defined and investigated which also are bound to play a key role in subsequent later developments. One such is the co-commutative coalgebra structure on AS and WS.

In this connection it becomes necessary to extend homomorphisms and even other simply linear maps from the subalgebra AS of WS to all of WS. Although here so far this is easily done simply because WS is the double dual WS $=$ $(AS)^{**}$, it seemed advisable to devise a very general method of extending certain linear maps from an ultraweakly dense subalgebra to the whole W^*-algebra, which is of independent interest by itself.

The reader is cautioned against confusing the symbol "$A(G)$" as used in [6; p. 182, p. 209] with the totally different algebra "AG"used here.

1. **Properties of Arbitrary W^*-algebras.** This section is of independent interest from a W^*-point of view. First, some facts are summarized in a form and in notation in which they later will be used frequently. For a W^*-algebra N and a subalgebra $D \subset N$, a generally applicable method of extending a homomorphism defined at first only on D, from D to all of N is given. In the usual ultraweak topology σ on N, N is not complete as a topological space. Another topology $\mathscr{S}$ has to be used in which N is complete. Since the extended map is to remain also a ring homomorphism, continuity of multiplication in the four non-norm topologies on N has to be considered.

1.1. If A is any Banach algebra with an involution and an approximate bounded identity, then the norm dual A^* is an A^{**}-bimodule, where A^{**} has the Arens multiplication. There is a natural embedding $\eta : A \to A^{**}$. The duality between any space A and any set such as A^* of linear functionals on A will be denoted by $\langle \, , \rangle : A^* \times A \to \mathbf{C}$. However, write $[\, ,] : A^{**} \times A^* \to \mathbf{C}$ for emphasis in order to distinguish this particular special case. Let a, b, $x \in A$, $f \in A^*$, and F, $G \in A^{**}$. Thus $[\eta a, f] = \langle f, a \rangle$. For any function f whatever on any multiplicative semigroup A, afb is defined as $\langle afb, x \rangle = \langle f, bxa \rangle$, and similarly for af, fb. Furthermore

$$Ff \in A^*: \quad \langle Ff, a \rangle = [F, fa]$$

$$fF \in A^*: \quad \langle fF, a \rangle = [F, af]$$

$$FG \in A^{**}: \quad [FG, f] = [F, Gf].$$

Always A^* will be endowed with the involution $\langle f^*, a\rangle = \langle f, a^*\rangle^-$. If in addition A is a C*-algebra, then $[F^*, f] = [F, f^*]^-$ defines an involution on A^{**}, and in this case FG is also the same as $[FG, f] = [G, fF]$. Let E be the functor $EA = A^{**}$. For a C*-algebra, $A \subset EA$ is its W^*-enveloping algebra.

1.2. The smallest topology on any topological space A making any set of complex-valued functions P on A continuous will be written as $\sigma(A, P)$. If P is any subset of a vector space, $\langle P\rangle$ denotes all finite complex linear combinations of elements of P. In this case $\sigma(A, P) = \sigma(A, B)$, where $B = \langle P\rangle$.

For A and A^* as before in 1.1, let

$$A^{*+} = \{f \in A^* \mid \text{if } 0 \leqq a \in A, \text{ then } 0 \leqq f(a)\}$$

and suppose $P \subseteq A^{*+}$ is any subset closed under multiplication by positive real scalars. The set of semi-norms obtained from

$$a \to p(a^*a),\ a \in A,$$

with $p \in P$ ranging over any subset such as P induces a topology $\mathscr{S}(A, P)$ on A. Write $\mathscr{S}(A, P) = \mathscr{S}(A, B)$. If in addition to $p(a^*a)$, also $a \to p(aa^*)$ are used, we obtain $\mathscr{S}^*(A, P) \equiv \mathscr{S}^*(A, B)$ with $\mathscr{S}(A, B) \leqq \mathscr{S}^*(A, B)$. In case A does not vary, abbreviate all of the above as $\sigma(B)$, $\mathscr{S}(B)$, and $\mathscr{S}^*(B)$. Since $A^* \subset (EA)^*$, replacement of A by EA gives extensions of the previous topologies $\sigma(EA, B)$, $\mathscr{S}(EA, B)$, and $\mathscr{S}^*(EA, B)$ to EA.

1.3. For any W^*-algebra M with predual M_*, write $\mathscr{S} = \mathscr{S}(M, M_*)$, $\mathscr{S}^* = \mathscr{S}^*(M, M_*)$, $\sigma = \sigma(M, M_*)$, and $\tau = \tau(M, M_*)$ where the latter is the Mackey topology on M of uniform convergence on absolutely convex relatively $\sigma(M_*, M)$—compact subsets of M.

1.4. For a C*-algebra A, take $M = EX = A^{**}$. Then $M_* = A^*$, $\mathscr{S} = \mathscr{S}(A^{**}, A^*)$, and $\mathscr{S}^* = \mathscr{S}^*(A^{**}, A^*)$. The completion of any topological vector space $(A, \mathscr{S})$ as a set is a topological vector space that will be denoted by $(A, \mathscr{S})^-$ with a topology that will also be denoted by $\mathscr{S}$. It is known that $(A, \mathscr{S})^- = (A, \mathscr{S}^*)^- = A^{**}$.

Some topological facts that are to be used later about the various topologies on a W^*-algebra are summarized below.

LEMMA 1.5. *Suppose M is any W^*-algebra with predual M_* and the topologies σ, $\mathscr{S}$, $\mathscr{S}^*$, and τ as in* 1.3. *If $K \subset M$ is any norm bounded set, then*

(i) $\sigma \subseteq \mathscr{S} \subseteq \mathscr{S}^* \subseteq \tau$.

(ii) *Involution is σ, $\mathscr{S}^*$, and τ-continuous, (but not in $\mathscr{S}$ in general).*

(iii) *The hermitian (or self adjoint) elements of M as well as the positive cone M^+ is closed in any of* $\sigma \subseteq \mathcal{S} \subseteq \mathcal{S}^* \subseteq \tau$.

(iv) $\mathcal{S}^*$ *and* τ *agree on* K.

(v) *If in addition* $M = A^{**}$ *for some* C*-*algebra as in* 1.4 *and* $D \subset A$ *is an involutive (not necessarily complete) norm dense subalgebra such that the set of norm continuous positive linear functionals* D^{*+} *on* D *is precisely* $D^{*+} = A^{*+}$, *then* $D \subset A^{**}$ *is dense in any one of* $\sigma \subseteq \mathcal{S} \subseteq \mathcal{S}^* \subseteq \tau$.

Proof. (i) That $\sigma \subseteq \mathcal{S} \subseteq \mathcal{S}^* \subseteq \tau$ follows from [13; p. 20, Theorem 1.8.9].

(ii) It is clear that involution is σ and $\mathcal{S}^*$-continuous while τ-continuity is shown in [13; p. 19, Proposition 1.8.5].

(iii) In any locally convex Hausdorff topological vector space, such as M with the σ-topology, the closure of any real convex set is the same for any topology between $\sigma \subseteq \tau$, because the dual of M, σ is M_*. Since by [13; p. 14, Lemma 1.7.1], M^+ as well as the self adjoint elements of M are σ-closed, they are also closed in all of the above topologies.

(iv) Conclusion (iv) is given in [13; p. 21, Remark 1.8.9].

(v) Since $(D, \| \|)^- = (A, \| \|)$, we have $(D, \mathcal{S})^- = (A, \mathcal{S})^-$, while $(A, \mathcal{S})^- = A^{**}$. By (iii), D is dense in all of $\sigma \subset \mathcal{S} \subset \mathcal{S}^* \subset \tau$.

Lemma 1.6. *With the same notation and hypotheses as in* 1.5, *multiplication has the following continuity properties*:

(vi) *Multiplication* $M \times M \to M$ *is separately continuous in all topologies* $\sigma \times \sigma \to \sigma$, $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$, $\mathcal{S}^* \times \mathcal{S}^* \to \mathcal{S}^*$, *and* $\tau \times \tau \to \tau$.

(vii) $K \times M \to M$ *is jointly continuous in* $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ *but not* $M \times K \to M$).

(viii) $K \times K \to K$ *is jointly continuous in* $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$, $\mathcal{S}^* \times \mathcal{S}^* \to \mathcal{S}^*$, *and* $\tau \times \tau \to \tau$ *(but not* $\sigma \times \sigma \to \sigma$).

Proof. (vi) Conclusion (vi) follows:

for σ : from [13 | p. 18, Theorem 1.7.8];
for $\mathcal{S}$: from [13 | p. 21, Proposition 1.8.12];
for $\mathcal{S}^*$: —same proof as for $\mathcal{S}$ works; and
for τ : from [13 | p. 19, Proposition 1.8.12].

(vii) This is shown in [13 | p. 21, Proposition 1.8.5].

(viii) Conclusion (vii) for $\mathcal{S}$ comes from (vii); while the right hand analogue of (vi) establishes it for $\mathcal{S}^*$. Finally for τ, it follows from 1.5 (iv).

Lemma 1.7. *If U is the unitary group of a W*-algebra M, then*

(i) *all the topologies* $\sigma \subseteq \mathcal{S} \subseteq \mathcal{S}^* \subseteq \tau$ *agree on* U;

(ii) (U, σ) *is a topological group.*

PROOF. (i) Fix $F \in U$. If $F, G \in U$, *then* $(G^* - F^*)(G - F) = -(G^* - F^*)F - F^*(G - F)$. Since multiplication by a fixed element such as F or F^* and involution are σ-continuous (by 1.6 (vi) and 1.5 (ii)), the map $U \to M$, $G \to (G^* - F^*)(G - F)$ is σ-continuous. Thus $\mathscr{S}^* \mid U \subseteq \sigma \mid U$, while always $\sigma \subseteq \mathscr{S}^*$. Consequently $\sigma \mid U = \mathscr{S}^* \mid U$. By 1.5 (iv), $\mathscr{S}^* \mid U = \tau^* \mid U$ and therefore $\sigma, \mathscr{S}, \mathscr{S}^*$, and τ all agree on U.

(ii) By (1.5) (ii), involution or inversion on U is continuous, while by 1.6 (viii) multiplication is jointly continuous on U. Hence U is a topological group.

For involutive algebras $D \subset M$, D^+ is defined as the set of all finite sums of d^*d, $d \in D$. Thus both containments $\{d^*d \mid d \in D^+\} \subseteq M^+ \cap D$ are in general proper.

LEMMA 1.8. *Suppose M is any W^*-algebra and $D \subset M$ a not necessarily norm closed, self-adjoint and $\sigma(M, M_*)$-dense subalgebra. If K is the unit ball of M, then $\{d^*d \mid d \in D\} \cap K \subseteq M^+ \cap K$ is dense in any one of the topologies $\sigma \subseteq \mathscr{S} \subseteq \mathscr{S}^* \subseteq \tau$, and in particular, so is $D^+ \cap K$.*

PROOF. Let $x = y^*y$, $y \in M^+ \cap K$. Since $D \subset M$ is σ-dense, by Kaplansky's density theorem ([13; p. 22, Theorem 1.9.1]), $D \cap K \subseteq M \cap K$ is τ-dense. Thus there is a net $\{t(\alpha)\} \subseteq D \cap K$ with $t(\alpha) \to y$ in τ. Since involution and multiplication are τ-continuous on K, $t(\alpha)^*t(a) \to x$ in τ. But since $\sigma \subseteq \mathscr{S} \subseteq \mathscr{S}^* \subseteq \tau$, convergence in τ implies convergence in all of these topologies. Hence the closures of both of the sets $\{d^*d \mid d \in D\} \subseteq D^+ \cap K$ are equal to $M^+ \cap K$ in any one of the topologies $\sigma, \mathscr{S}, \mathscr{S}^*$, and τ.

COROLLARY 1.9. *With $D \subset M$ as in 1.7, if $x \in M^+$ with $\|x\| = 1$, then there exists a net $\{t(\alpha)\} \subseteq D$ with $\|t(\alpha)\| \leqq 1$ such that $t(\alpha)^*t(\alpha) \to x$ in any one of the topologies $\sigma \subseteq \mathscr{S} \subseteq \mathscr{S}^* \subseteq \tau$.*

LEMMA 1.10. *A positive $\mathscr{S}$-continuous real linear map $T: M \to N$ of W^*-algebras M and N is also σ-continuous.*

PROOF. A linear map $T: M \to N$ is normal if (i) $Tm \geqq 0$ if $m \geqq 0$ and (ii) T preserves least upper bounds of uniformly order bounded directed sets of self adjoint elements ([**12**; p. 1.52 Definition 10.2]) A normal linear map $T: M \to N$ is σ-continuous by [**12**; p. 1.53, Proposition 10.3]. Alternatively the latter can also be proved directly by using [**13**; p. 28, Theorem 1.13.2] to show that the adjoint T^* of T satisfies $T^*N_* \subseteq M_*$. Thus let $\mathscr{F} \subseteq M$ be any upper directed net of self adjoint elements (indexed by itself) bounded above by a self adjoint element. It may be assumed without loss of generality that $\mathscr{F}$ is also bounded below, that $\|m\| \leqq 1$, and that $-e \leqq m \leqq e$ for all $m \in \mathscr{F}$ (see [**16**; p. 7, Lemma 1]). Any upper directed net $\mathscr{F}$ of self adjoint elements which are bounded above by a self adjoint element in a W^*-algebra converges in $\mathscr{S}$ to $x = \text{l.u.b.}\,\mathscr{F}$ ([**16**; *p.* 7, Lemma 1]). Since T is positive, $T(m^*) = (Tm)^* = Tm$ if $m = m^*$. Thus $\{Tm \mid m \in \mathscr{F}\} \to y \equiv \text{l.u.b.}\ (T\mathscr{F})$ in $\mathscr{S}$ in N. Since T is $\mathscr{S}$-continuous, also $Tm \to Tx$; so $Tx = y$. Thus T is normal and hence σ-continuous.

REMARK 1.11. Any topology, such as $\sigma, \mathscr{S}, \mathscr{S}^*$, or τ induced by a set of seminorms is obtained from a uniform structure. The $\mathscr{S}$-uniform structure is $\{\{(x, y) \in M \times M \mid p((x-y)^*(x-y)) < 1\};\ 0 \leqq p \in M_*^+\}$.

Note that in the next theorem $T(D \cap M^+) \subseteq N^+$ is a conclusion, not a hypothesis.

THEOREM 1.12. *Suppose that M, N are W^*-algebras, $D \subset M$ an involution closed σ-dense subalgebra (which is neither assumed norm closed nor to contain the identity of M), and $T: D \to N$ an $\mathscr{S}$-continuous, positive (i.e., $T(d^*d) \geqq 0$ for all $d \in D$), complex (real, or complex conjugate) linear map. Then T extends to a complex (real, or complex conjugate) linear*

(i) *unique positive map $\bar{T}: M \to N$ (which is involution preserving if T is) such that*

(ii) *$\bar{T}$ is continuous in both the σ or $\mathscr{S}$ topologies on M and N.*

(iii) *Furthermore, if T is an algebra homomorphism, then $\bar{T}$ is a homomorphism of W^*-algebras.*

PROOF. (i) and (ii). Note that M is $\mathscr{S}$-complete and denote the $\mathscr{S}$-closure or completion of any set such as $D^+ = \{\Sigma d^*d \mid d \in D\}$ by $\text{cl}(D^+)$.

