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A CHARACTERIZATION OF "HEISENBERG GROUPS"; WHEN IS A PARTICLE FREE?

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ABSTRACT. Let H be a connected Lie subgroup of a connected Lie group G. Sufficient conditions are given to ensure that the smallest closed normal subgroup of G containing H is a generalized Heisenberg group. The conditions are that there should exist an irreducible representation T of G and a nonzero vector φ in the space of T such that (a) $g \rightarrow (T_g(\varphi), \varphi)$ vanishes at infinity, and (b) the set of functions $h \rightarrow |(T_h(T_g(\varphi)), T_g(\varphi))|$ should be an equicontinuous family of functions on H. Applications are made to theoretical quantum mechanics to conclude that a particle is free of external forces providing some state is appropriately transformed under the action of the symmetry group.

1. **Introduction.** In this paper we shall concern ourselves primarily with a Lie subgroup H of a Lie group G. There are two properties of representations T of G which we shall explore. The first is that T should "vanish at infinity", (matrix elements vanish at infinity). The second is that there should exist a nonzero vector φ in the space $H(T)$ of T such that the set $[T_g(\varphi)]$ of vectors is an "absolutely equicontinuous family of vectors for $T|_H$." (This means that the set of functions $h \rightarrow |(T_h(T_g(\varphi)), T_g(\varphi))|$ should be an equicontinuous family of functions of H.) Let us discuss these properties briefly.

Whether a representation vanishes at infinity or not is of interest to other researchers, for other reasons, and it is not very well understood at all. We prove below, Theorem 2.4, that many standard representations do vanish at infinity, e.g., faithful irreducible representations of connected compact extensions of vector groups. We also indicate a proof below that every locally compact group possesses a separating family of representations which vanish at infinity. On the other hand, there are groups none of whose irreducible representations vanish at infinity.

Our second property is perhaps more interesting, perhaps because it is satisfied so seldom. There are a number of other conditions which imply the existence of an absolutely equicontinuous family of vectors. If $T|_H$ is uniformly continuous, then every vector φ generates an absolutely equicontinuous family. If, for some vector φ, the set $[T_g(\varphi)]$ forms a precompact subset of the space $H(T)$, then $[T_g(\varphi)]$ is an absolutely equicontinuous family. If the set of functions $h \rightarrow (T_h(T_g(\varphi)), T_g(\varphi))$ (without

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the absolute values) is an equicontinuous family of functions on H, then φ generates an absolutely equicontinuous family. The important point is that all three of these conditions are strictly stronger than absolute equicontinuity. These three rarely occur. For instance, in [5] it is shown that if H has a separating family of uniformly continuous representations, then H is the direct product of a compact group with an abelian group. On the other hand the weaker property of absolute equicontinuity does occur, when H is a Heisenberg group for example, and from our experience it appears to be a property both delicate and important.

In many applications, for example in Section 5 where we prove a theorem in theoretical quantum mechanics, H is a one-parameter group. In that case we can describe absolute equicontinuity in a more concrete way. Thus if T is a unitary representation of the real line, (as $T|_H$ would be), then by Stone's theorem we know that there exists a projection-valued measure $\underline{p}$ on the Borel subsets of the line such that $T_t = \int e^{it\lambda}\, d\underline{p}(\lambda)$. If ψ is any vector in $H(T)$, then $(T_t(\psi), \psi) = \int e^{it\lambda}\, d\mu_\psi(\lambda)$, where μ_ψ is the measure on R determined by $\underline{p}$ and ψ. Now the assumption that φ generates an absolutely equicontinuous family of vectors is the same as assuming that the set of functions $t \to |\int e^{it\lambda}\, d\mu_g(\lambda)|$, (where μ_g is the measure determined by $\underline{p}$ and the vector $T_g(\varphi)$.), is an equicontinuous family of functions on the real line. But these functions are equicontinuous if and only if the functions $t \to |\int e^{it\lambda}\, d\mu_g(\lambda)|^2$, which equals $|\hat{\mu}_g|^2(t)$, which equals $[\mu_g * \mu_g^*]\hat{}(t)$, are equicontinuous. Consequently, φ generates an absolutely equicontinuous family of vectors if and only if this set of Fourier transforms of measures, (probability measures if φ is a unit vector), forms an equicontinuous family of functions on R. We do not know of any particular criterion ensuring this, but it appears to be a condition with which one could compute.

These properties of the representations of G will have again implications about the structure of G and of H. Compactness and commutativity will again play a central role, but this structure is not so simple as a direct product. Compare [5]. Indeed our main result, Theorem 3.1, asserts that if a representation T satisfies both of the above properties, then the smallest closed normal subgroup H'' of G containing H is a compact extension of a vector group, by which we mean that H'' contains a compact normal subgroup K for which the quotient group H''/K is a vector group. Mathematically this is simply a statement that certain representational hypotheses imply certain structural conclusions. This may very well be of theoretical interest in itself, but the author feels that it is in the contrapositive direction that this result will be useful. If one knows that H'' is not a compact extension of a vector group, then information about $T|_H$ is available by our theorem. Some examples of this sort are given in Section 4.