Since a uniformly continuous map of uniform spaces extends to a uniformly continuous map of their completions, it follows that T extends to a uniformly continuous $\bar{T}: (M, \mathscr{S}) \to (N, \mathscr{S})$. Then since $\bar{T}$ is $\mathscr{S}$-continuous, $\bar{T}(\text{cl}(D^+)) \subseteq \text{cl}\, T(D^+)$. But $T(D^+) \subseteq N^+$, and $N^+ \subset N$ is $\mathscr{S}$-closed ([**13**; p. 14, Lemma 1.7.1]). Thus $\text{cl}\, T(D^+) \subseteq N^+$. Since the closure of D in M in any topology between σ and τ is the same, $D \subset M$ is σ-dense. Now by 1.8 or 1.9, $\text{cl}(D^+) = M^+$. Thus $\bar{T}(M^+) \subseteq N^+$, or $\bar{T}$ is a positive $\mathscr{S}$-continuous map. Hence by 1.10 $\bar{T}$ is also σ-continuous. Since $D \subset M$ is dense, any two continuous extensions of T to M agree

(ii) Since ring multiplication is separately $\mathscr{S}$-continuous, it follows that $\bar{T}$ is a ring homomorphism.

2. **Enveloping Algebras.** Starting with an algebra R over the complexes **C** with an involution, and a set P of positive functionals on R, a semi-norm "$\|\cdots\|$" is constructed on R. After forming $D = R/I$, where $I \subset R$ is the ideal of elements of zero norm, the completion of D yields a C^*-algebra A.

2.1. Consider an algebra R over the complexes $\mathbf{C}$ with a conjugate linear involution "*". The positive cone R^+ of any algebra R will always be defined as the set of all sums of elements of the form d^*d. Suppose that $\{u(\lambda)\}$ is a net of self adjoint ($u(\lambda)^* = u(\lambda)$) elements indexed by some upper directed index set $\{\lambda\}$. In case R has an identity e, assume that $e^* = e$ and that $u(\lambda) = e$ identically for all λ.

2.2. A complex linear functional p on R is *positive* if for all $d \in R$, $p(d^*d) \geqq 0$. Denote this by $0 \leqq p$. Define $\|p\|$ by $\|p\| = \lim p(u)\lambda)$ if the limit exists. For $p \geqq 0$ consider the following axioms or properties which are to hold for all $d \in R$:

(a1) $p(u(\lambda)) \to \|p\|$; $0 \leqq p(u(\lambda)) \leqq \|p\|$ for all λ;
(a2) $p(u(\lambda)^2) \to \|p\|$;
(a3) $p(u(\lambda)d) \to p(d)$, $p(du(\lambda)) \to p(d)$;
(a4) $p(d^*u(\lambda)d) \to p(d^*d)$; $0 \leqq p(d^*u(\lambda)d) \leqq p(d^*d)$ for all λ;
(a5) $p(d^*u(\lambda)^2d) \to p(d^*d)$;
(a6) $p(b^*u(\lambda)d^*du(\lambda)b) \to p(b^*d^*db)$ for all $b, d \in R$.

Note that (a1)–(a6) are completely trivial if $e \in R$.

REMARK 2.3. Let $0 \leqq p$ satisfy (a1)–(a5) and let $a, b \in R$ be arbitrary. Then $p(b^*a)$, $p(b^*u(\lambda)a)$, and $p(b^*u(\lambda)^2a)$ are three sesqui-linear positive forms. It follows from the polarization identity that $p(au(\lambda)b) \to p(ab)$ and $p(au(\lambda)^2b) \to p(ab)$ for all $a, b \in R$.

2.4. Suppose that P is some given set of positive $\mathbf{C}$-linear functionals on R closed under multiplication by positive real scalars. From now on it will always be assumed that in addition to (a1)–(a5), P also satisfies

(A1) $\qquad dpd^* \in P$ for any $p \in P$, $d \in R$.

Note that if (a1)–(a4) and A1 hold for P, then (a5) also holds. For $d \in R$ define $\|d\|$ only in case it is finite by

$$\|d\|^2 = \sup\{p(d^*d) \mid p \in P, \|p\| = 1\}$$

Sometimes it will be necessary to assume also that

(A2) $\qquad \exists\|d\|$ for all $d \in R$.

For P as above (satisfying (a1)–(a5) and A1), define I as $I = \{d \in R \mid \|d\| = 0\}$.

The proof of the next lemma can be considerably simplified ([5; p. 22, Proposition 2.1.5]) if R is a Banach algebra. Inequalities like 2.5 (vii) below will be assumed to hold in case $\|a\|$ is undefined, i.e., infinite.

LEMMA 2.5. *If* $0 \leqq p$ *satisfies* (a1)–(a5), *then the following hold for all* $a, b \in R$:

(i) $\overline{p(a^*)} = p(a)$;

(ii) $|p(a)|^2 \leqq \|p\| \min[p(a^*a), p(aa^*)]$;

(iii) $\|p\| = 0 \Longleftrightarrow p = 0$;

(iv) $\|bpb^*\| = p(b^*b)$; $p(b^*b) = 0 \Longleftrightarrow p(b^*Rb) = (0)$.

(v) $|p(b^*ab)|^2 \leqq p(b^*b)(bpb^*)(a^*a)$

If $\|\cdots\|$ *on* R *is defined by some* P *as in* 2.4 *with* $p \in P$, *then*:

(vi) $|p(b^*ab)|^2 \leqq p(b^*b)^2\|a\|^2$

(vii) *If* $\{a(i)\} \subset R$ *with* $\|a(i)\| \leqq 1$ *is a net such that* $|p(a(i))| \to \|p\|$, *then also* $p(a(i)^*a(i)) \to \|p\|$.

PROOF. (i) Since $p(b^*a) = \overline{p(a^*b)}$ for all, $a, b \in R$, (i) follows by (a3) with $b = u(\lambda)$.

(ii) and (iii) By the Cauchy-Schwartz inequality, $|p(u(\lambda)a)|^2 \leqq p(a^*a)p(u(\lambda)^2)$. Taking limits on both sides, we get (ii) $|p(a)|^2 \leqq p(a^*a) \|p\|$ by (a2). Conclusion (iii) now follows from (ii).

(iv) By (a1) and (a4), $\|bpb^*\| = \lim bpb^*(u(\lambda)) = p(b^*b)$. Now by (iii), $p(b^*b) = 0$ implies that $p(b^*ab) = 0$, for all $a \in R$.

(v) If $p(b^*b) = 0$, (v) holds. So let $p(b^*b) \neq 0$ and set $f = bpb^*/p(b^*b)$. By (iv), $\|f\| = 1$. Application of (ii) to f gives $|f(a)|^2 \leqq \|f\|f(a^*a)$, and hence (v)

$$|p(b^*ab)|^2 \leqq p(b^*b)p(b^*a^*ab).$$

(vi) Again, if $p(b^*b) = 0$, then (vi) holds by (v). Let $p(b^*b) \neq 0$. If $\|a\|$ is not finite, (v) holds. Since $f = bpb^*/p(b^*b) \in P$ and $\|f\| = 1$, (vi) now follows from (v).

(vii) If $\|p\| = 0$, then $p = 0$, and (vii) holds. Let $\|p\| \neq 0$. Then $p/\|p\|$ is of norm one, and

$$p(a(i)^*a(i)) \leqq \|p\| \, \|a(i)\|^2 \leqq \|p\|$$

by the definition of "$\|a(i)\|$" in 2.4. By (ii) and by the last estimate, $|p(a(i))|^2 \leqq \|p\| \, p(a(i)^*a(i)) \leqq \|p\|^2$. Dividing by $\|p\|$ and taking limits, we get

$$\begin{aligned}\|p\| &\leqq \liminf p(a(i)^*a(i)) \\ &\leqq \limsup p(a(i)^*a(i)) \leqq \|p\|.\end{aligned}$$

LEMMA 2.6. *With* P *satisfying* (a1)–(a5), *for any* $p \in P$, *define* $|||p|||$ *by* $|||p||| = \sup\{p(b)/\|b\| \mid b \in R;\ \exists\|b\|$, *i.e., is finite*; $\|b\| \neq 0\}$. *Then for any* $p \in P$, $|||p||| = \|p\|$.

PROOF. If $\|p\| = 0$, then also $p = 0$, and $|||p||| = 0$. So take $\|p\| \neq 0$. Then by (a1), $\|u(\lambda)\| \leqq 1$. Thus $|p(u(\lambda))| \leqq |||p|||\ \|u(\lambda)\| \leqq |||p|||$. Again, by (a1) $\|p\| \leqq |||p|||$.

By 2.4, $|p(b^*b)| \leqq \|p\|\ \|b\|^2$, while 2.5 (ii) shows that $|p(b)|^2 \leqq \|p\|p(b^*b) \leqq \|b\|^2\|p\|^2$ for all $b \in R$. Consequently, $|||p||| \leqq \|p\|$. Hence $\|p\| = |||p|||$ or $\|\cdots\| = |||\cdots|||$.

PROOF 2.7. *For a positive functional q on R satisfying* (a1)–(a5), *let L be the set of all positive real multiples of $\{bqb^* \mid b \in R\}$. Then L satisfies* (a1)–(a5) *and* A1.

PROOF. (a1): By (a4) for q we have $0 \leqq bqb^*(u(\lambda)) \leqq q(b^*b)$, and by (a4) for q that $q(b^*b) = \lim bqb^*(u(\lambda)) \equiv \|bqb^*\|$. The rest are obtained as follows:

(a2): by (a5) for q;
(a3): by (a3) and 1.4;
(a4); (a5), and A1: from (a4), (a5), and A1.

REMARK 2.8. With q and L as above in 2.5, A2 holds for L if and only if for each $d \in R$ the following is finite:

$$\|d\|^2 = \sup\{q(b^*d^*db)/q(b^*b) \mid b \in R,$$
$$q(b^*b) \neq 0\}.$$

From now on the notation "$\lhd$" will indicate ideals in a ring.

2.9. For q and L as in 2.7, in addition to (a1)–(a5), A1 assume also that (a6) holds for q and A2 for L. If N is the left ideal $N = \{b \in R \mid q(b^*b) = 0\} \subset R$, then R/N is a pre-Hilbert space with inner product $(a + N \mid b + N) \equiv q(b^*a)$, whose completion yields a Hilbert space. Each $d \in R$ gives a linear map πd: $R/N \longrightarrow R/N$ by $\pi d(a + N) = da + N$, $a \in R$. (So far A2 was not needed.) If $\|d\|$ is as in 2.4 and $\|\pi d\|$ is the operator norm in the C*-algebra $L(H)$ of all bounded operators on H, then

$$\|d\|^2 = \sup\{q(b^*d^*db)/q(b^*b) \mid b \in R/N\}$$
$$= \|\pi d\|^2.$$

By A2, both are finite and πd extends to $\pi d \in L(H)$. Thus $\pi : R \longrightarrow L(H)$ is an involutive isometric ring homomorphism with kernel

$$I = \{d \in R \mid \|d\| = 0\} \lhd R.$$

Note that also $I = \{d \in R \mid dR \subset N\}$ and $I = \{d \mid q(b^*d^*db) = 0\ \forall b \in R\}$. Here for the first time (a6) is required to show that for any

$b \in R$, $bu(\lambda) + N \to b + N$ in the Hilbert space norm in R/N. In particular, if $d \in I$, then $N = du(\lambda) + N \to d + N$, and hence $d \in N$. Thus $I \subseteq N$.

It follows from $\|d\| = \|\pi d\|$, from kernel $\pi = I$, and $R/I \cong \pi R \subset L(H)$, that $I^* = I$ and that the seminorm on R induces a C*-algebra norm on R/I by $\|d + I\| = \|d\|$, except that R/I need not be complete. Since by 2.6 (ii) L vanishes on I, L can be viewed as functionals on R/I. It follows either from 2.6 or from 2.5 (vi) that the members of L are continuous on R and R/I.

The previous considerations immediately generalize from one functional q as above to a set P of any size.

THEOREM 2.10. *Suppose P satisfies axioms* (a1)–(a6), A1, *and* A2 (2.2) *and that R, $\|\cdots\|$ on R, and I are as previously* (*see* 2.1, 2.4, 2.9). *Then R is an algebra with an algebra semi-norm, and in particular for a, $b \in R$ the following hold*

(*i*) $\|ab\| \leqq \|a\|\,\|b\|$;

(ii) $\|a^*\| = \|a\|$;

(iii) $\|a^*a\| = \|a\|^2$:

(iv) $I \lhd R$; $I = \{a \in R \mid \|a\| = 0\}$;

(v) $I^* = I$;

(vi) *The completion of R/I in the norm $\|a + I\| = \|a\|$ is a* C*-*algebra.*

(vii) *The functionals of P vanish on I and hence induce functionals on R/I; the members of P are continuous on R and R/I.*

COROLLARY 2.11. *Under the hypotheses of the previous theorem with $D = R/I$, the following hold*

(i) $\{u(\lambda)\}$ (or $\{u(\lambda) + I\}$) *is a bounded, self adjoint approximate identity for R (D).*

(ii) *D and P satisfy* (a1)–(a6) A1, *and* A2 (*with respect to* $\{u(\lambda) + I\}$).

(iii) *The set I defined for D* (*by* 2.4) *is zero.*

PROOF. Conclusions (ii) and (iii) are clear, while for (i) it suffices to show that for $d \in R$, both $\|u(\lambda)d - d\| \to 0$ and that $\|du(\lambda) - d\| \to 0$ in R. The first follows from (a5), the second from (a6).

COROLLARY 2.12. *With the same notation and hypotheses as in the previous theorem, if* $\mathbf{A}$ *is the* C*-*algebra obtained by completing R/I in its norm* (2.10 (vi)), *then*:

(i) $\exists\, 1 = e \in \mathbf{A} \Longleftrightarrow \{u(\lambda) + I\} \subset R/I \subset \mathbf{A}$ *is a Cauchy net.*

(ii) $\exists\, 1 = e \in \mathbf{A} \Rightarrow \|u(\lambda) + I - e\| \to 0$.

REMARKS 2.13. 1. There does not seem to be a straightforward way of proving 2.10 (ii) and 2.10 (iii) directly without 2.9.

2. Alternatively, perhaps the condition 2.10(ii) could perhaps be used to replace some of the axioms (a1)–(a6), A1, and A2.

3. The second parts of axioms (a1) and (a4) could be omitted at the expense of having to abandon 2.6.

3. **Functorial and Universal Properties of Enveloping Algebras.** The completion $\mathbf{A}$ of the algebra R/I of the previous section is the C*-enveloping algebra of R, while $\mathbf{A}^{**}$ is its W^*-enveloping algebra. Both $\mathbf{A}$ and $\mathbf{A}^{**}$ are uniquely characterized by their universal mapping properties. Eight functors are associated with R. The notation of the previous two sections is continued without further explanations.

3.1. Under the above hypotheses (a1)–(a6), A1, and A2, define $D = R/I$. The completion of D in its norm is a C*-algebra $\mathbf{A}$. Set $\mathbf{W} = E\mathbf{A} = \mathbf{A}^{**}$. The linear functionals induced by P on D, $\mathbf{A}$, as well as $\mathbf{A}^{**}$ will for all of these be denoted by P. All finite, complex linear combinations of any subset such as P of any vector space will be denoted $\langle P \rangle$. For P as above define $B = \langle P \rangle$. Again, view $B \subseteq \mathbf{A}^*$ so that $P \subseteq \mathbf{A}^{*+}$ is in the positive cone of the norm dual of $\mathbf{A}$. Let $\bar{\mathbf{B}}$ be the norm closure of B in $\mathbf{A}^*$. Thus $P \subseteq B \subseteq \bar{\mathbf{B}} \subseteq \mathbf{A}^*$. The algebras $\mathbf{A}$ and $\mathbf{W}$ will be called the C* and W^*-enveloping algebras of R.

3.2. Sometimes I; D, $\mathbf{A}$, $\mathbf{W}$; P, B, $\bar{\mathbf{B}}$, $\mathbf{A}^*$ will be regarded as functors, in which case an index R will be written after each of these, i.e.

$$IR;\ DR \subseteq \mathbf{A}R \subseteq \mathbf{W}R;\ PR \subset BR \subseteq \bar{\mathbf{B}}R \subseteq \mathbf{A}^*R.$$

Regarding PR as functionals on D, by 2.11 (ii) we have

$$I(DR) = 0;\ D(R/IR) = DR,$$

and hence the functors D, $\mathbf{A}$, $\mathbf{W}$; P, B, $\bar{\mathbf{B}}$, and $\mathbf{A}^*$ agree on R and DR. Consequently, $DR = D(R/IR)$ and $\mathbf{A}R = \mathbf{A}DR$ will sometimes be identified, and similarly for the other functors in place of $\mathbf{A}$.

3.3. Consider an (i) involutive, normed, not necessarily complete algebra D over $\mathbf{C}$ with norm $|\cdots|$ ($|ab| \leqq |a|\,|b|$ and $|a^*| = |a|$ for all a, $b \in D$). Assume that (ii) D has an approximate, bounded two sided identity, and that the continuous positive linear functionals $P = D^{*+}$ of D separate the points of D. Then (a1)–(a6), A1, and A2 hold, the ideal I for D is zero and $\| \, \|$ is a norm. If in addition, (iv) D is complete in $| \, |$, then it is known that $\mathbf{A}^{*+} = P$ and $\bar{\mathbf{B}} = B = \mathbf{A}^*$. If (iv) holds, then $\| \, \| \leqq | \, |$ and $\mathbf{A}$ is the C*-enveloping algebra of D. Since $\mathbf{A} = (D, \| \, \|)^-$, it follows from 1.4 that

$$(D, \mathscr{S})^- = (D, \mathscr{S}^*)^- = \mathbf{A}^{**}.$$

3.4. Suppose R, Q are algebras as in 2.1; PR, PQ their associated linear functionals; and $\{u(\lambda)\} \subset R$, $\{w(\gamma)\} \subset Q$ the approximate identities satisfying (a1)–(a6), A1, and A2. All such algebras and involution preserving, norm continuous, algebra homomorphisms $\phi : R \longrightarrow Q$ form a category. Since $IR = \{d \in R \mid \|d\| = 0\}$, $\phi IR \subseteq IQ$.

LEMMA 3.5. *For R, Q as above suppose $\phi : R \to Q$ is* merely an algebraic homomorphism such that $\phi^*(PQ) = \{q\phi \mid q \in PQ\} \subseteq PR$. *Then ϕ will be norm decreasing provided any one of the conditions* (i), *or* (ii) *holds*:

(i) $\|q\phi\| \leqq \|q\| \quad \forall q \in PQ$;

(ii) $\{\phi(u(\lambda))\} \subseteq \{w(\gamma)\}$.

PROOF. (i) If $d \in R$, $q \in PQ$, then

$$\|\phi d\|^2 = \sup\{q((\phi d)^*\phi d)) \mid \|q\| \leqq 1\}.$$

But since $q((\phi d)^*\phi d) = q(\phi(d^*d))$, and $\|q\phi\| \leqq \|q\| \leqq 1$, and $q\phi \in PR$, it follows that $\|\phi d\|^2 \leqq \|d\|^2$. (ii) Condition (ii) guarantees (i).

THEOREM 3.6. *Suppose R, Q are algebras with associated positive functionals PR, PQ in the category in* 3.4, *and that $\phi : \mathrm{R} \longrightarrow Q$ is an algebra homomorphism as in* 3.5 (i). *Then the following hold*:

(i) *I, D, $\mathbf{A}$, $\mathbf{W}$; P, B, $\overline{\mathbf{B}}$, and $\mathbf{A}^*$* (*see* 3.2) *are functors.*

(ii) *$D\phi$, $\mathbf{A}\phi$, and $\mathbf{W}\phi$ are ring homomorphisms in the above category* (*see* 3.4; *all algebras DR, $\mathbf{A}R$, $\mathbf{W}R$ have the same associated functionals PR*); *$\mathbf{A}\phi$ is a $\mathbf{C}^*$ and $\mathbf{W}\phi$ a W^*-map.*

(iii) *There is a commutative diagram where all the vertical maps except $R \longrightarrow DR$ and $Q \longrightarrow DQ$ are natural inclusions, and all horizontal maps are positive.*

$$\begin{tikzcd}
IR \arrow[r] \arrow[d, hook] & IQ \arrow[d] \\
R \arrow[r, "\phi"] \arrow[d] & Q \arrow[d] \\
DR \arrow[r, "D\phi"] \arrow[d] & DQ \arrow[d] \\
\mathbf{A}R \arrow[r, "\mathbf{A}\phi"] \arrow[d] & \mathbf{A}Q \arrow[d] \\
\mathbf{W}R \arrow[r, "\mathbf{W}\phi"] & \mathbf{W}Q
\end{tikzcd}
\qquad
\begin{tikzcd}
PQ \arrow[r, "P\phi"] \arrow[d] & PR \arrow[d] \\
BQ \arrow[r, "B\phi"] \arrow[d] & BR \arrow[d] \\
\overline{\mathbf{B}}Q \arrow[r, "\overline{\mathbf{B}}\phi"] \arrow[d] & \overline{\mathbf{B}}R \arrow[d] \\
\mathbf{A}^*Q \arrow[r, "\mathbf{A}^*\phi"] & \mathbf{A}^*R
\end{tikzcd}$$

PROOF. (ii) Since $\phi IR \subseteq IQ$, the latter induces $D\phi$, which in turn extends to a norm continuous $\mathbf{C}^*$-homomorphism $\mathbf{A}\phi$. Define $\mathbf{A}^*\phi$ to be the usual adjoint $\mathbf{A}^*\phi \equiv (\mathbf{A}\phi)^*$ of $\mathbf{A}\phi$ and let $\bar{\mathbf{B}}\phi$, $B\phi$, and $P\phi$ be its restrictions and corestrictions. Finally set $\mathbf{W}\phi = (\mathbf{A}^*\phi)^*$. The rest is clear.

COROLLARY 3.7. *In the above theorem,* $\{u(\lambda) + I\}$ *is a self adjoint, bounded, approximate identity for* DR, $\mathbf{A}R$, *and* $\mathbf{W}R$. (*It converges in norm to* 1, *in case* $1 \in DR$, *or* $1 \in \mathbf{A}R$; *always* $1 \in \mathbf{W}R$.)

COROLLARY 3.8. *If in the previous theorem* $Q = DR$ *and* $\phi : R \to DR$ *is the natural projection, then all horizontal maps except* ϕ *in the diagram* (3.5 (iii)) *are isomorphisms.* (*i.e.,* $D\phi$, $\mathbf{A}\phi$, $\mathbf{W}\phi$; $P\phi$, $B\phi$, $\bar{\mathbf{B}}\phi$, *and* $\mathbf{A}^*\phi$).

From now on homomorphisms of algebras and semigroups will mean involution preserving (unless stated otherwise). In the next theorem α and β need not be identity preserving maps. In fact, the C^*-algebra N need not even contain an identity.

THEOREM 3.9. (*Universal mapping property*). *Consider an involutive algebra* R *over* $\mathbf{C}$ *as in* 1.2 *and* P *a set of functionals on* R *satisfying* (a1)–(a6), A1, A2. *For a* C^**-algebra* N, *and a* W^**-algebra* $M = M_*{}^*$ *with the topology* $\sigma \equiv \sigma(M, M_*)$ (*see* 1.3), *where* M_* *is the predual of* M, *if* $\alpha : R \to N$ *and* $\beta : R \to M$ *are involutive algebra homomorphisms and* N^* *is the norm dual of* N, *then*:

(i) *Conditions* (*a*) *and* (b) *are equivalent, i.e.,* (*a*) $\Longleftrightarrow$ (b):

(a) α (*or* β) *is norm continuous with the norm topology on* N (*or* M);

(b) $\{q\alpha \mid q \in N^*\} \subseteq \mathbf{A}^*R$ (*or* $\{q\beta \mid q \in M^*\} \in \mathbf{A}^*R$) *respectively.*

Now, in addition assume (i) (a) *or* (b) *throughout. Then* α *and* β *extend to unique* C^* *and* W^**-maps* (*i.e.,* $\bar{\beta}$ *is* σ*-continuous*):

$$\text{(ii)}\quad \begin{tikzcd} R \arrow[rr] \arrow[dr, "\alpha"'] & & \mathbf{A}R \arrow[dl, dashed, "\bar{\alpha}"] \\ & N & \end{tikzcd} \qquad \text{(iii)}\quad \begin{tikzcd} R \arrow[rr] \arrow[dr, "\beta"'] & & \mathbf{W}R \arrow[dl, dashed, "\bar{\beta}"] \\ & M & \end{tikzcd}$$

(iv) *Both* $\mathbf{A}R$ *and* $\mathbf{W}R$ *are uniquely determined by* (i) *and* (ii) *up to an automorphism leaving* R/IR (*see* 3.2 *and* 2.4) *in*

$$R \to R/IR \subset \mathbf{A}R \subset \mathbf{W}R$$

element-wise fixed.

PROOF. (i) Clearly, (a) $\Rightarrow$ (b); the converse (b) $\Rightarrow$ (a) is a consequence of $\|y\|^2 = \sup\{f(y^*y) \mid 0 \leqq f \in N^{*+}, \|f\| = 1\}$ for any $y \in N$.

(ii) Since $\mathbf{A}R = (R, \|\ \|)^-$, α extends.

(iii) By (i), β extends to a C^*-map $\mathbf{A}R \to M$, (see [3; 3.2]) and by the universal property of E, the latter lifts to a unique W^*-map $\bar{\beta} : E(\mathbf{A}R) \to M$.

(iv) follows from (i) and (ii).

COROLLARY 3.10. *Suppose R is an involutive algebra over* $\mathbf{C}$ *with* $\{u(\lambda)\} \subset R$ *as in* 2.1 *and P, L any two given sets of positive functionals on R satisfying* (a1)–(a6), A1, A2. *Let* $\mathbf{W}(R, P)$ *and* $\mathbf{W}(R, L)$ *be the W^*-enveloping algebras formed with respect to P and L. If $L \subset P$, then there is a commutative diagram of continuous maps*

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & \mathbf{W}(R, P) \arrow[d] \\ & \mathbf{W}(R, L) \end{tikzcd}$$

COROLLARY 3.11. *For R as in the last corollary and $q : R \to \mathbf{C}$ a positive linear functional satisfying* (a1)–(a6), *and* 2.8:

(i) *There exists a homomorphism* $\psi : R \to M = M_*{}^*$ *into a W^*-algebra M.*

(ii) *Furthermore,* $q = f\psi$ *for some* $0 \leqq f \in M_*{}^+$.

(iii) kernel $\psi = \{b \in R \mid q(d^*b^*bd) = 0 \quad \forall d \in R\}$.

(iv) *If $q \in P$, then ψ is norm continuous.*

(v) *If $q \in P$, then ψ factors through* $\mathbf{W}(R, P)$ *by a W^*-map.*

PROOF. The set $L = \{dqd^* \mid d \in R\}$ satisfies (a1)–(a6), A1, and A2. Set $M = \mathbf{W}(R, L)$ and let $\psi : R \to M$ be the canonical map.

4. **The Enveloping Algebras of Semigroups.** Consider a semitopological semigroup S where the multiplication need be only separately continuous, and with a given continuous involution "*" on S with $(st)^* = t^*s^*$ for $s, t \in S$.

The general constructions should be at least sufficiently flexible to handle as special cases, for example, multiplicative semigroups S of linear contraction operators on a Hilbert space. For these the possibility must be allowed for that some, but not necessarily all, elements of S may be multiplied by some complex or real scalars. In representing such semigroups S, it may be possible to map S into a C*-algebra so that the scalar multiplication in the C*-algebra does not agree with the naturally given scalar multiplication in S, and so that the zero element of S maps onto a non-zero element. The latter perhaps might be the case for the special case when the algebra is the so-called universal enveloping algebra. However, the general construction is sufficiently flexible so that the naturally given partial algebraic operations defined on some subsets of S, or polynomial identities, could be preserved in the representation of S in a C*-algebra. For example, the self-adjoint elements of S could be closed under real convex linear combinations.

4.1. Since it is always possible to adjoin an identity element e to S so that multiplication remains separately continuous, without loss of generality assume from now on that $e = e^* = 1 \in \mathrm{S}$. Provided it exists, $\theta = \theta^* \in \mathrm{S}$ will denote the zero of S.

4.2. The elements of the ordinary semigroup algebra CS are finitely nonzero valued functions $\alpha, \beta : \mathrm{S} \to \mathrm{C}$ with pointwise addition, scalar multiplication, but the following product and involution, α^*:

$$\alpha\beta(u) = \Sigma\{\alpha(s)\beta(t) \mid (s, t) \in \mathrm{S} \times \mathrm{S},\ u = st\}.$$

$$\alpha^*(s) = \alpha(s^*)^- = \bar{\alpha}(s^*),$$

where $\bar{\alpha}$ is the complex conjugate. Alternatively CS consists of all finite, formal, C-linear combinations $\alpha = \Sigma\alpha(s)s$ out of S with $\alpha^* = \overline{\Sigma\alpha(s)s^*}$. Identify $\mathrm{S} = \{1s \mid s \in \mathrm{S}\} \subset \mathrm{CS}$ and thus $e \in \mathrm{S} \subset \mathrm{CS}$ becomes the identity of CS. Even if there is a zero $\theta \in \mathrm{S}$, $\theta = 1\theta \neq 0 \in \mathrm{CS}$. Note that $\alpha(s) \in \mathrm{C}$ while $\alpha s \in \mathrm{CS}$.

4.3. A complex valued function p on S is positive definite if its linear extension (i.e., $p(\alpha) = \Sigma\alpha(s)p(s)$) is a positive linear functional, i.e., if for all $\alpha \in \mathrm{CS}$,

$$p(\alpha^*\alpha) = \Sigma\{\bar{\alpha}(t)\alpha(s)p(t^*s) \mid (s, t) \in \mathrm{S} \times \mathrm{S}\} \geqq 0.$$

Sums, products, positive real multiples, and complex conjugates $\bar{p}$ of positive definite functions are positive definite.

4.4. Set $R = \mathrm{CS}$. Suppose that $P \equiv PR$ is some definite given set of continuous positive definite functions p on S (or functionals on R) satisfying A1 (of 2.4) and such that

(1) $|p(s)| \leqq p(e)$ for all $s \in \mathrm{S}$. Thus $R = \mathrm{CS}$ and P satisfy all the axioms (a1)–(a6), A1, and A2 trivially.

4.5. Hence also 2.5 (iv) and (vi) apply, and they may be used to show that the set

$$\Omega\mathrm{S} \equiv \{(s, t) \in \mathrm{S} \times \mathrm{S} \mid \forall\, p \in P,$$
$$p((s - t)^*(s - t)) = 0\}$$

is a semigroup congruence. When S is fixed, abbreviate $\Omega \equiv \Omega\mathrm{S}$ and $I = IR$. Define $\pi : \mathrm{S} \to \mathrm{S}/\Omega$, by $\pi s \equiv \{t \in \mathrm{S} \mid (s, t) \in \Omega\}$, where S/Ω has the biggest topology making π continuous. Since $(s, t) \in \Omega$ if and only if $(s^*, t^*) \in \Omega$, also S/Ω has a continuous involution. There is a linear extension $\pi : \mathrm{CS} \to \mathrm{C}(\mathrm{S}/\Omega)$ to the semigroup algebras.

4.6. In case $\mathrm{S} = G$ is a group, even for a non-continuous positive definite function p, 4.4(1), i.e., $|p(g)| \leqq p(e)$ and $p(g^{-1}) = \overline{p(g)}$ holds for all $g \in G$. As usual, define $K \lhd G$ as the normal subgroup $K = \{st^{-1} \mid (s, t) \in \Omega\}$. Note that

$$K = \{k \in G \mid (k, e) \in \Omega\}$$
$$= \{k \in G \mid \forall\, p \in P,\ p(k) = p(e)\}.$$

Hence $K \subset G$ is closed.