Suppose T is an irreducible unitary representation of a Heisenberg group G. Then, factoring out its kernel Z, we may think of T as a representation of G/Z. We know all of the irreducible representations of G, and so we know that G/Z either is a circle group T^1 or Z is isomorphic with the group of integers, in which case G/Z is an extension of an even-dimensional Euclidean space R^{2j} by a circle group. Always then G/Z is a compact extension of a vector group. It seems natural to think of this class of groups as "Generalized Heisenberg Groups" at least in so far as their representations are concerned.

Finally, in Section 5, we make an application to quantum mechanics. Using Theorem 3.1 we can give conditions under which a one-parameter subgroup of the symmetry group of a system is a group of pure translations. In other words, given certain hypotheses about the representation of the symmetry group, (that representation coming from the quantum mechanical model), there must exist some translational symmetry. For example, if there is an external force acting on the system, then it must be acting perpendicularly to some fixed direction. Clearly, if enough of these hypotheses hold then there can be no external force.

2. **Special Properties of Representations.** By a *representation* we shall mean either a unitary representation or a multiplier representation. If T is a representation, then $H(T)$ denotes the Hilbert space in which T acts.

DEFINITION 2.1. Let G be a locally compact group. A representation T of G is said to *vanish at infinity* if for each pair (v, w) of vectors in $H(T)$ the function $g \longrightarrow (T_g(v), w)$ vanishes at infinity.

This definition is made in the same spirit as are those of integrable and square-integrable representations. Indeed, since the functions $g \longrightarrow (T_g(v), w)$ are all uniformly continuous, it follows that a representation which is either integrable or square-integrable must necessarily vanish at infinity. We have the following expected and routine proposition.

PROPOSITION 2.2.

(i) *If T is a cyclic representation of a locally compact group G, and if there exists a cyclic vector v such that the function $g \longrightarrow (T_g(v), v)$ vanishes at infinity, then T vanishes at infinity.*

(ii) *If T vanishes at infinity, then its kernel must be compact.*

(iii) *No finite dimensional representation of a noncompact group can vanish at infinity.*

(iv) *Each cyclic subrepresentation of the regular representation of a* *locally compact group vanishes at infinity. In particular, every discrete series representation vanishes at infinity.*

PROOF. Part (iii) is perhaps the only statement not immediately provable. If T is an n-dimensional representation of G which vanishes at infinity, and if $[v_i]$ is an orthonormal basis for $H(T)$, then there exists a compact subset C of G such that every matrix entry $(T_g(v_j), v_i)$ of the operator T_g has absolute value less than $n^{-1/2}$ whenever g is outside C. But if there were such an element g, then T_g would not be a unitary operator. Therefore $C = G$.

We remark that from (iv) it follows that every locally compact group possesses a separating family of (probably reducible) unitary representations which vanish at infinity. On the other hand, the real line is an example of a group none of whose irreducible unitary representations vanish at infinity. Exactly which irreducible unitary representations of which groups do vanish at infinity is a question of considerable interest and complication. We content ourselves here with the one result, Theorem 2.4 below.

LEMMA 2.3. *Let G be a connected Lie group which contains a compact normal subgroup N for which G/N is a vector group. Let Z denote the center of N. Then the quotient G/Z is a direct product of a semisimple compact group with a vector group.*

PROOF. Let p be a Borel cross-section of G/N into G. (See [7].) We show first that the element $p(x)p(y)[p(x + y)]^{-1}$ of N is actually an element of Z. Thus let L be an irreducible unitary representation of N. By the Mackey procedure, [9], we know that for each x in G/N there exists a unitary operator U_x on the space of L such that $L_{[p(x)n[p(x)]^{-1}]} = U_x$-$L_n U_x^{-1}$ for all n in N. Then by a direct computation, and using the fact that U_{x+y} is a scalar multiple of $U_x U_y$, we find that $L_{[p(x)p(y)[p(x+y)]^{-1}]}$ commutes with every operator L_n and is therefore a scalar. Since this is true for every irreducible unitary representation L of N, we have that $p(x)p(y)[p(x + y)]^{-1}$ belongs to Z.

Modulo Z then, p is an isomorphism of G/N into G/Z, and G/Z is the semidirect product of the normal semisimple compact group N/Z with the vector group G/N. The Lie algebra $\mathscr{A}$ of G/Z is then a direct sum of a semisimple Lie algebra $\mathscr{B}$, the Lie algebra of N/Z, with an abelian algebra $\mathscr{C}$, the Lie algebra of G/N. Now since every derivation of $\mathscr{B}$ is inner, and since $\mathscr{B}$ is an ideal, there exists a linear mapping Φ of $\mathscr{C}$ into $\mathscr{B}$ such that $[Y, X] = [\Phi(Y), X]$ for all Y in $\mathscr{C}$ and all X in $\mathscr{B}$. By the Jacobi identity we see also that Φ preserves brackets. Defining $\psi(Y) = Y - \Phi(Y)$, we find that ψ is an isomorphism of $\mathscr{C}$ with another abelian subalgebra $\mathscr{D}$ of $\mathscr{A}$. Finally, the mapping $X + Y \rightarrow X + \psi(Y)$ is an isomorphism of the direct sum $\mathscr{B} \oplus \mathscr{C}$ onto the sum $\mathscr{B} \oplus \mathscr{D}$. Also $\mathscr{B}$ and $\mathscr{D}$ are orthogonal ideals in $\mathscr{A}$. The global result now follows routinely.

THEOREM 2.4. *Let G be a connected Lie group which contains a compact normal subgroup N for which G/N is a vector group. Then every irreducible representation T of G vanishes at infinity modulo its kernel.*

PROOF. By passing to the usual group extension, we may as well assume that T is unitary. (Because the usual group extension is an extension by a torus, a multiplier representation vanishes at infinity if and only if the corresponding unitary representation vanishes at infinity.) Now $T|_N$ is a multiple of a fixed irreducible unitary representation L of N. Hence the restriction of T to the center Z of N is a scalar. Modulo Z then, T is a multiplier representation, and it will suffice to show that every irreducible multiplier representation of G/Z vanishes at infinity modulo its kernel. By the above lemma, G/Z is the direct product of a compact group with a vector group, and any irreducible multiplier representation of such a direct product is the outer Kronecker product of irreducible multiplier representations of the factors. It follows then that we need only verify that every irreducible multiplier representation of a vector group vanishes at infinity modulo its kernel.

Let W be an irreducible multiplier representation of Euclidean space R^n. Then, by [2], there exists a vector subgroup M of R^n, for which R^n/M is an even-dimensional space R^{2j}, such that W is equivalent to the representation $\Phi(S \cdot \pi)$, where Φ is a character of R^n, π is a projection of R^n onto R^n/M and S is the multiplier representation of R^{2j} acting in $L^2(R^j)$ and defined by $[S_{(q,p)}(f)](p') = e^{i\lambda(q,p')}f(p + p')$, where λ is a nonzero real number. Now W will vanish at infinity modulo its kernel if S vanishes at infinity.

Fix an element f of $L^2(R^j)$ with compact support C_1. For each p in R^j define $f_p(p') = f(p + p')\bar{f}(p')$. Then $p \rightarrow f_p$ is a continuous map of R^j into $L^1(R^j)$. It follows from this fact, together with the Riemann-Lebesgue lemma, that for any $\epsilon > 0$ and any compact subset C' of R^j there exists a compact subset C^* of R^j such that $|f_p\hat{\ }(\lambda q)| < \epsilon$ whenever p belongs to C' and q is outside C^*. The fact that S vanishes at infinity follows now by taking $C' = C_1 - C_1$. This completes the proof.

The other property of representations that we wish to consider here is somewhat more subtle.

DEFINITION 2.5. Let H be a connected Lie subgroup of a connected Lie group G and let T be a multiplier representation of G. A set $[\psi]$ of vectors in $H(T)$ is called an *absolutely equicontinuous family of vectors for T restricted to H* if the set of functions $h \rightarrow |(T_h(\psi), \psi|$ is an equicontinuous family of functions in the topology of H. A vector φ is called an *absolutely equicontinuous vector for T restricted to H* if the set $[T_g(\varphi)]$, for g an element of G, forms an absolutely equicontinuous family of vectors for $T|_H$.

It is implicit here that if a nonzero absolutely equicontinuous vector exists, then the multiplier is at least locally continuous. As remarked in the introduction, there are various other conditions which imply the existence of an absolutely equicontinuous vector. We shall see in the course of the proof of the next theorem that an absolutely equicontinuous vector can exist without any of those conditions holding. Observe too that, since the topology of a Lie subgroup is stronger than the relative topology it inherits from G, we have that if a vector is absolutely equicontinuous for $T|_G$ then it is absolutely equicontinuous for $T|_H$ for all Lie subgroups H.

THEOREM 2.6.

(i) *If T is a finite dimensional representation of G, then every vector is absolutely equicontinuous for $T|_G$.*

(ii) *If T is an irreducible unitary representation of a Heisenberg group G, then every vector is an absolutely equicontinuous for $T|_G$.*

(iii) *Suppose G is a compact extension of a vector group. (See the introduction for the definition of "extension.") If T is an irreducible representation of G, then every vector is an absolutely equicontinuous vector for $T|_G$.*

PROOF. Finite dimensional representations are uniformly continuous, and this proves (i). As we mentioned in the introduction, every irreducible unitary representation of a Heisenberg group is really an irreducible representation of a compact extension of a vector group. So part (ii) follows from part (iii).