4.7. Starting with an arbitrary semigroup S as in 4.1 and functionals P as in 4.4, the machinery of the previous sections applied to $R = \mathrm{CS}$ and P produces $D = R/I \subset \mathbf{A} \subset \mathbf{W}$ and $P \subset B \subseteq \overline{B} \subseteq \mathbf{A}^*$, where the index R in the functors has been, and will be, omitted. When $S = G$ is a group, also write $K = KG \lhd G$. Define IS by $IS \equiv IR \equiv I(\mathrm{CS})$. Thus when $S = G$, $IG = I(\mathrm{C}G)$.

Let $s \in S \cap IS$ and $y \in S$ be arbitrary. By 2.5(ii), $p(s) = 0$. Since $IS \lhd \mathrm{CS}$ (2.9), $ys,\ sy \in IS$. (This can also be verified directly by 2.5(v) and 2.4 A1). Thus $p((s - y)^*(s - y)) = p(y^*y)$ because s^*y, y^*s, and $s^*s \in IS$. Since for any $s,\ t \in S \cap IS$ with $(s, t) \in \Omega$, it follows that $S \cap IS \subseteq \pi s = \pi t$ are all in the same equivalence class modulo Ω. Conversely, for any $s \in S \cap IS$ and any $y \in S$, the above shows that $(s, y) \in \Omega$ if and only if $y \in S \cap IS$. Hence $\pi s \subseteq S \cap IS$. Thus $\pi s = S \cap IS \in S/\Omega$ is a single element.

For any $y \in S$, $ys,\ sy \in IS$ and $\pi s\, \pi y = \pi y\, \pi s = \pi ys = \pi s$. Thus if $S \cap IS \neq \Phi$ and $s \in S \cap IS$, then $\pi s = S \cap IS \in S/\Omega$ is a zero element for the semigroup S/Ω, even in case S does not have a zero.

However if S already has a zero element $\theta \in S$, and under the additional assumption that $p(\theta) = p(\theta^*\theta) = 0$ for any $p \in P$, it follows that $\theta \in IS$ and $\pi\theta = S \cap IS = \overline{\theta}$ is the zero element of S/Ω.

Lemma 4.8. *For a semigroup S as before* (4.4(1)), *the following hold*:

(i) $\Omega S = \{(s, t) \in S \times S \mid s - t \in IS\}$;

(ii) $S/\Omega S \to \{s + IS \mid s \in S\}$, $\pi s \to s + IS$ (*where* $s \in S$) *is an involution preserving isomorphism of* (*multiplicative*) *semi-groups*;

(iii) $S \cap IS \neq \Phi \Rightarrow \exists$ *a zero in* $S/\Omega S$;

(iv) *The kernel of the homomorphism* $\psi : \mathrm{C}(S/\Omega S) \to \mathrm{CS}/IS$ *induced by* (ii) *is* πIS.

(v) *For* $S = G$ *a group,*

$$KG = \{k \in G \mid k - e \in IG\}$$
$$= \{k \in G \mid\ p \in P,\ p(k) = p(k^*) = p(e)\}.$$

Hence $G \cap IG = \Phi$.

Proof. (i), (ii), and (iv), are clear; (iii) follows from 4.7.

(v) By 4.4, $|p(k)| \leqq p(e)$, $|p(k^*)| \leqq p(e)$. For $k - e \in IG$, $2p(e) - p(k)$

$- p(k^*) = 0$, and $p(k^*) = p(k)^-$. Thus both $p(k)$, $p(k^*)$ are real, and (v) follows.

4.9. For semigroups S, T as in 4.4 with associated functionals PR, PQ where $R \equiv CS$, $Q \equiv CT$ and an (involutive), continuous, (not necessarily identity preserving) homomorphism $\phi : S \to T$, assume that $\{q\phi \mid q \in PQ\} \subseteq PR$. Hence also ϕ is σ, $\mathscr{S}$, and $\mathscr{S}^*$-continuous. By 3.5, not only ϕ, but also its extension $\phi : R \to Q$ is norm decreasing. Then $\phi IR \subseteq IQ$, and the induced homomorphism $\phi : DR \to DQ$ is also norm continuous:

4.10. Let $\mathscr{O}$ be the original topology on S. Any topology $\mathscr{O}$ on S in duces a quotient topology $\Omega\mathscr{O}$ on S/Ω. Define S/I to be a multiplicative subsemigroup $S/I \equiv \{s + I \mid s \in S\}$ of D. By 4.7 (ii), identify $S/\Omega = S/I \subset \mathbf{W}$, where $\mathbf{W} = \mathbf{W}CS$. The smallest topology on S and S/Ω making $S \to S/\Omega \to \mathbf{W}$ continuous induces the norm, σ, $\mathscr{S}$, $\mathscr{S}^*$, and τ topologies from $\mathbf{W}$ onto S/Ω and S. Alternatively, S maps onto $\pi : S \to S/\Omega \subset \mathbf{W}$ and the open sets in S are simply defined to be inverse images under π of the respective open sets in $\mathbf{W}$. Since S maps into the unit ball of $\mathbf{W}$, 2.5 (iv) shows that $\mathscr{S}^*$ and τ agree on S/Ω. Then $\mathscr{O}$ and $\Omega\mathscr{O}$ contain the induced σ, $\mathscr{S}$, and $\mathscr{S}^*$-topologies. Lastly, $\sigma \subseteq \mathscr{S} \subseteq \mathscr{S}^* \subseteq \tau \subseteq \mathscr{O}$ are all contained in the norm topology on S (and similarly for $\Omega\mathscr{O}$).

4.11. For S, $R = CS$, $P = PR$, and $D = R/I$ as before, the elements of P can also be viewed as positive linear functionals on D, and hence also on $S/\Omega \subset D$. Set $P(\mathbf{C}(S/\Omega)) \equiv \{p\pi \mid p \in P\}$ and identify $P(\mathbf{C}(S/\Omega)) = P$.

Furthermore, since $\Omega\mathscr{O}$ is the smallest topology on S/Ω making π continuous, the elements of $P(\mathbf{C}(S/\Omega))$ are continuous and satisfy 4.4 (1). Since $\pi : S \to S/\Omega$ is epic, its adjoint $\pi^* : P(\mathbf{C}(S/\Omega)) \to P$, $h \to h\pi$ is monic. Thus the above identification of $P(\mathbf{C}(S/\Omega))$ with P is given more precisely by the isomorphism π^* as $\pi^* P(\mathbf{C}(S/\Omega)) = P$.

Although every result of the previous section now could be stated for the special case of a semigroup algebra $R = CS$ and further information extracted from the additional semigroup as well as topological structure, this will only be done for two theorems.

THEOREM 4.12. *Consider a not necesssarily identity preserving morphism* $\phi : S \to T$ *of involutive semi-topological semigroups* S, T *as in* 4.1 *and* 4.2 (1); *their semigroup rings* $R = CS$, $Q = CT$; *their functionals* PR, PQ; *and the* $\mathbf{C}^*$ *and* W^**-enveloping algebras* $\mathbf{A}R$, $\mathbf{A}Q$, $\mathbf{W}R$, $\mathbf{W}Q$ *in the norm and* σ*-topologies. If* $\{h\phi \mid h \in PQ\} \subseteq PR$, *then*

(i) $S \to \mathbf{A}R$, $S \to \mathbf{W}R$ *are continuous,*

(ii) *there is a commutative diagram where all the horizontal maps are continuous homomorphisms*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d] & T \arrow[d] \\ \mathbf{A}R \arrow[r] \arrow[d] & \mathbf{A}Q \arrow[d] \\ \mathbf{W}R \arrow[r, "\mathbf{W}\phi"'] & \mathbf{W}Q \end{tikzcd}$$

COROLLARY 4.13. *With the same hypotheses as in he last theorem, the multiplication and involution on the image* $S/\Omega S$ (*see* 4.5, 4.8 (ii)) *of* S *in* $S \to S/\Omega S \subset \mathbf{A}R \subset \mathbf{W}R$ *has the following continuity properties*:

(i) *involution*: *norm*, σ, *and* $\mathscr{S}^*$-*continuous*;

(ii) *multiplication separately continuous*: *all topologies—norm*, σ, $\mathscr{S}$, $\mathscr{S}^*$ ($\mathscr{S}^*$ *and* τ *agree on* $S/\Omega S$);

(iii) *multiplication jointly continuous*: *norm*, $\mathscr{S}$, $\mathscr{S}^*$.

(v) *The original topology of* S (*or* $S/\Omega S$) *contains the induced* σ, $\mathscr{S}$, *and* $\mathscr{S}^*$-*topologies* (*see* 4.10).

Below, the possibility that $\Omega = \Omega S$ is trivial and $S \cong S/\Omega$, while $ICS \neq 0$ should not be ruled out.

COROLLARY 4.14. *With* S, $\Omega = \Omega S$, $R = \mathbf{C}S$, *and* PR *as in the previous theorem* (*i.e., satisfying* 4.2 (1)); *with* S/IR, $\pi : S \to S/\Omega$ (4.5), *and* $\psi : \mathbf{C}(S/\Omega) \to R/IR$ (4.8 (ii), (iv)) *as previously, define* $Q \equiv \mathbf{C}(S/\Omega)$, *and* $PQ = \pi^*(PR) \cong PR$ (4.11). *Then the following hold*:

(i) $\pi IR = IQ$, $\pi(IR \cap S) = (IQ) \cap (S/\Omega)$;

(ii) kernel $\psi = IQ$;

(iii) S/IR, $(S/\Omega)/IQ$, *and* $(S/\Omega)/\Omega(S/\Omega)$ *are all isomorphic*;

(iv) *There is a commutative diagram of continuous morphisms*; *every horizontal map is onto*; *isomorphisms are indicated by double lines*:

$$\begin{tikzcd} (IR) \cap S \arrow[rr] \arrow[d] & & (IQ) \cap \dfrac{S}{\Omega} \arrow[d] \\ \dfrac{S}{IR} \arrow[r, equal] \arrow[d] & \dfrac{S}{\Omega} \arrow[r, equal] & \dfrac{S/\Omega}{IQ} \arrow[d] \\ DR \arrow[rr, equal] & & DQ \end{tikzcd}$$

$$
\begin{tikzcd}
\mathit{IR} \arrow[r] \arrow[d] & \mathit{IQ} \arrow[d] \arrow[dr] & \\
\mathrm{C}R \arrow[r] \arrow[d] & \mathrm{C}Q \arrow[d] \arrow[dr, "\psi"] & (0) \arrow[d] \\
\mathit{DR} \arrow[r, equal] \arrow[d] & \mathit{DQ} \arrow[r, equal] \arrow[d] & \mathit{DR} \arrow[d] \\
\mathbf{W}R \arrow[r, equal] & \mathbf{W}Q \arrow[r, equal] & \mathbf{W}R
\end{tikzcd}
$$

Definition 4.15. For any involutive, semitopological semigroup (with $1 = e \in \mathrm{S}$), PS denotes the real cone of all continuous, positive definite functions on S satisfying

$$|p(s)| \leqq p(e) \qquad \text{for all } s \in \mathrm{S}$$

(then all the other axioms a1–a6, A1, A2 also hold.) Application of the usual functors to CS gives

$$\mathrm{S}/\Omega S \subset DS = \mathrm{CS}/IS \subset \mathbf{A}S \subset \mathbf{W}S;$$

$$PS \subset BS \subseteq \bar{\mathbf{B}}S \subseteq \mathbf{A}^*S,$$

where the index "CS" will be abbreviated to "S" only for the above special PS. The algebras $\mathbf{A}S$ and $\mathbf{W}S$ are called the *universal* $\mathbf{C}^*$ and W^*-enveloping algebras of S.

The significance of the universal algebras is that in Theorem 4.12, the additional hypothesis besides the standard axioms may be omitted.

Theorem 4.16. *With* S *as in* 4.4 (1), *let* $\alpha : \mathrm{S} \to N$ *and* $\beta : \mathrm{S} \to M$ *be norm and* σ*-continuous morphisms into a* $\mathbf{C}^*$ *and* W^**-algebra* N *and* M (α *and* β *are not assumed identity preserving*). *Then*

(i) α *and* β *extend to unique* $\mathbf{C}^*$ *and* W^**-maps* $\mathbf{A}\alpha$ *and* $\mathbf{W}\beta$ *giving commutative diagrams of continuous maps*:

$$
\begin{tikzcd}
\mathrm{S} \arrow[rr] \arrow[dr, "\alpha"'] & & \mathbf{A}S \arrow[dl, "\mathbf{A}\alpha"] \\
& N &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{S} \arrow[rr] \arrow[dr, "\beta"'] & & \mathbf{W}S \arrow[dl, "\mathbf{W}\beta"] \\
& M &
\end{tikzcd}
$$

(ii) *The above universal property* (i) *determines* AS *and* WS *uniquely up to automorphisms leaving the image* $S/\Omega S$ *of* S *elementwise fixed in* $S \to S/\Omega S \subset \mathbf{A}S \subset \mathbf{W}S$.

LEMMA 4.17. *For a semigroup* S, $R = \mathbf{C}S$, *and any* $P = PR$ *whatever satisfying property* (i) *of* 4.4, *consider the following point separation properties* (a) *and* (b) *of* P:

(a) $\forall n = 0, 1, 2, \cdots; \ \forall y, x(1), \cdots, x(n) \in S; \ \forall \lambda \in \mathbf{C}$: $[y \neq \lambda y^*$; $\forall i, x(i) \neq y, y^*$, *or* $e] \Rightarrow [\exists p \in P$ *such that* $p(y) \neq \lambda p(y^*); p(x(i)) = 0.]$

(b) (1) *P is closed under multiplication.*

(2) $\forall y \in S \Rightarrow \exists \ p \in P, p(y) \neq 0$

(3) $\forall y, x \in S; \forall \lambda \in \mathbf{C}: [y \neq \lambda y^*; \ x \neq y, y^*,$ *or* $e] \Rightarrow [\exists p \in P, p(y) \neq \lambda p(y^*), p(x) = 0]$.

Then (i) (b) $\Rightarrow$ (a);

(ii) (a) $\Rightarrow IR = 0$, Ω *is the diagonal*, $S = S/\Omega$, *and* $DR = \mathbf{C}R$.

PROOF. For $n = 0$, $\lambda = 0$ we have (a) $\Longleftrightarrow$ (b) (2). Secondly, (b) (2) and (b) (3) $\Longleftrightarrow$ (a) for $n = 0$ and $n = 1$.

(i) If $n = 2$ in (b), take p in (b) (3) with $p(y) \neq \lambda p(y)^- \neq 0$, and $p(x(1)) = 0$. Then take $q \in P$ by (b) (3) with $q(x(2)) = 0$, and $q(y) \neq \mu q(y^*)$ for $\mu = \lambda p(y^*)/p(y)$. Thus $pq(y) \neq \lambda pq(y^*)$. The rest is clear.

4.18. Suppose that S, $R \equiv \mathbf{C}S$, PR and T, $Q \equiv \mathbf{C}T$, PQ are two semigroups, each one satisfying 4.4(1) and 4.17(a). Define RQ as $RQ \equiv \mathbf{C}(S \times T)$ and define $(PR)(PQ)$ to be

$$(PR)(PQ) = \{fg : S \times T \to \mathbf{C} \mid f \in PR, \ g \in PQ,$$
$$fg(s, t) = f(s)g(t) \quad \text{for } s \in S, \ t \in T\}.$$

Assume that $P(RQ)$ is some given set of functionals such that 4.4(1) holds for $S \times T$ and with $(PR)(PQ) \subseteq P(RQ)$. Then

(i) $P(RQ)$ satisfies 4.17(a) and $S \times T \to W(RQ)$ is monic.

(ii) $(PR)(PQ)$ satisfies 4.4(1), 4.17(a), and 4.17(b).

REMARK 4.19. Suppose that S is a semitopological semigroup with some of the following possible additional structure.