Of course as before we need only prove (iii) for unitary representations. Let T be an irreducible unitary representation of a group G having a compact normal subgroup K for which G/K is a vector group. By the Mackey machine, [9], there exists an irreducible finite dimensional unitary representation L of K, a continuous multiplier ω on the vector group G/K, a $\bar{\omega} \cdot \pi$-representation M of G which extends L, and an irreducible ω-representation S of G/K such that T is equivalent to the tensor product $M \otimes (S \cdot \pi)$. Because ω is continuous, both M and S are strongly continuous multiplier representations. Since M is finite dimensional it is actually uniformly continuous, and so it follows that we need only verify that every vector φ in $H(S)$ is absolutely equicontinuous for $S|_{G/K}$. Now according to [2], we can describe the multiplier representation S as follows. There exists a closed vector subgroup N of G/K and a character χ of G/K such that S is equivalent to the multiplier representation $\chi(V \cdot \pi')$, where π' is projection of G/K onto $(G/K)/N$, which is an even-dimensional space R^{2j}, and where V is the unique irreducible multiplier representation of R^{2j} associated with the cocycle $[(q_1, p_1), (q_2, p_2)] \to e^{i(q_2, p_1)}$. We have that V acts in $L^2(R^j)$ and is defined by

$$[V_{(q,p)}(\varphi)](p') = e^{i(q,p')}\varphi(p + p').$$

Because of the absolute value signs in the definition of absolutely equicontinuous vectors, the character χ is of no consequence, and we need only verify that every vector in $H(V)$ is an absolutely equicontinuous vector for $V|_{(R^{2j})}$. But this is trivial since

$$|(V_{(q,p)}(V_{(x,y)}(\varphi)), V_{(x,y)}(\varphi))| = |(V_{(q,p)}(\varphi), \varphi)|,$$

i.e., there is but one function in the family and so that family is obviously equicontinuous. We remark that (q, p) and (x, y) do commute but the operators $V_{(q,p)}$ and $V_{(x,y)}$ do not commute. Indeed V is a multiplier representation. However the absolute value signs wipe out the multiplier. Without the absolute value signs we have

$$(V_{(q,p)}(V_{(x,y)}(\varphi)), V_{(x,y)}(\varphi)) = e^{i(x,p)}(V_{q,p)}(\varphi), \varphi).$$

As x varies, this is not an equicontinuous family of functions on R^{2j} unless $\varphi = 0$.

We have proved part (iii), and we have seen that absolute equicontinuity is definitely weaker than is mere equicontinuity for a vector φ. It is likewise clear that V is not uniformly continuous, and that $[V_{(x,y)}(\varphi)]$ does not constitute a precompact subset of $L^2(R^j)$.

In the next section we shall be interested in a rather delicate interplay between the two properties of representations we have introduced here. We have just seen that every irreducible representation of a generalized Heisenberg group has plenty of absolutely equicontinuous vectors. It follows directly from the Mackey machine and Corollary 2.4 that every irreducible representation of a generalized Heisenberg group vanishes at infinity. These groups will be our prototypes.

EXAMPLE 2.7. Let G be the three-dimensional Heisenberg group. For each real number λ let T^λ be the representation of G defined on $L^2(R)$ by

$$[T^\lambda_{(t,q,p)}(\varphi)](p') = e^{i\lambda t}e^{i\lambda qp'}\varphi(p + p').$$

Now if $\lambda \neq 0$, then T^λ is irreducible and so every vector in $H(T^\lambda)$ is an absolutely equicontinuous vector. However T^λ does not vanish at infinity, its kernel being isomorphic with the group of integers. If we define T to be the direct integral representation $T = {}_0^1 \int T^\lambda \, d\lambda$, then one sees easily that every vector in $H(T)$ is still an absolutely equicontinuous vector for $T|_G$. Furthermore T vanishes at infinity. It is a subrepresentation of $\int_R T^\lambda \, d\lambda$, and this representation is quasi-equivalent to the regular representation of G. Since the multiplicity of the regular representation is infinity, T must be a subrepresentation of that regular representation and therefore vanishes at infinity.

We have then an example of a group, which is not a compact extension of a vector group, but which has a representation vanishing at infinity and possessing many absolutely equicontinuous vectors. It is of course not irreducible.

3. **A Characterization of Generalized Heisenberg Groups.** As indicated at the end of the last section, it will be the class of compact extensions of vector groups in which we shall be interested here. It is known that a connected Lie group G is a compact extension of a vector group, (G contains a compact normal subgroup K for which G/K is a vector group), if and only if G is an *IN-group*, (G contains a compact neighborhood of its identity which is invariant under all inner automorphisms). Although the invariant neighborhood concept has been quite popular and productive, it seems perhaps more appropriate, from the point of view of representation theory, to think in terms of the group extension notion. A representation of G/K obviously extends to a representation of the extension group G. We shall make use of the equivalence of these two properties throughout.

The following theorem is the main result of this paper.

THEOREM 3.1. *Let H be a connected Lie subgroup of a connected Lie group G. Suppose T is a representation of G which vanishes at infinity on G and for which there exists a nonzero absolutely equicontinuous vector φ for $T|_H$. Then:*

(i) *The smallest normal subgroup H' of G which contains H is a Lie subgroup and is in fact a compact extension of a simply connected two-step nilpotent Lie group. ("Two-step" nilpotent is to mean that the first or second commutator subgroup is trivial.)*

(ii) *If T is irreducible, in fact if T is a primary representation, then the smallest closed normal subgroup H'' of G containing H is a compact extension of a vector group.*

REMARK. We are dealing here with a rather subtle interplay between the two ideas introduced in the last section. If we think of H as a kind of "winding line" (nonclosed subgroup) in G, then it is clear that T may very well fail to vanish at infinity in the topology of H. Also, the equicontinuity of the functions $h \to |(T_h(T_g(\varphi)), T_g(\varphi))|$ on H is a far weaker assumption than is the hypothesis of their equicontinuity on all of G.

PROOF. From the equicontinuity hypothesis, there exists a compact connected symmetric neighborhood V of the identity in H such that $|(T_{(g^{-1}hg)}(\varphi), \varphi)| > \|\varphi\|^2/2$ for all g in G and all h in V. Because T vanishes at infinity, we have that the set C, which is the closure in G of the set of all $g^{-1}hg$ for h in V and g in G, is compact. Obviously C is invariant under all inner automorphisms, and clearly C belongs to H''. One begins to imagine that we shall be able to prove that H'' is an IN-group, which would give part (ii) even without the assumption of irreducibility. However part (ii) is false without some extra assumption as example 2.7 shows. Hence we cannot show that H'' is an IN-group so easily. However, there is a germ of validity to the argument suggested above. If H' is already closed, then C belongs to H'. Also, $\bigcup C^n$ is a normal subgroup of G containing H, whence $H' = \bigcup C^n$. By the Baire category theorem some C^n is a compact invariant neighborhood of the identity in H', and so the theorem is proved in this special case. We shall use this case later on, but for now we remark that H' may very well not be closed, and the proof in general seems to be much more complicated.

The argument we shall give goes as follows: We replace the "representational hypothesis" by a kind of "structural hypothesis." Arguing then by contradiction we reduce to three special cases. We translate back then into a different representational hypothesis, and arrive at our desired contradiction.

The fact that H' is a Lie subgroup of G is a consequence of a reasonably familiar kind of argument in Lie theory. We include an outline.

LEMMA 3.2. *Let H be a connected Lie subgroup of a connected Lie group G. Then the smallest normal subgroup H' of G containing H is a Lie subgroup, and its Lie algebra is the smallest ideal in the Lie algebra $\mathscr{G}$ of G containing the Lie algebra $\mathscr{H}$ of H.*

PROOF. Let $\mathscr{I}$ denote the smallest ideal in $\mathscr{G}$ containing $\mathscr{H}$, and let I denote the analytic subgroup of G with Lie algebra $\mathscr{I}$. Then I is a normal subgroup and H' belongs to I.

First of all, the linear span of all the elements Y in $\mathscr{G}$ of the form $Y = e^{(ad_{X_1})} \cdots e^{(ad_{X_j})}(W)$, where $X_1, \cdots, X_j$ belong to $\mathscr{G}$ and W belongs to $\mathscr{H}$, is $\mathscr{I}$. Indeed these linear combinations can be shown to belong to $\mathscr{I}$ by induction on j. Also the linear span of these elements is an ideal in $\mathscr{G}$ because of the formula $[Z, Y] = \lim_{t\to 0}(1/t)(e^{(ad)}tZ)(Y) - y)$. And finally, this linear span contains $\mathscr{H}$.

Letting $[Y_i]$ be a basis of $\mathscr{I}$ of elements of the above form, and recalling the equation $\exp(e^{(ad_X)}(W)) = \exp(X)\exp(W)\exp(-X)$ for sufficiently small X and W, we see that $\exp(tY_i)$ is conjugate to some element of H for sufficiently small t and hence for all t.

Define a mapping φ of $\mathcal{J}$ into G as follows: $\varphi(\Sigma_i t_i Y_i) = \Pi_i \exp(t_i Y_i)$. We have that φ is an analytic mapping of the Euclidean space $\mathcal{J}$ into the manifold I. The range of φ is contained in the group H'. However, by the Campbell-Baker-Hausdorff formula, we see that the differential of φ at the origin is nonsingular. (It is in fact the identity.) And so, since $\mathcal{J}$ and I have the same dimension, φ maps a neighborhood of the origin in $\mathcal{J}$ onto a neighborhood of the identity in I. Hence H' contains a neighborhood of the identity in I, whence $H' = I$, both being connected groups.