(1) Complex multiplication: $D \equiv \{\lambda \in \mathbf{C} \mid \ |\lambda| < 1\}$, there is a separately continuous map $D \times S \to S$, $(\lambda, s) \to \lambda s \in S$ such that $(\lambda c)s = \lambda(cs)$, $\lambda(st) = (\lambda s)t = s(\lambda t)$, $(\lambda s)^* = \bar{\lambda} s^*$; $\lambda, c \in \mathbf{C}$; $s, t \in S$.

(2) Multiplication by $\{\lambda \in \mathbf{R} \mid -1 \leqq \lambda \leqq 1\}$.

(3) $\exists \ \theta \in S$.

First, form $R = \mathbf{C}S$, PS, AS, and WS as in 4.15. Let $Q \subset PS$ be that subset which preserves one or several of (1)–(3); e.g., $p(\lambda s) = \lambda p(s)$ for $s \in S$, $\lambda \in \mathbf{C}$ and/or $p(\theta) = 0$. Assume that Q satisfies 4.4.

Next apply the process of 2.10 and 3.2 to CS with Q (not P) to yield its C*-enveloping algebra $\mathbf{A}(R, Q)$ and W*-enveloping algebra $\mathbf{W}(R, Q)$ with respect to Q. It follows from 4.16 (see also 3.10) that there are commutative diagrams of norm and σ-continuous maps respectively.

$$\begin{tikzcd} \mathrm{S} \arrow[rr] \arrow[dr] & & \mathbf{A}\mathrm{S} \arrow[dl] \\ & \mathbf{A}(R, Q) & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{S} \arrow[rr] \arrow[dr] & & \mathbf{W}\mathrm{S} \arrow[dl] \\ & \mathbf{W}(R, Q) & \end{tikzcd}$$

It is conjectured that an analogous result holds for semigroups S having the following possible additional structure which would first have to be rigorously defined:

(4) Closed under convex real linear combinations;

(5) Closed under a Jordan $(xy + yx)/2$, or a Lie product $(xy - yx)/2$ $x, y \in \mathrm{S}$.

(6) Only the self adjoint elements are closed under (4), and/or (5).

5. **Tensor Products and Multiplicative Categories.** The results needed about various tensor products are briefly summarized and four multiplicative categories are introduced.

5.1. If A and B are C*-algebras, their usual algebraic tensor product is denoted by $A \bigcirc B$ and their categorical C*-tensor product by $A \,\bar{\bigcirc}\, B$, where the latter is simply the closure of $A \bigcirc B$ with respect to a certain ([3; p. 440, 2.1]) C*-crossnorm ρ. If both A and B have identities, then write $A \otimes B = A \bigcirc B$ and $A \,\bar{\otimes}\, B = A \,\bar{\bigcirc}\, B$, the distinction being that for "—$\bar{\otimes}$—", $A, B, \subset A \,\bar{\otimes}\, B$ while the latter need not necessarily hold for "—$\bar{\bigcirc}$—". Then $A \,\bar{\bigcirc}\, B$ is uniquely determined by the universal property that any C*-maps $\alpha : A \longrightarrow D$ and $\beta : B \longrightarrow D$ with elementwise commuting images in a C*-algebra D extend to a unique C*-map $\phi : A \,\bar{\bigcirc}\, B \longrightarrow D$ such that $(\alpha a)(\beta b) = \phi(a \bigcirc b)$ for all $a \in A$, $b \in B$.

5.2. The norm ρ induces always a dual norm ρ^* on $A^* \bigcirc B^*$ ([3; 2.3]); the ρ^*-completion of the latter will be denoted by $A^* \,\bar{\bigcirc}\, B^*$. If $1 \in A$ and also $1 \in B$, write $A^* \,\bar{\otimes}\, B^* \equiv A^* \,\bar{\bigcirc}\, B^*$.

5.3. We give an alternative description of $A \,\bar{\bigcirc}\, B$. If γ denotes the greatest crossnorm ([8; p. 6]), then let $A \,\hat{\bigcirc}\, B$ denote the γ-completion of the algebraic tensor product of two Banach spaces A, B. Then it can be shown directly that $A \,\hat{\bigcirc}\, B$ has the universal property of $A \,\bar{\bigcirc}\, B$, and thus $A \,\hat{\bigcirc}\, B \neq A \,\bar{\bigcirc}\, B$. The latter also follows from [8; p. 38].

5.4. If in addition A and B are W^*-algebras with preduals A_*, B_* then $A_* \Delta B_* \subset (A \bar{\otimes} B_*$ is defined as the set of all linear functionals $T \in (A \bar{\otimes} B)^*$ such that $T(- \otimes b) \in A_*$ and $T(a \otimes -) \in B_*$ for all $a \in A$, $b \in B$. The annihilator $(A_* \Delta B_*)^\perp \subset (A \bar{\otimes} B)^{**}$ of $A_* \Delta B_* \subset A \bar{\otimes} B^*$ is a direct summand and a non-Neumann subalgebra of $(A \bar{\otimes} B)^{**}$. Then the W^*-tensor product of A and B is denoted by $A \nabla B$ and is defined as

$$A \nabla B = (A \bar{\otimes} B)^{**}/(A_* \Delta B_*)^\perp$$

Note that by its very definition, $A \nabla B = (A_* \Delta B_*)^*$. Again, $A \nabla B$ is uniquely determined by the usual universal property for tensor products.

For any subspaces $V \subset A_*$, $W \subset B_*$ (not assumed closed in ρ^*), $V \Delta W \subset A \bar{\otimes} B)^*$ is defined the same way. Also, write $A_* \bar{\otimes} B_* \equiv A_* \bar{\bigcirc} B_*$ for preduals A_* and B_*.

5.5. The functor E is multiplicative or product preserving, i.e., for C*-algebras A, B, we have $E(A \bar{\otimes} B) \cong EA \nabla EB$, or $\cong (A \bar{\otimes} B)^{**} \cong (A^* \Delta B^*)^*$. Consequently, $(A \bar{\otimes} B)^* \cong A^* \Delta B^*$.

The basic definitions and facts used about multiplicative categories may be found in [8] and [14]. Four multiplicative categories will play a useful role.

5.6. First is the category (Sgrps, $\times$) of involutive semitopological semigroups (satisfying an additional hypothesis 6.2) with identity, direct product, and morphisms that need not preserve the identity.

Secondly, $(\mathbf{C}^*, \bar{\otimes})$ is the category of C*-algebras (i) with identity; (ii) but with not necessarily identity preserving morphisms and the product "$\bar{\otimes}$". The map $S \to \mathbf{A}S$ as in 4.15 gives a functor $\mathbf{A} : \text{Sgps} \to \mathbf{C}^*$.

The W^*-algebras with normal (i.e., σ-continuous), involutive, not necessarily identity preserving homomorphisms form a multiplicative category $(\mathbf{W}^*, \nabla)$. The functor $E : (\mathbf{C}^*, \bar{\otimes}) \to (W^*, \nabla)$ is multiplicative. Set $\mathbf{W} = EA$.

Lastly, the preduals M_*, N_* of W^*-algebras $M = M_*^*$, $N = N_*^*$ form a multiplicative category $(\mathscr{D}, \Delta)$ that is contravariantly isomorphic to (W^*, ∇).

Let F be the functor that assigns to spaces M_*, their duals $FM_* = M_*^*$. The morphisms of $\mathscr{D}$ are maps $\psi : N_* \to M_*$ such that their adjoints $\psi^* : M \to N$ are W^*-maps, and $F\psi = \psi^*$. Then $F(M_* \Delta N_*) = (FM_*) \nabla (FN_*)$, and $F : (\mathscr{D}, \Delta) \to (W^*, \nabla)$ is a multiplicative, contravariant equivalence of categories. Furthermore, so is $F^{-1} : (W^*, \nabla) \to (\mathscr{D}, \Delta)$, where $F^{-1}M$ is the predual $F^{-1}M = M_*$ of M.

For a proof of 5.7, see [4; p. 469].

5.7. For any algebras (with or without identities) over a field (of arbitrary characteristic) and any ideals $I \lhd A$, $J \lhd B$,

(i) $A \bigcirc B/(I \bigcirc B + A \bigcirc J) \cong A/I \bigcirc B/J$;

(ii) for C*-algebras A and B, (i) also holds for the C*-tensor product "$-\bar{\bigcirc}-$" in place of "$-\bigcirc-$".

5.8. In case A, B are any C*-algebras with or without identities, then define $A^1 = A$ when $1 \in A$; otherwise when $1 \notin A$, then $A^1 = \mathbf{C} \times A$ is the C*-algebra obtained by adjoining the identity $(1, 0)$ to A and $A \cong \{0\} \dotplus A \lhd A^1$ as usual. If $1 \notin A$, $1 = (1, 0) \in A^1$ and $\lambda = (\lambda, 0) \in A^1$ for $\lambda \in \mathbf{C}$ may be written without ambiguity. Now $A \otimes B$ is defined as $A \otimes B = A^1 \bigcirc B + A \bigcirc B^1$ (see [3; p. 449, 5.4]), while its ρ-closure is $A \bar{\otimes} B = A^1 \bar{\bigcirc} B + A \bar{\bigcirc} B^1$ ([3: p. 450, 5.5]). There are natural embeddings A, $B \subset A \otimes B$ as $A \cong A \bigcirc 1 \subset A \bar{\otimes} B$, $B \cong 1 \otimes B \subset A \bar{\otimes} B$ and $A \bar{\bigcirc} B \lhd A \bar{\otimes} B$ is a closed ideal. By [3; 5.7 Theorem II], $E(A \bar{\otimes} B) = EA \bigtriangledown EB$. Hence there are natural embeddings $A \subset A \bar{\otimes} B \subset EA \bigtriangledown EB$, similarly for B, and $A \subset EA \subset EA \bigtriangledown EB$, where $A \bar{\otimes} B \cap EA = A$.

LEMMA 5.9. *For any* C*-*algebras* A, B, (*with or without identities*) *and any* $0 \leqq f \in A^{**+}$, $0 \leqq g \in B^{**+}$, *there exists a* W^*-*algebra* $M = M_*{}^*$ *and* C*-*homomorphisms* $\phi : A \longrightarrow M$, $\psi : B \longrightarrow M$ *such that there also exists a* $0 \leqq h \in M_*$ *with* $\|g\|f = h\phi$ *and* $\|f\|g = h\psi$.

PROOF. There is a C*-map $\alpha : A \longrightarrow L(H)$ into the bounded operators on a Hilbert space H such that for some $x \in H$, $f(a) = (\alpha a x \mid x)$, $\|x\|^2 = \|f\|$, and $\alpha A x \subseteq H$ is dense, $a \in A$. There are similar objects $\beta : B \longrightarrow L(K)$, $y \in K$ for B and g. Form the ordinary Hilbert space tensor product $H \otimes K$, set $M = L(H \otimes K)$, and take $\xi = x \otimes y \in H \otimes K$. Now define $\phi : A \longrightarrow M$ by $\phi a = \alpha a \otimes 1 \in M$, and $\psi : B \longrightarrow M$ by $\psi b = 1 \otimes \beta b$. Then defined h as $h(m) = (m\xi \mid \xi)$ for $m \in M$. Thus

$$h\phi a = (\alpha a x \mid x)(y \mid y) = f(a)\|g\|,$$

$$h\psi b = (x \mid x)(\beta b y \mid y) = \|f\|g(b) \quad a \in A,\ b \in B.$$

PROPOSITION 5.10. *Consider* C*-*algebras* A, B (*with or without identities*) *and any* $0 \neq f \in A^*$, $0 \neq g \in B^*$, *where as in* 5.8 *we have*:

$$0 \neq f \in (EA)_* = A^*;\ 0 \neq g \in (EB)_* = B^*;$$

$$A,\ B \subset A \bar{\otimes} B \subset EA \bigtriangledown EB; \text{ and}$$

$$A \subset EA,\ B \subset EB.$$

Then there exist $f \bar{\otimes} g \in (A \bar{\otimes} B)^*$ *and* $f \bigtriangleup g \in (EA \bigtriangledown EB)_* = A^* \bigtriangleup B^*$ *such that their restrictions satisfy the following*:

(i) $f \overline{\otimes} g \mid A = f, f \overline{\otimes} g \mid B = g,$

(ii) $f \triangle g \mid EA = f, f \triangle g \mid EB = g,$

(iii) $f \triangle g \mid A \overline{\otimes} B = f \overline{\otimes} g,$

(iv) *If* $1 \in A$ *and* $1 \in B$, *then* $f \overline{\otimes} g$ *and* $f \triangle g$ *are unique.*

PROOF. Let $\phi : A \rightarrow M$, $\psi : B \rightarrow M$, and $h : \mathrm{M} \rightarrow \mathbf{C}$ be as previously in 5.9. By the universal property of E ([3; 3.2]), ϕ factors through EA by $E\phi = \phi^{**} : EA \rightarrow M$. Set $\phi = E\phi$. By the universal properties of $-\overline{\otimes}-$ and $-\nabla-$ ([3; 5.5] and [3; 4.8 Theorem I]), ϕ and ψ give unique maps $\phi \overline{\otimes} \psi$ and $E\phi \nabla E\psi$. Set $E\phi \nabla E\psi = \phi \nabla \psi$. First assume that $0 < f$, and $0 < g$. Thus except for f, g, and h all maps are $\mathbf{C}^*$ and W^*-maps respectively in the following commutative diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[dr, "\phi"] \arrow[ddr, "\|g\|f"'] & A \overline{\otimes} B \arrow[d, "\phi \overline{\otimes} \psi"] & B \arrow[l] \arrow[dl, "\psi"'] \arrow[ddl, "\|f\|g"] \\
& M \arrow[d, "h"] & \\
& \mathbf{C} &
\end{tikzcd}$$

$$\begin{tikzcd}
EA \arrow[r] \arrow[dr] & EA \nabla EB \arrow[d] & EB \arrow[l] \arrow[dl] \\
A \arrow[u] \arrow[r, "\phi"'] \arrow[dr, "\|g\|f"'] & M \arrow[d, "h"] & B \arrow[u] \arrow[l, "\psi"] \arrow[dl, "\|f\|g"] \\
& \mathbf{C} &
\end{tikzcd}$$

Set $\mu = 1/(\|f\| \; \|g\|)$. Thus $f \otimes g = \mu h(\phi \otimes \psi)$, $f \overline{\otimes} g = \mu h(\phi \overline{\otimes} \psi)$, and $f \Delta g = \mu h(\phi \nabla \psi)$. In general, f or g is a complex linear combination of four positive linear functionals. Hence $f \otimes g$, $f \overline{\otimes} g$, and $f \triangle g$ in the general case will be a complex linear combination of sixteen or less terms of the above kind with different μ's and h's.

If $1 \in A$ and $1 \in B$, then $A \bigcirc B = A \otimes B \subset A \overline{\otimes} B$ is norm dense, and $f \bigcirc g$ has a unique extension $f \overline{\otimes} g$. Again, because $A \overline{\otimes} B \subset EA \nabla EB$ is σ-dense ([3; p. 451, 5.8]), also $f \Delta g$ is unique.

6. **The Coalgebra of a Semigroup.** Although CS is a so-called bialgebra with an anti-pode (see [8], [14]), it does not seem to be possible to extend all of these operations to σ-continuous maps on WS.

NOTATION 6.1. Throughout this section S and T will be involutive semi-topological semigroups with an identity. Since only universal enveloping algebras will be considered, 4.4 (1) holds automatically as a consequence of definition 4.15.

Define $S \bigcirc T$ as the multiplicative subsemigroup

$$S \bigcirc T = \{s \bigcirc t \mid s \in S,\ t \in T\} \subset \mathbf{C}S \bigcirc \mathbf{C}T.$$

Thus $S \times T \cong S \bigcirc T$, $(s, t) \longrightarrow s \bigcirc \mathrm{t}$ induces a ring isomorphism $\mathbf{C}(S \times T) \cong \mathbf{C}S \bigcirc \mathbf{C}T$.