From our earlier remarks we see that it is exactly the case when I is not a closed subgroup of G that we must consider. Denote by G^* the simply connected covering group of G with covering map θ. Let I^* denote the closed normal subgroup of G^* with Lie algebra $\mathcal{J}$. If D denotes the kernel of θ, then D is discrete and central in G^*, and $\theta^{-1}(H'')$ is the closure of I^*D. There exists a compact connected symmetric subset C^* of G^* for which $\theta(C^*) = C$. Unfortunately C^* does not inherit all the nice properties of C. The reader should again think of H' as a winding line on a two-dimensional torus and think of C as a line segment on that winding line. The set C^* is no longer necessarily invariant, but it is always invariant modulo D, i.e., $(g^*)^{-1}C^*(g^*)$ is contained in C^*D for all g^* in G^*. On the other hand $\bigcup(C^*)^n$ is again a connected subgroup of G^*, and since $\theta \bigcup(C^*)^n) = \bigcup C^n$, which is a normal subgroup of G between H' and H'', we have that I^* is contained in $\bigcup((C^*)^nD)$. By the Baire theorem once again we have that for some n $(C^*)^nD$ contains a neighborhood of the identity in I^*. Since D is countable, some $(C^*)^n$ itself contains such a neighborhood. Finally H' is exactly $\theta(I^*)$, which is $I^*/(I^* \cap D)$, and it is this Lie group we will show is a compact extension of a simply connected two-step nilpotent Lie group. For shorthand in this proof, let us call such a nilpotent group an SCTSN-group. The proof to part (i) of Theorem 3.1 follows then directly from the lemma below, where we have changed notation for convenience.

We remark that we cannot simply pass to the covering group G^*, where the smallest normal subgroup containing H would be closed, and prove the theorem using the easy argument mentioned at the beginning of the proof. Indeed, on G^* the representation T will very likely no longer vanish at infinity.

LEMMA 3.3. *Let I be a closed normal connected subgroup of a connected Lie group G. Suppose D is a central subgroup of G whose inter-*

section with I is trivial and for which G is the closure of ID. Suppose finally that C is a compact connected symmetric subset of G for which $C \cap I$ contains a neighborhood of the identity in I and for which C is invariant modulo D, i.e., $g^{-1}Cg$ is contained in CD for all g in G. Then I is a compact extension of an SCTSN-group.

Proof. We remark that if J is a closed connected subgroup of I, then the hypotheses of this lemma apply to the group J in the group G_J which is the connected component of the closure of JD with respect to the central subgroup $D \cap G_J$ and the compact set $C \cap G_J$. If J is normal in I, then J is normal in all of G, and the hypotheses of the lemma apply to the group I/J in the group G/J with respect to the central subgroup D/J and the compact set C/J.

We assume, by way of contradiction, that I is a group satisfying the hypotheses of the lemma but which is not a compact extension of an SCTSN-group. We assume further that I has smallest possible dimension with this contrary property. We shall arrive eventually at a contradiction.

1. I contains no compact normal subgroups of positive dimension. Indeed if K were such a compact normal subgroup, then I/K would have smaller dimension than I, the hypotheses of the lemma apply to I/K, and so I/K would be a compact extension of an SCTSN-group, and consequently so would I.

2. I is not semisimple. To see this, let S denote the "radical" of G, i.e., the maximum (closed) solvable subgroup of G. Then S contains D since S always contains the center of G. If I were semisimple, then I/S would be a semisimple subgroup of the semisimple group G/S, and therefore I/S would be closed in G/S. (See the exercises at the end of Chapter II of [4].) Now, because $G = \overline{ID}$ we have that $G/S = \overline{ID}/S$ which is contained in $\overline{(ID/S)}$ which equals $\overline{I/S}$ which is I/S. But now C/S is a compact invariant neighborhood of the identity in I/S, whence I/S is compact, being a semisimple IN-group. But then I itself must be compact, being at worst a covering group of a compact semisimple Lie group. But by 1, I cannot be compact.

3. Let N be a maximal closed normal connected abelian subgroup of I. Because I is not semisimple, $\dim N > 0$. Since I contains no compact normal subgroups of positive dimension, N must be a vector group. Since the dimension of I/N is less than the dimension of I, there must exist a closed normal subgroup L of I containing N such that L/N is compact and I/L is an SCTSN-group. Now L must be connected. For in any event the connected component of the identity in L is a normal subgroup L' of I for which L/L' is finite. But I/L' is a covering group

of the simply connected group I/L, and therefore L/L' must be the one-element group, or L is connected.

Now if L lies properly between N and I, then L itself is a compact extension of an SCTSN-group. But that compact normal subgroup of L is a "characteristic" subgroup of L, (invariant under all automorphisms of L), and is therefore normal in all of I. But by 1 there are no compact normal subgroups of I of positive dimension so that L is a finite extension of an SCTSN-group. But we just argued in the preceding paragraph that that can happen only when the normal finite subgroup is trivial. Hence L is itself an SCTSN-group. Therefore L is diffeomorphic with a Euclidean space which is impossible if L/N is a nontrivial compact group. Therefore L either is N or it is all of I.