It will be assumed throughout this section unless explicitly stated otherwise that (in addition to property (1) of 4.4) the following holds:

$$(2) \qquad IS = 0,\ IT = 0, \text{ and } I(S \times T) = 0$$

It will be convenient to identify $\mathbf{C}S \equiv DS$ and

$$\mathbf{C}(S \times T) = \mathbf{C}S \bigcirc \mathbf{C}T = D(S \times T).$$

If S is fixed, the index S will be omitted in the following algebras and spaces of functionals

$$D \equiv DS \subset \mathbf{A} \equiv \mathbf{A}S \subset \mathbf{W} \equiv \mathbf{W}S;$$

$$P \equiv PS \subset B \equiv BS \subseteq \bar{\mathbf{B}} \equiv \bar{\mathbf{B}}S \subseteq \mathbf{A}^* \equiv \mathbf{A}^*S.$$

The same objects for the semigroup $S \times S$ are denoted by $D^2 \subset \mathbf{A}^2 \subset \mathbf{W}^2$ and $P^2 \subset \mathbf{B}^2 \subseteq \bar{\mathbf{B}}^2 \subseteq \mathbf{A}^{*2}$, likewise $\sigma^2 \subseteq \mathscr{S}^2 \subseteq \mathscr{S}^{*2} \subseteq \tau^2$ will be the topologies on $\mathbf{W}^2$.

6.2. If S, T; PS, PT satisfy 4.17 (a) or (b) then $IS = 0$, $IT = 0$, and also $I(S \times T) = 0$ by 4.18 (i). For a continuous map $\phi : S \longrightarrow T$, $\phi^*(PT) \subseteq PS$. Thus the class of all such semigroups and maps form a category closed under direct products satisfying our hypotheses 4.4 (1) and 6.1 (2).

6.3. The complex numbers $\mathbf{C}$ with $\mathbf{C}^{*+} = \mathbf{R}^+$ satisfy $D\mathbf{C} = \mathbf{AC} = \mathbf{WC} = \mathbf{C}$; since $\mathbf{R}^+$ are exactly all the positive functionals on the trivial one element semigroup $\{1\}$, also

$$D\{1\} = \mathbf{A}\{1\} = \mathbf{W}\{1\} = \mathbf{C}.$$

Lastly, $\sigma, \mathscr{S}, \mathscr{S}^*$, and τ on $\mathbf{C}$ all are equal to the norm topology.

6.4. There are three isometric involutions i, j, and ij: $D \to D$; $\alpha \to i\alpha$, $\alpha \to j\alpha$, and $\alpha \to ij\alpha$ defined by

$$i\alpha(s) = \alpha(s^*),\ j\alpha(s) = \bar{\alpha}(s),\ ij\alpha$$
$$= ji\alpha = \alpha^* \quad s \in S;\ \alpha \in D$$

In particular, $is = s^* = ijs$ and thus ij is the main, natural involution to be used throughout. Note that $i\alpha = \alpha i$ is the composition of the two functions $i : D \to D$, $\alpha : D \to \mathbf{C}$.

6.5. Since, first i, j are isometries; secondly, $iD = jD = D$; and thirdly $D \subset \mathbf{A}$ is norm dense, it follows that there are isometric extensions $i, j : \mathbf{A} \to \mathbf{A}$ with $i\mathbf{A} = j\mathbf{A} = \mathbf{A}$.

Next, the adjoints $i^* : \mathbf{A}^* \to \mathbf{A}^*$, $j^* : \mathbf{A}^* \to \mathbf{A}^*$ define involutions on $\mathbf{A}^*$; the double adjoints

$$i = i^{**},\ j = j^{**},\ ij = (ij)^{**} : \mathbf{W} \to \mathbf{W}$$

on $\mathbf{W} = \mathbf{A}^{**}$.

For any isometric maps $i, j : A \to A$ of any Banach spaces A, i^*, j^* are also isometric, provided that i and j are onto. Since $i^2 = j^2 = 1$, also $i^{*2} = j^{*2} = 1$ are idempotent, and hence $i^*A^* = j^*A^* = A^*$. Repetition of this shows that i, j, and ij are isometric involutions of $\mathbf{W}$ with $i\mathbf{W} = j\mathbf{W} = \mathbf{W}$.

6.6. To avoid confusion later, the above process can be summarized for the standard involution ij or "*" as follows. The involutions on $\mathbf{A}$ are given first, and the ones on $\mathbf{A}^*$ and $\mathbf{W}$ will be always completely determined from their restrictions to $\mathbf{A}$ by

$$\langle f^*, a\rangle = f(a^*)^-,$$
$$[F^*, f] = F(f^*)^- \quad a \in \mathbf{A},\ f \in A^*,\ F \in \mathbf{W}.$$

Note that the above imply that $\|f^*\| = \|f\|$ and $\|F^*\| = \|F\|$, provided $\|a^*\| = \|a\|$ or all $a \in \mathbf{A}$.

6.7. Suppose that $\gamma : S \to S \times T$ is any semigroup map such that $\pi\gamma = 1$ is the identity on S for the natural projection $\pi : S \times T \to S$ onto the first factor. Since γ is one to one and π onto, their adjoints $\gamma^*(P(S \times T)) = \{h\gamma \mid h \in P(S \times T)\} = PS$ and $\pi^*(PS) \subset P(S \times T)$ are epic and monic. The latter implies by 3.5, that the linear extensions $\gamma : DS \to D(S \times T)$ and $\pi : D(S \times T) \to DS$ are norm decreasing. Now Theorem 3.6 applied to γ and π yield the sequences of maps below, which will be shown to be one to one on the left, non-exact in general in the middle, and onto on the right:

$$\begin{array}{ccccc}
S & \xrightarrow{\gamma} & S \times T & \xrightarrow{\pi} & S \\
DS & \longrightarrow & D(S \times T) & \longrightarrow & DS \\
\mathbf{A}S & \longrightarrow & \mathbf{A}(S \times T) & \longrightarrow & \mathbf{A}S \\
\mathbf{W}S & \xrightarrow[\mathbf{W}\gamma]{} & \mathbf{W}(S \times T) & \xrightarrow[\mathbf{W}\pi]{} & \mathbf{W}S
\end{array}$$

$$\begin{array}{ccccc}
PS & \xrightarrow{\pi^*} & P(S \times T) & \xrightarrow{\gamma^*} & PS \\
BS & \longrightarrow & B(S \times T) & \longrightarrow & BS \\
\bar{\mathbf{B}}S & \longrightarrow & \bar{\mathbf{B}}(S \times T) & \longrightarrow & \bar{\mathbf{B}}S \\
\mathbf{A}^*S & \xrightarrow[\mathbf{A}^*\pi]{} & \mathbf{A}^*(S \times T) & \xrightarrow[\mathbf{A}^*\gamma]{} & \mathbf{A}^*S
\end{array}$$

6.8. Since $D\gamma$, $D\pi$ are norm decreasing with $D\pi D\gamma = 1$, so are also $\mathbf{A}\gamma$, $\mathbf{A}\pi$ and $\mathbf{A}\pi\mathbf{A}\gamma = 1$. Thus $\mathbf{A}\gamma$ is an isometry as well as the restriction of $\mathbf{A}\pi$ to $\gamma\mathbf{A}S$. Note that

$$\{(s, t) - (s, 1) \in D(S \times T) \mid s \in S, t \in T\} \subset \text{kernel } \pi$$

will not be in the image of γ if T has two or more elements. Since $\mathbf{A}\gamma$ is one to one, the range of $\mathbf{A}^*\gamma$ is dense in the weak-$\mathbf{A}$ topology, and $\mathbf{W}\gamma$ is again one to one. Again, $\mathbf{W}\pi\gamma = 1$ shows that $\mathbf{W}\gamma$ is an isometry, while $\mathbf{W}\pi$ is isometric on $\gamma(\mathbf{W}S)$. Because $\mathbf{A}\gamma$, $\mathbf{A}\pi$ are norm decreasing, $\mathbf{A}^*\pi$, $\mathbf{A}^*\gamma$ are norm increasing. Since $\gamma\pi = 1$, $\mathbf{A}^*(\gamma\pi) = (\mathbf{A}^*\gamma)(\mathbf{A}^*\pi) = 1$, $\mathbf{A}^*\pi$ is an isometry, and so is $\mathbf{A}^*\gamma$ on the image of $\mathbf{A}^*\pi$.

6.9. For simplicity, abbreviate $\mathbf{W}\gamma \equiv \gamma$, $\mathbf{A}\gamma \equiv \gamma$, $D\gamma \equiv \gamma$, $\mathbf{A}^*\gamma \equiv \gamma^*$, etc. Because a double adjoint γ^{**} is always σ-continuous, $\mathbf{W}S \cong \gamma\mathbf{W}S \subset \mathbf{W}(S \times T)$ is a W^*-subalgebra, i.e., the restriction of the σ-topology of $\mathbf{W}(S \times T)$ to $\mathbf{W}S \cong \gamma(\mathbf{W}S)$ is the natural σ-topology of $\mathbf{W}S$. For $1 = (1, 1) \in S \times T$, $\pi(1, 1) = 1 \in S$ implies that $\mathbf{A}^*\pi$ preserves functionals that are equal to one at the identity. The isometry π^* gives $\mathbf{A}^*S \cong \pi^*\mathbf{A}S \subset \mathbf{A}^*(S \times T)$.

6.10. The special case $\gamma : S \longrightarrow S \times T$, $\gamma s = (s, 1)$ gives natural embeddings $S \subset DS \subset D(S \times T)$ as well as $\mathbf{W}S \subset \mathbf{W}(S \times T)$ by the identification $S = \gamma S \subset \mathbf{C}(S \times \{1\}) \subset D(S \times T)$ Similarly, $T \longrightarrow S \times T$, $t \longrightarrow (1, t)$, gives $\mathbf{W}T \subset \mathbf{W}(S \times T)$. Later, in this special case only, γ will be replaced by an inclusion as above.

6.11. Set $T = \mathrm{S}$ and $\gamma s = (s, s)$; for $s \in S$ define δ and ϵ by $\delta = \gamma$, $\epsilon s = 1$. Theorem 3.6 gives isometric (on left) and epic (in the middle, on the right) ring homomorphisms $\delta \equiv \mathbf{W}\delta \equiv \delta^{**}$, $\epsilon \equiv W\epsilon = \epsilon^{**}$.

$$\begin{array}{ccccccc}
\mathrm{S} & \xrightarrow{\delta} & \mathrm{S} \times \mathrm{S} & \xrightarrow{\pi} & \mathrm{S} & \xrightarrow{\epsilon} & \{1\} \\
D & \longrightarrow & D^2 & \longrightarrow & D & \longrightarrow & \mathbf{C} \\
\mathbf{A} & \longrightarrow & \mathbf{A}^2 & \longrightarrow & \mathbf{A} & \longrightarrow & \mathbf{C} \\
\mathbf{W} & \xrightarrow[\delta]{} & \mathbf{W}^2 & \xrightarrow[\pi]{} & \mathbf{W} & \xrightarrow[\epsilon]{} & \mathbf{C}
\end{array}$$

Only on D, the restrictions $\delta = \delta|D$, $\epsilon = \epsilon|D$ satisfy

$$\epsilon\colon D \to \mathbf{C},\ \epsilon\alpha = \sum \{\alpha(s) \mid s \in \mathrm{S}\}$$

$$\delta\colon D \to D^2,\ \delta\alpha(s, t) = 0 \quad \text{if } t \neq s$$

$$\delta\alpha(s, t) = \alpha(s) \quad \text{if } t = s \quad \alpha \in D;\ s, t \in \mathrm{S};$$

$\epsilon(\mathrm{S}) = \{1\}$, $\epsilon \in P\mathrm{S} \subset \mathbf{A}^*\mathrm{S}$

$\delta|D$ is monic; $\delta D \subseteq D^2$

(i) $\epsilon^2\delta = \epsilon$

(ii) δ preserves all three involutions:

$$\delta i = i^2\delta,\ \delta j = j^2\delta,\ \delta(ij) = (ij)^2\delta;$$

(iii) $\bar{\epsilon} i = \bar{\epsilon}$; $\epsilon j = \epsilon$.

6.12. Next, by 6.8, $\delta : \mathbf{W} \to \mathbf{W}^2$ is an isometry. Secondly, since $D \subset \mathbf{A}$, $D^2 \subset \mathbf{A}^2$ are norm dense, also $\delta\mathbf{A} \subseteq \mathbf{A}^2$. Thirdly, the density of $D \subset \mathbf{A}$, $D^2 \subset \mathbf{A}^2$, the continuity of ϵ, ϵ^2, δ, i, i^2, j^2 on $\mathbf{A}$, $\mathbf{A}^2$; the (separate) continuity of multiplication on $\mathbf{A}$, $\mathbf{A}^2$ imply that the equations (i)–(iii) remain true on $\mathbf{A}$ also. Lastly, by the same method, steps (1), (2), and (3) imply that (i)–(iii) remain valid on $\mathbf{W}$:

(1) $\mathbf{A} \subset \mathbf{W}$, $\mathbf{A}^2 \subset \mathbf{W}^2$ are σ, σ^2-dense.

(2) All the functions in (i)–(iii) are continuous on $\mathbf{W}$, $\mathbf{W}^2$ (not merely on $\mathbf{A}$, $\mathbf{A}^2$).

(3) Algebraic operations involved in (i)–(iii) are separately σ, σ^2-continuous.

REMARKS 6.13. 1. $\sigma \subseteq \mathscr{S} \subseteq \mathscr{S}^* \subseteq \tau$ are given by uniform structures; multiplication is separately continuous in all of them.

2. In general, $\mathbf{W}$ is not σ or τ-complete as a uniform topological space.

3. $\mathbf{W}$ is $\mathscr{S}$ and $\mathscr{S}^*$-complete as a topological space.

4. $\mathbf{W}$ is the completion of the uniform topological space A in $\mathscr{S}$ or $\mathscr{S}^*$.

5. Any uniformly continuous function of uniform spaces extends to a uniformly continuous function on the completion of these uniform spaces.

6. Now in 6.12, $\mathscr{S}$ or $\mathscr{S}^*$ could be used in place of σ, and 6.11 (2) weakened to require uniform continuity only on $\mathbf{A}$, $\mathbf{A}^2$.

7. All the functions in 6.12 (2) above (i.e., i, j, δ, and ϵ) are easily seen to be uniformly $\mathscr{S}$ (or $\mathscr{S}^*$)-continuous on D.

8. Assume now that $D \subset \mathbf{W}$ is $\mathscr{S}$-dense. Then by Remark 2, all our functions immediately extend to $\mathscr{S}$-continuous functions on $\mathbf{W}$. Theorem 1.12 shows that they are all (i.e., i, j, δ, ϵ) also σ-continuous. By Remark 1, δ and ϵ remain ring homomorphisms on $\mathbf{W}$.

9. The above, more general method of extending functions to all of $\mathbf{W}$, did not depend on the very specialized nature of $\mathbf{W} = \mathbf{A}^{**}$.