3.A. $L = I$. By the theorem of Iwasawa on vector extensions of compact groups, see [11], we may write I as a semidirect product $I = NK$, where K is a compact group. If $\dim K > 1$, let T denote a closed one-dimensional torus in K, an consider the subgroup NT in I. Its dimension is less than that of I, the hypotheses of the lemma apply to NT, so NT must be a compact extension of an SCTSN-group. Because NT is clearly not simple connected, there must be a compact normal subgroup M of NT of positive dimension. But T must belong to M since the quotient of NT by M is simply connected. It follows that $M = T$ and therefore T commutes with N, i.e, the semidirect product is direct in this case. But this implies that the normal subgroup K' of K consisting of those elements of K which commute with N has positive dimension. But then K' would be a compact normal subgroup of all of I with positive dimension, and this we have ruled out in 1. Therefore K has dimension 1.

Now the vector space N decomposes into a direct sum of irreducible subspaces (one or two dimensional) under the action of the one-dimensional torus K. Since I is certainly not abelian, some one of these subspaces, say N_1, is two-dimensional. The hypotheses of this lemma apply to the group $I_1 = N_1K$. This is of course the semidirect product of the complex plane with the circle group, where an element λ of the circle acts on a complex number z by $\lambda(z) = \lambda^j z$ for some nonzero integer j. We show below, in 4.A, that the hypotheses of the lemma cannot apply to this group, and we reach a contradiction in this case.

3.B. $L = N$. In this case I is at worst a three-step solvable group. Let J denote a closed normal subgroup of I of codimension 1, and choose a closed one-parameter subgroup A of I so that $I = JA$. Now $\dim J < \dim I$, and so J is a compact extension of an SCTSN-group. That compact normal subgroup of J would, as before, be a characteristic subgroup and hence normal in I which is not the case, so J is itself nilpotent and simply connected. Let Z denote the center of J. Then Z

is a nontrivial vector group. In addition Z is setwise invariant under A. However, Z may not decompose into a direct sum of irreducible subspaces, because A very likely is not compact but is the real line. However there is at least one irreducible subspace Z_1 under the action of A. We shall examine the group Z_1A. There are some subcases.

3.B.i. Suppose $\dim Z_1 = 1$ and that Z_1A is not commutative. Then A must be the real line, and the group $I_2 = Z_1A$ is the semidirect product of the reals with the reals where the real number t acts on a real number b by $t(b) = e^{at}b$ for $a \neq 0$. We show in 4.B.i that the hypotheses of the lemma cannot apply to this group, arriving again at a contradiction.

3.B.ii. Suppose $\dim Z_1 = 2$. Then Z_1A is necessarily nonabelian. In this case the group I_3 is the semidirect product of the complex plane with the real line where a real number t acts on complex number z by $t(z) = e^{ibt}e^{at}z$, where $b \neq 0$ and a is any real number. In 4.B.ii we see that the hypotheses of the lemma cannot apply to this group either.

3.B.iii. The only remaining case is when $\dim Z_1 = 1$ and Z_1A is commutative. But then Z_1 commutes with everything in I, i.e., I contains a center Z' of positive dimension. So I/Z' is a compact extension of an SCTSN-group. As usual there can be no compact normal subgroups of positive dimension of I/Z'', so I/Z' is itself a two-step nilpotent Lie group. Then I, being a central extension of I/Z', is at worst a three-step nilpotent group, and therefore so is G. Letting Z'' denote the center of G, we know that D belongs to Z'' and so I/Z'' is a dense normal subgroup of G/Z''. But G/Z'' is simply connected, and so $I/Z'' = G/Z''$. But we have now that C/Z'' is a compact invariant (D having been factored out) neighborhood of the identity in I/Z''. Therefore I/Z'' is an IN-group, and is then a compact extension of a vector group. There can be no compact normal subgroup of I/Z'' for the usual reasons, and so I/Z'' is a vector group, whence I, being a central extension of a vector group, is a two-step nilpotent Lie group. This of course is already a contradiction to our assumptions on I.

To complete the proof of the lemma we need to verify that the hypotheses of the lemma cannot apply to three particular groups, I_1, I_2, and I_3.

4. Because $G = \overline{ID}$, it follows that every irreducible representation of G restricts to an irreducible representation of I. It then follows, using the Mackey machine, that every irreducible representation of I extends to an irreducible representation of G, (perhaps not uniquely). We now return to a hypothesis on I which is a "representational" hypothesis. For any irreducible unitary representation W of I, (we think of W

as a representation of G as well), and any vector φ in $H(W)$, we have that the set $[W_{(g^{-1}cg)}(\varphi)]$ for g in I and c in $C \cap I$ is precompact in the space $H(W)$. Indeed $[W_{(g^{-1}cg)}(\varphi)]$ is contained in $[W_{CD}(\varphi)]$, which is contained in $[\lambda W_C(\varphi)]$, for $|\lambda| = 1$. It is this property of the representations of I which we show cannot hold for I_1, I_2 and I_3.

4.A. Consider the irreducible unitary representation W of I_1 which is induced from the character $x + iy \longrightarrow e^{ix}$ of the complex plane. Then $H(W)$ is $L^2(T^1)$ and $[W_{(z,\lambda)}(\varphi)](\lambda') = e^{iRE(\lambda'^j z)}\varphi(\lambda'\lambda)$.