THEOREM 6.14. *Consider involutive semitopological semigroups* S, T *as in* 6.1 (*satisfying* 4.4 (1), 6.1 (2)); *the functors* D, $\mathbf{A}$, $\mathbf{W}$, $\mathbf{A}^*$ *that assign to* S *its semigroup, universal* $\mathbf{C}^*$ *and* S^**-enveloping algebras, and* $\mathbf{A}^*S = (\mathbf{A}S)^*$ *the norm dual* (4.15); *and the* (*tensor*) *products* $-\bigcirc-$, $-\bar{\otimes}-$, $-\nabla-$, $-\Delta-$ (5.1, 5.4). *Under the following identification*

$$S \times T = S \bigcirc T, \quad (s, t) = s \bigcirc t \quad s \in S,\ t \in T:$$

of the two semigroups, there is an inclusion diagram.

$$
\text{(i)}\quad
\begin{array}{ccccccccc}
S \times T & \subset & D(S \times T) & \subset & \mathbf{A}(S \times T) & \subset & \mathbf{W}(S \times T) & = & (\mathbf{A}^*(S \times T))^* \\
\| & & \| & & \| & & \| & & \| \\
S \bigcirc T & \subset & D(S \bigcirc T) & \subset & \mathbf{A}S \bar{\otimes} \mathbf{A}T & \subset & \mathbf{W}S \nabla \mathbf{W}T & = & (\mathbf{A}^*S \Delta \mathbf{A}^* T)
\end{array}
$$

$$
\begin{array}{c}
A^*(S \times T) \\
\| \\
\mathbf{A}^*S \Delta \mathbf{A}^*T
\end{array}
$$

If X, Y *are semigroups exactly of the same kind as* S, T *above, and*

$\phi : S \to X$ $\psi : T \to Y$ any involutive semigroup homomorphisms (not assumed to be identity preserving), then

(ii) $\exists!$ *(unique)* *W^*-map* $\mathbf{W}\phi \bigtriangledown \mathbf{W}\psi = \mathbf{W}(\phi \times \psi) : \mathbf{W}S \bigtriangledown \mathbf{W}T \to \mathbf{W}X \bigtriangledown \mathbf{W}Y$ *and a commutative diagram of algebra homomorphisms*

$$\begin{array}{ccccccc} S \times T & \subset DS \bigcirc DT & \subset \mathbf{A}S \,\bar{\otimes}\, \mathbf{A}T & \subset & \mathbf{W}S \bigtriangledown \mathbf{W}T \\ {\scriptstyle \phi \times \psi}\downarrow & \downarrow & \downarrow & & \downarrow {\scriptstyle \mathbf{W}(\phi \times \psi) = (\mathbf{A}^*\phi \times \psi))^*} \\ X \times Y & \subset DX \bigcirc DY & \subset \mathbf{A}X \,\bar{\otimes}\, X \,\bar{\otimes}\, \mathbf{A}Y & \subset & \mathbf{W}X \bigtriangledown \mathbf{W}Y \end{array}$$

$$\begin{array}{c} \mathbf{A}^*S \,\Delta\, \mathbf{A}^*T \\ \uparrow {\scriptstyle A^*(\phi \times \psi)} \\ \mathbf{A}^*X \,\Delta \mathbf{A}^*Y \end{array}$$

PROOF. (i) By 5.8, it suffices to prove that $\mathbf{A}(S \times T) = \mathbf{A}S \,\bar{\otimes}\, \mathbf{A}T$. First proof. Since $DS \subset \mathbf{A}\ S$ is dense, it follows that $D(S \times T) = DS \bigcirc DT \subset \mathbf{A}S \bigcirc \mathbf{A}T$ is dense. Second proof. Use of Theorem 4.16 easily shows that $\mathbf{A}(S \times T)$ has the universal property determining the $\mathbf{C}^*$-tensor product.

(ii) By 4.14, $S \times T$ satisfies 4.13(3). By 6.1(2), $D(S \times T) = \mathbf{C}(S \times T)$. Now Theorem 4.12 gives a unique extension of $\phi \times \psi : S \times T \to X \times Y$ to $\mathbf{A}(S \times T) \to \mathbf{A}(X \times Y)$. The rest follows from (i).

THEOREM 6.15. *Consider a semigroup S satisfying* 6.1 (2) *with $T \equiv S$, its semigroup algebra $D = \mathbf{C}S$, the two involutions i, j as well as the customary standard involution ij:*

$$i\alpha = \bar{\alpha};\ j\alpha(s) = \alpha(s^*),\ \alpha \in D,\ s \in S;$$

and the semigroup homomorphisms

$$\delta : S \to S \times S \qquad \epsilon : S \to \{1\}$$
$$S \to (s, s)$$

Let $\mathbf{A} \subset \mathbf{W}$ be the $\mathbf{C}^*$ and W^*-enveloping algebras of S and denote the corresponding identical objects for $S \times S$ by squares

$$i^2,\ j^2,\ \delta^2,\ \epsilon^2;\ D^2 \subset \mathbf{A}^2 \subset \mathbf{W}^2.$$

The categorical algebraic, $\mathbf{C}^*$, and W^*-tensor products are denoted by $- \bigcirc -$, $- \bar{\otimes} -$, and $- \bigtriangledown -$. Then all of the above five functions extend to $\mathbf{W}$ (where they are denoted as before) such that

(i) $i, j, ij : \mathbf{W} \to \mathbf{W};\ \delta : \mathbf{W} \to \mathbf{W}^2;\ \epsilon : \mathbf{W} \to \mathbf{C}$ *are σ-continuous.*

There is a commutative diagram below, where:

(ii) *i, j: $\mathbf{W} \to \mathbf{W}$ are isometric involutions of $\mathbf{W}$ onto itself;*

(iii) *$\delta : \mathbf{W} \to \mathbf{W}^2$, $\epsilon : \mathbf{W} \to \mathbf{C}$ are (involutive) homomorphisms of W^*-algebras with δ an isometry;*

(iv) *algebraic equations satisfied on D remain valid on $\mathbf{W}$; i.e., δ preserves all three involutions*

$$\delta i = i^2\delta, \quad \delta j = j^2\delta, \quad \delta ij = i^2 j^2 \delta; \text{ and}$$

$$\epsilon i = \bar{\epsilon}, \quad \epsilon j = \epsilon;$$

$$\epsilon = \epsilon^2 \delta;$$

(v) $D^2 \cong D \bigcirc D$, $\mathbf{A}^2 \cong \mathbf{A} \mathbin{\bar{\otimes}} \mathbf{A}$, $\mathbf{W}^2 \cong \mathbf{W} \triangledown \mathbf{W}$.

(vi)

$$\begin{tikzcd}
S \arrow[d, "\delta"'] & \subset D \arrow[d] & \subset \mathbf{A} \arrow[d] & \subset \mathbf{W} \arrow[d, "\delta"] \arrow[dr, "\epsilon"] & \\
S \times S & \subset D^2 & \subset \mathbf{A}^2 & \subset \mathbf{W}^2 \arrow[r, "\epsilon^2"'] & \mathbf{C}
\end{tikzcd}$$

Corollary 6.16. *With the notation and hypotheses of the previous theorem, set $\mathbf{W} \triangledown \mathbf{W}^2 \equiv \mathbf{W}^3 \equiv \mathbf{W}^2 \triangledown \mathbf{W}$, let $1 \bigcirc \delta : D \bigcirc D \to \mathbf{W} \triangledown \mathbf{W}^2$ be induced by $1 \times \delta : S \times S \to S \times S \times S$, and let $\tau : \mathbf{W}^2 \to \mathbf{W}^2$ be the W^*-map extending the transpose*

$$\tau : S \times S \to S \times S, \ \tau(a, b) = (b, a), \ a, b \in S.$$

Then there are commutative diagrams of W^*-maps

(i)

$$\begin{tikzcd}
\mathbf{W}^2 \arrow[r, "1 \triangledown \delta"] \arrow[d, "\delta \triangledown 1"'] & \mathbf{W}^3 \arrow[d, "\delta \triangledown 1"] \\
\mathbf{W}^3 \arrow[r] & \mathbf{W}^4
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{W}^2 \arrow[rr, "\tau"] \arrow[dr, "1 \triangledown \delta"'] & & \mathbf{W}^2 \arrow[dl, "\delta \triangledown 1"] \\
& \mathbf{W}^3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{W}^2 \arrow[rr, "\tau"] \arrow[dr, "\epsilon^2"'] & & \mathbf{W}^2 \arrow[dl, "\epsilon^2"] \\
& \mathbf{C} &
\end{tikzcd}$$

(ii) *The diagrams in* (i) *hold for* $\mathbf{A}$, $- \bar{\otimes} -$ *and for* D, $- \bigcirc -$ *in place of* $\mathbf{W}$, $- \triangledown -$.

(iii) *Furthermore, the diagram for W^* restricts to the one for $\mathbf{C}^*$, and the $\mathbf{C}^*$-diagram by restriction to the one for involutive rings.*

Theorem 4.12 together with the last theorem give the next corollary.

Corollary 6.17. *Suppose both S and T are semigroups as in the previous theorem (i.e., with S, S and T, T satisfying* 6.1 (2)) *and $\phi : S \to T$*

is a continuous involutive semigroup homomorphism. If k, ∂ are the maps for T corresponding to ϵ, δ (for S) then there is a diagram of W^*-maps

(i) *where the rectangle is commutative.*

(ii) *Furthermore, if $\phi e = 1 \in T$ is the identity for T where $e = 1 \in S$, then the whole diagram commutes.*

(iii) *Conclusions* (i) *and* (ii) *remain valid if first,* $\mathbf{W}$, $-\nabla-$ *is replaced by* $\mathbf{A}$, $-\bar{\otimes}-$ *and second, by* D, $-\bigcirc-$.

$$\begin{tikzcd}
 & \mathbf{W}S \arrow[dl, dashed, "\epsilon^2"'] \arrow[r, mapsto, "\delta"] \arrow[dd, "\mathbf{W}\phi"] & \mathbf{W}S \nabla \mathbf{W}S \arrow[dr, dashed, "\epsilon^2"] \arrow[dd, "\mathbf{W}(\phi \times \phi)"] & \\
\mathbf{C} & & & \mathbf{C} \\
 & \mathbf{W}T \arrow[ul, dashed, "k"] \arrow[r, mapsto, "\partial"'] & \mathbf{W}T \nabla \mathbf{W}T \arrow[ur, dashed, "k"'] &
\end{tikzcd}$$

The next theorem emphasizes one important property of the functors and tensor products.

THEOREM 6.18. *Consider the multiplicative categories* (sgrps, x), ($\mathbf{C}^*$, $\bar{\otimes}$), (W^*, ∇), *and* ($\mathscr{D}$, Δ) *of semigroups as in* 6.2, $\mathbf{C}^*$*-algebras,* W^**-algebras, and their preduals* (5.4) (*all of these with not necessarily identity preserving maps*). *Then there is a commutative diagram of functors* $\mathbf{A}$, E, $\mathbf{W}$, $\mathbf{A}^*$ *and* F (4.15) *of multiplicative categories* (*i.e.,* (*tensor*) *products in the domain category are mapped into products in the image category of the same corresponding objects*):

$$\begin{tikzcd}
 & (\text{Sgrps}, x) \arrow[dl, "\mathbf{A}"'] \arrow[dr, "\mathbf{A}^*"] \arrow[dd, dashed, "\mathbf{W}"] & \\
(\mathbf{C}^*, \bar{\otimes}) \arrow[dr, "E"'] & & (\mathscr{D}, \Delta) \arrow[dl, "F"] \\
 & (W^*, \nabla) &
\end{tikzcd}$$

The functor F^{-1}, being contravariant, will turn coalgebras into algebras. Finally all the separate parts have been constructed—i.e., 5.10, 6.15, (v), 6.15 (vi), and 6.17—so that they can be assembled together to produce what long has been our objective—the next theorem.

THEOREM 6.19. *With the notation of the last theorem* (6.15 *and* 6.1) *and the same hypotheses as in the last theorem* (6.15), *for any f, $g \in \mathbf{A}^*$, form $f \bigcirc g : D^2 \to \mathbf{C}$, while $f \bar{\otimes} g \in \mathbf{A}^{2*}$ and $f \Delta g \in \mathbf{AA}^{*2} = \mathbf{A}^{*2} = \mathbf{A}^* \Delta \mathbf{A}^*$ have already been formed* (5.10). *There is a commutative* *diagram of also commutative normed algebras with an isometric involution* $((ij)^*)$ *and multiplication* $\delta^* : \mathbf{A}^* \times \mathbf{A}^* \rightarrow \mathbf{A}^*$:

(i)
$$\begin{tikzcd} B^2 \arrow[d, "\delta^*"'] & \subseteq & \overline{\mathbf{B}}^2 \arrow[d] & \subset & \mathbf{A}^{*2} \arrow[d, "\delta^*"] \arrow[dr, "\epsilon^{2*}"] & \\ & & & & & \mathbf{C} \\ B & \subseteq & \overline{B} & \subset & \mathbf{A}^* \arrow[ur, "\epsilon^*"'] & \end{tikzcd}$$

(ii) $\mathrm{S} \subset \mathbf{W} \equiv (\mathbf{A}^*\mathrm{S})^*$ *are multiplicative linear functionals on the Banach algebra* $\mathbf{A}^*$, *i.e.*, $(\delta^*(f \Delta g))(s) = f(s)g(s)$ *for* $s \in \mathrm{S}$.

(iii) *The restriction of* $f \Delta g$ *to* $\mathbf{A}^2$ *is*: $f \Delta g \mid \mathbf{A}^2 = f \mathbin{\overline{\otimes}} g$

(iv) $f, g \in \mathbf{B} \Rightarrow f \Delta g \mid \mathrm{S} \times \mathrm{S} = f \bigcirc g$

(v) *When regarded as* $\epsilon \in P \subset \mathbf{B} \subset \mathbf{A}^*$, ϵ *is the identity element of* $\mathbf{A}^*$, *i.e.*, $\delta^*(\epsilon \Delta f) = \delta^*(f \Delta \epsilon) = f$.

Proof. Remark: For β, $\alpha = \Sigma\alpha(s)s \in D$, $f(\alpha) = \Sigma\alpha(s)f(s)$ although $f \notin \overline{\mathbf{B}}$ in general. Thus

$$(f \nabla g)(\alpha \bigcirc \beta) = \Sigma\{\alpha(s)f(s)\beta(t)g(t) \mid (s, t) \in \mathrm{S} \times \mathrm{S}\},$$

and again $f \nabla g \notin \overline{\mathbf{B}}^2$ in general.

Conclusions (i)–(iv) are clear. (v) For $\alpha \in D$, $\delta\alpha = \Sigma\alpha(s)s \bigcirc s$ and $\delta^*(\epsilon \Delta f)(\alpha) = (\epsilon \Delta f)(\delta\alpha) = \Sigma\alpha(s)((\epsilon \Delta f)(s \bigcirc s)) = \Sigma\alpha(s)\epsilon(s)f(s) = f(\alpha)$. Hence $\delta^*(\epsilon \Delta f) = f$.

Remark 6.20. In view of 5.6, this whole section could be generalized to the case where 6.1 (2) is replaced by the assumption that $I(\mathrm{S} \times T) = I\mathrm{S} \bigcirc \mathbf{C}T + \mathbf{C}\mathrm{S} \bigcirc IT$. Since the functors D, $\mathbf{A}$, $\mathbf{W}$; P, $\mathbf{B}$, $\overline{\mathbf{B}}$, $\mathbf{A}^*$ agree on S and $\mathrm{S}/\Omega\mathrm{S}$, it could be assumed without loss of generality that $\mathrm{S} = \mathrm{S}/\Omega\mathrm{S}$; the more general case would then follow.

7. **Locally Compact Group.**

The previous general construction is now specialized for the case of a locally compact group.

7.1. Consider a locally compact topological group G with left invariant Haar measure and modular function Δ, i.e., $\int f(tx)\,dt = \Delta(x)^{-1} \int f(t)\,dt$ for all $x \in G$ and $f \in L^1 = L^1(G)$. The set of all bounded complex Borel measures on G will be denoted by $M(G)$ with $L^1(G) \subset M(G)$. For μ, $\nu \in M(G)$ and a measurable set $E \subset G$ with characteristic function k, define $\mu * \nu$ and $\mu^* \in M(G)$ by

(1) $(\mu * \nu)(E) = \int k(st)\,d\mu(s)\,d\nu(t)$,

(2) $\mu^*(E) = \mu(E^{-1})^-$.

For $a \in G$, $\bar{a} \in M(G)$ is defined by $\bar{a}(x) = 0$ if $x \neq a$ and $\bar{a}(x) = 1$ if $x = a$. The left and/or right translate of a function $f \in L^1(G)$ by a, $b \in G$ is denoted by afb, where $(afb)(x) = f(bxa)$. Then formulas (1) and (2) become

$$(3) \qquad (\bar{a} * \nu)(E) = \nu(a^{-1}E) \qquad (\nu * \bar{b})(E) = \nu(Eb^{-1})$$
$$\bar{a} * f = fa^{-1} \qquad f * \bar{b} = \Delta(b)^{-1}b^{-1}f$$
$$\bar{a} * \bar{b} = (ab)^{-}.$$

7.2. In the special case 3.3 of the general framework of 2.2, set $R = D = L^1(G)$, take $P = P(G)$ as the continuous positive definite functions on G, and $|\cdots|$ as the L^1-norm. By 3.3, $\mathbf{A}R$ is the ordinary $\mathbf{C}^*$-enveloping algebra of G; $\mathbf{A}R$ will be denoted by $\mathbf{C}^*(G) \equiv \mathbf{A}L^1(G)$. Hence $\mathbf{W}R$ is the ordinary W^*-enveloping algebra of G, to be denoted henceforth by $W^*(G) \equiv \mathbf{W}R$. The completion of $L^1(G) = D$ in either $\mathscr{S}$ or $\mathscr{S}^*$ is

$$(D, \mathscr{S})^{-} = (D, \mathscr{S}^*)^{-} = W^*(G).$$

Furthermore, $(\mathbf{A}^*R)^+ = P$, and $(\mathbf{C}^*G)^* = B(G) = (W^*(G))_*$.

7.3. Any strongly (weakly or ultra-weakly) continuous representation of G as unitary operators on a Hilbert space extends uniquely to $L^1(G)$ and uniquely to one of $\mathbf{C}^*(G)$. Conversely, the unitary representations of $\mathbf{C}^*(G)$ come from those of G. Thus subject to containing $L^1(G)$, the latter universal property uniquely characterizes $\mathbf{C}^*(G)$.

7.4. Embed $G \subset M(G)$ by $a \to \bar{a}$; write $a = \bar{a}$, and hence $\mathbf{C}G \subset M(G)$. If the duality between $M(G)$ in the total variation norm and the bounded continuous functions on G is denoted by $\langle , \rangle$, then

$$\langle \mu^*, f \rangle = \langle \mu, f^* \rangle^{-},$$
$$\langle \mu * \nu, f \rangle = \int f(st)\, d\mu(s)\, d\nu(t).$$

Each $a \in G$ induces a Hilbert space isometry

$$L^2(G) \to L^2(G),\ \xi \to \bar{a} * \xi = \xi a^{-1}.$$

Let $V(G)$ denote the double commutant $V(G)$ of these isometries; then $L^1(G) \subset M(G) \subset V(G)$ (see 7.1, (3)). Let $\mathscr{K}(G) \subset B(G)$ be the set of those functions having compact support. The predual $V(G)_*$ of $V(G)$ can be identified in $V(G)_* \subset BG$ as the closure of $\mathscr{K}(G)$, as a subset $\mathscr{K}(G) \subset B(G)$, in the norm derived from $B(G) = (W^*(G))_*$. Since $V(G)_*$ is translation invariant, its annihilator $(V(G)_*)^{\perp}$ is both an ideal $(V(G)_*^{\perp} \lhd W^*(G)$ as well as a W^*-subalgebra of $W^*(G)$. Hence $(V(G)_*)^{\perp}$ is a direct summand of $W^*(G)$; its complementary summand is

$$W^*(G)/(V(G)_*)^{\perp} \cong (V(G)_*)^* \cong V(G).$$

Consequently

$$V(G)_* = B(G) \Longleftrightarrow V(G)_*^{\perp} = 0$$

$$\Longleftrightarrow G \text{ is compact} \Longleftrightarrow V(G) = W^*(G).$$

7.5. In the context of 6.1 and 6.2 with $S = G$ and $PS = P(G)$ as before, all the hypotheses hold (6.1 (2), 4.15, and 6.2). Since $P(G)$ induces the same norm on $\mathbf{C}G$ as that obtained as a subset $\mathbf{C}G \subset W^*(G)$, $\mathbf{A}G$ is the norm closure of $\mathbf{C}G$ in $W^*(G)$. Since $\mathbf{C}G \subset W^*(G)$ is σ-dense, $W^*(G) = \mathbf{WC}G \equiv \mathbf{W}G$. The latter can be independently established by observing that the universal property determining $\mathbf{WC}G$ is also satisfied by $\mathbf{W}^*(G)$. Since $(\mathbf{W}G)_* = \mathbf{A}^*G$, it follows that $\mathbf{A}^*G = BG = B(G)$.

7.6. Embed $V(G) \dashrightarrow \mathbf{W}G$ as the unique complement of $(V(G))^{*\perp}$. There is a diagram of natural inclusions that is a commutative diagram provided the latter map $(V(G) \dashrightarrow \mathbf{W}G$ is removed.

$$\begin{tikzcd}
 & \mathbf{W}G & \\
\mathbf{A}G \arrow[ur] & V(G) \arrow[u] & \mathbf{C}^*(G) \arrow[ul] \\
G \arrow[u] \arrow[r] & M(G) \arrow[u] \arrow[r] & L^1(G) \arrow[u]
\end{tikzcd}$$

7.7. For any topological group G with a locally compact topology $\mathscr{O}$, let G_{d} be G with the discrete topology, and set $\ell^1(G_{\mathrm{d}}) \equiv L^1(G_{\mathrm{d}})$. Its usual $\mathbf{C}^*$-algebra $\mathbf{C}^*(G_{\mathrm{d}}) = \mathbf{A}\ell^1(G_{\mathrm{d}})$ is obtained by use of the set $P(G_{\mathrm{d}})$ of all (also discontinuous) positive definite functions on G ([**6**; p. 188, (1.18)]).

Since $\mathbf{C}G_{\mathrm{d}} \subset \ell^1(G_{\mathrm{d}})$ is dense in the L^1 or ℓ^1-norm, which is smaller than the C^*-norm 2.4, it follows that $\mathbf{A}G_{\mathrm{d}} \equiv \mathbf{A}G_{\mathrm{d}} = \mathbf{A}\ell^1(g_{\mathrm{d}})$, or $\mathbf{A}G_{\mathrm{d}} = \mathbf{C}^*(G_{\mathrm{d}})$.

Let $\mathbf{A}G$ be as in 4.15. Since $P(G_{\mathrm{d}}) \supseteq P(G)$, by 3.9(b) there is a $\mathbf{C}^*$-map $\mathbf{C}^*(G_{\mathrm{d}}) \to \mathbf{A}G$. By 7.5 the algebras given by 4.15 are equal to those given by 7.2, i.e., $\mathbf{W}G_{\mathrm{d}} = \mathbf{W}^*(G_{\mathrm{d}})$ and $\mathbf{W}G = W^*(G)$. The σ-topologies induce on $G \subset \mathbf{W}G_{\mathrm{d}}$ the discrete and on $G \subset \mathbf{W}G$ the topology $\mathscr{O}$. Thus if $\mathscr{O}$ is nondiscrete, $\mathbf{W}G \neq \mathbf{W}G_{\mathrm{d}}$. But since $\mathbf{W}G = (\mathbf{A}G)^{**}$ and $\mathbf{W}G_{\mathrm{d}} = \mathbf{C}^*(G_{\mathrm{d}})^{**}$, it follows that also $\mathbf{A}G \neq \mathbf{C}^*(G_{\mathrm{d}})$.

Conclusion (iii) below follows from the fact that the identity $e \in \mathbf{C}^*(G)$ if and only if G is discrete (see [**1**; p. 457, Corollary 1 to Theorem 2]).

7.8. A locally compact group $(G, \mathscr{O})$ and its discretization G_{d} satisfy the following:

(i) $\mathbf{A}G_d = \mathbf{C}^*(G_{\mathrm{d}})$;

(ii) $\mathbf{A}G = \mathbf{C}^*(G_{\mathrm{d}}) \Longleftrightarrow (G, \mathscr{O})$ is discrete $\Longleftrightarrow \mathbf{W}G = W^*(G_{\mathrm{d}})$;

(iii) $AG = \mathbf{C}^*(G) \Longleftrightarrow (G, \mathcal{O}) = G_d$ is discrete.

8. **Locally Compact Abelian Group.** These groups provide easy examples of W^*-algebras A, B with $A_* \bar{\otimes} B_* \subset A_* \Delta B_*$ properly.

NOTATION 8.1. For a locally compact abelian group G, $\hat{G}$ denotes its character group with $(\hat{G})\hat{} = G$. For $f \in L^1(G)$ and $\mu \in M(\hat{G})$, as usual, the Fourier transforms are functions $\hat{f}: \hat{G} \to \mathbf{C}$ and $\hat{\mu}: G \to \mathbf{C}$. As before, the $\mathbf{C}^*$-norm from $(\mathbf{W}G)_* = \mathbf{B}(G)$ on $\mathbf{B}(G)$ is $\| \;\|$. The greatest and least crossnorms on the tensor product of two Banach spaces will be denoted by "γ" and "λ" respectively.

Some facts to be used later are summarized below.

8.2. If X, Y are locally compact Hausdorff spaces and $M(X)$, $M(Y)$ the bounded complex regular Borel measures with the total variation norm, then the measures absolutely continuous with respect to some product measure on $X \times Y$ can be identified as exactly as the γ-closure

$$\gamma - \mathrm{cl}\,(M(X) \bigcirc M(Y)) \subseteq M(X \times Y)$$

(See [**11**; p. 370, Theorem 2.2]). If X and Y are non-discrete, then it can be shown that the above inclusion is proper.

8.3. For $f \in L^1(G)$, $\mu \in M(\hat{G})$, and $b \in \mathbf{B}(G)$, the following hold:

(i) $\int f(g)\, d\hat{\mu}(g) = \int \hat{f}(\gamma)\, d\mu(\gamma) \quad g \in G, \gamma \in \hat{G}$.

(ii) The Fourier transform $M(\hat{G}) \to M(\hat{G})\hat{}$ is an isometry in the total variation norm.

(iii) $\|b\| = \sup\{\hat{b}(\gamma) \mid \gamma \in \hat{G}\}$.

(iv) $(M(\hat{G}))\hat{} = \mathbf{B}(G)$; furthermore $M(\hat{G}) \to M(\hat{G})\hat{} = \mathbf{B}(G)$ is an isometry for the total variation norm on $M(\hat{G})$ and the $\mathbf{C}^*$-norm $\| \;\|$ on $\mathbf{B}(G)$.

8.4. For locally compact abelian groups G, H also $\mathbf{W}G$ and $\mathbf{W}H$ are abelian as well as $\mathbf{A}G$ and $\mathbf{A}H$. But commutative $\mathbf{C}^*$-algebras carry only one unique $\mathbf{C}^*$-tensor product, and the unique $\mathbf{C}^*$-cross norm used to form the categorical $\mathbf{C}^*$-tensor product is the least cross norm λ. Then $\mathbf{W}G_* \bar{\otimes} \mathbf{W}H_*$ is the completion in the dual norm λ^* where

$$\mathbf{W}G_* \bar{\otimes} \mathbf{W}H_* \subseteq \mathbf{W}G_* \Delta \mathbf{W}H_* \subseteq (\mathbf{W}G \otimes \mathbf{W}H)^*.$$

But λ^* equals γ, the greatest cross-norm.

COUNTEREXAMPLE 8.5. For locally compact abelian non-discrete groups G and H, set $X = \hat{G}$ and $Y = \hat{H}$. Then $\mathbf{W}G_* = \mathbf{B}(G) \cong M(\hat{G})$ and similarly for H. Also

$$(\mathbf{W}(G \times H))_* = \mathbf{B}(G \times H) \cong M(\hat{G} \times \hat{H}).$$

Since $\mathbf{W}G_* \Delta \mathbf{W}H_* = \mathbf{W}(G \times H)_*$, always we have a proper inclusion $\mathbf{B}G \bar{\otimes} \mathbf{B}H \cong \gamma - \mathrm{cl}\,(M(\hat{G}) \bigcirc M(\hat{H})) \subset M(\hat{G} \times \hat{H}) \cong \mathbf{B}(G \times H)$, or $WG_* \bar{\otimes} WH_* \subset WG_* \triangle WH_*$ properly.

8.6. For any W^*-algebras M and N whatever, their non-categorical spatial tensor product will be denoted by $M \boxtimes N$. (See [13; p. 67, 1.22.10–1.22.11] and [12; p. 3.17, Definition 2.2 and Theorem 2.3]). If α is the greatest C^*-cross norm on $N \bigcirc N$, and α^* the induced dual norm on $M^* \bigcirc N^*$, then the predual of $M \boxtimes N$ is

$$M \boxtimes N* = \alpha^* - \mathrm{cl}(M* \bigcirc N*).$$

For M and N commutative, there is a unique C^*-cross norm $\alpha = \lambda = \rho$ and hence $\alpha^* = \rho^* = \gamma$([12; p. 62, 1.22.5]).

Now set $M = \mathbf{W}G$, $N = \mathbf{W}H$ as in 8.5. Then $(\mathbf{W}G \boxtimes \mathbf{W}H)* = WG* \bar{\otimes}\ \mathbf{W}H*$ by the above and $(\mathbf{W}G \boxtimes \mathbf{W}H)* \neq \mathbf{W}G* \Delta\ \mathbf{W}H*$ by 8.5. Thus $\mathbf{W}G \boxtimes\ WH \neq \mathbf{W}G \bigtriangledown \mathbf{W}H$.

References

1. C. Akeman and M. Walter, *Non-abelian Pontriagin duality,* Duke Math. Jour. **39** (1972), 451–463.

2. J. Dauns, *Multiplier rings and primitive ideals,* Trans. Amer. Math. Soc. **145** (1969), 125–158.

3. ———, *Categorical W*-tensor product,* Trans. Amer. Math. Soc. **166** (1972), 439–456.

4. ———, *Simple modules and centralizers,* Trans. Amer. Math. Soc. **166** (1972), 457–477.

5. J. Dixmier, *Les C*-algébrès et lurs représentations,* Gauthier-Villars, Paris (1969).

6. P. Eymard, *Algèbre de Fourier,* Bull. Soc. Math. France **92** (1964), 181–236.

7. A. Guichardet, *Catégorie des algèbres de von Neumann,* Bull. Sciences Math. **90** (1966), p. 41–64.

8. K. H. Hofmann, *The duality of compact semigroups and C*-algebras,* Lecture Notes in Math., Springer, New York **129** (1970), 1–141.

9. E. McCharen, Ph.D. Dissertation, Louisiana State Univ., Baton Rouge (1968).

10. ———, *A new topology on B*-algebras,* Proc. Amer. Math. Soc. **37** (1973), 77–83.

11. J. F. Rennison, *Arens products and measure algebras,* J. London Math. Soc. **44** (1969), 369–377.

12. S. Sakai, *The theory of W*-algebras,* Yale University Lecture Notes (1962).

13. ———, *C*-algebras and W*-algebras,* Springer, New York, Ergebnisse series **60** (1971).

14. E. M. Sweedler, *Hopf algebras,* Math. Lecture Note Series, W. A. Benjamin Inc., New York (1969).

15. M. Takesaki, *Duality and von Neumann algebras,* Lecture Notes in Mathematics, Springer, New York **247** (1972), 666–786.

16. D. Topping, *Lectures on von Neumann algebras,* Van Nostrand Mathematical Studies, London **36** (1971).

17. M. Walter, *Group duality and isomorphisms of Fourier and Fourier Stieljes algebras from a W*-algebra point of view,* Bull. Amer. Math. Soc. **76** (1970), 1321–1325.

18. ———, *The dual group of the Fourier-Stieltjes algebra,* Bull. Amer. Math. Soc. **78** (1972), 824–827.

19. ———, *W*algebras and non-abelian harmonic analysis,* Jour. of Functional Analysis **11** (1972), 17–38.

Tulane University, New Orleans, Louisiana 70118