Now if φ is the identically 1 function, c is an element $(0, \lambda)$ in $C \cap I_1$ with λ not a j'th root of unity, and g any element of the form $(z, 1)$, then

$$\begin{aligned}[W_{(g^{-1}cg)}(\varphi)](\lambda') &= [W_{((1-\lambda^j)z,\lambda)}(\varphi)](\lambda') \\ &= e^{iRE(\lambda'^j(1-\lambda^j)z)}.\end{aligned}$$

But this does not form a precompact set, since, letting z run through the numbers $n/(1 - \lambda^j)$, we obtain a sequence of unit vectors which converges pointwise to zero. This is a contradiction to the representational hypothesis on I_1, and shows that the hypotheses of the lemma cannot apply.

4.B.i. Let W be the irreducible unitary representation of I_2 induced from the character $b \longrightarrow e^{ib}$. $H(W)$ is $L^2(R)$. Let φ be a nonzero function in $L^2(R)$, let $c = (0, t)$ be an element in $C \cap I_2$ with $t \neq 0$, and let $g = (b, 0)$. Then

$$[W_{(g^{-1}cg)}(\varphi)](t') = [W_{((1-e^{at})b,t)}(\varphi)](t') = e^{i[e^{at'}(1-e^{at})b]}\varphi(t + t').$$

Again we can find a sequence of real numbers b giving a sequence of functions of constant L^2 norm but converging pointwise to zero, contradicting the representational hypothesis.

4.B.ii. The same kinds of arguments will work in this case as worked in the two preceding ones. If the parameter a in the multiplication formula in I_3 is zero, then I_3 maps homomorphically onto I_1. We could lift the representation of I_1 constructed in 4.A up to I_3 obtaining the same contradiction. Hence assume that $a \neq 0$. Again let W be the representation induced from the character $x + iy \longrightarrow e^{ix}$. Let φ be a nonzero function in $L^2(R)$, the space of W, let $c = (0, t)$ $t \neq 0$ be an element of $C \cap I_3$, and let $g = (z, 0)$. Then

$$[W_{(g^{-1}cg)}(\varphi)](t') = e^{iRE(e^{ibt'}e^{at'}(1-e^{ibt}e^{at})z)}\varphi(t + t').$$

Again the proper choice of z leads to a sequence constant in L^2 norm but converging pointwise to zero.

We have proved the lemma and consequently part (i) of the theorem. We turn next to the proof of part (ii).

Let K be a compact normal subgroup of H' for which H'/K is a simply connected nilpotent Lie group. Then K is compact and normal in G, and H'/K is dense in H''/K. Since the closure of a nilpotent group is nilpotent, we have that H''/K is nilpotent. There exists a torus L in the center of H''/K such that $(H''/K)/L$ is simply connected. Let K' be a compact normal subgroup of H'' such that $K'/K = L$.

We define H_1 to be the Lie subgroup $K'H$ of G. We claim first that the smallest normal subgroup H_1' of G containing H_1 is closed. Indeed H_1' is contained .in H'', and H_1'/K' is dense in the simply connected group H''/K'. Of course H_1'/K' is normal, and so $H_1'/K' = H''/K'$. Since K' belongs to H_1', we have that $H_1' = H''$ which is closed.

We shall show that H_1' is a compact extension of a vector group. By the first paragraph of the proof to Theorem 3.1, this will follow if the representation T has a nonzero absolutely equicontinuous vector for $T|_{(H_1)'}$. (Of course somewhere we must use the fact that T is irreducible.)

We know that there is a vector φ in $H(T)$ which is an absolutely equicontinuous vector for $T|_{H'}$. If it is also absolutely equicontinuous for $T|_{K'}$ then it will be absolutely equicontinuous for $T|_{(H_1)'}$. Since K' is normal in all of G, and since T is irreducible, (in fact primary would do), it follows from [9] that $T|_{K'}$ is a multiple of one fixed finite dimensional irreducible representation L of K'. Therefore $T|_{K'}$ is in fact uniformly continuous, and every vector, including φ, is absolutely equicontinuous.

This completes the proof to Theorem 3.1.

The example in 2.7 shows that without the assumption of irreducibility we could not conclude that H'' is a generalized Heisenberg group. We give next an example showing that H' need not be a generalized Heisenberg group even if T is irreducible, i.e., the closure hypothesis on H'' is necessary.

EXAMPLE 3.4. Let α be an irrational number, and define G^* to be the five-dimensional group, diffeomorphic with $T^1 \times R^4$, where multiplication is defined by

$$\begin{aligned}(\lambda_1, x_1, y_1, q_1, p_1)\,(\lambda_2, x_2, y_2, q_2, p_2) &\\ = (\lambda_1\lambda_2 e^{i(q_2 p_1)}, x_1 + x_2 + q_2 p_1, y_1 &\\ + y_2 + \alpha q_2 p_1, q_1 + q_2, p_1 + p_2).&\end{aligned}$$

Let I be the four dimensional subgroup consisting of the elements of the form $(\lambda, t, \alpha t, q, p)$, and let D be the central subgroup of G^* consisting of the elements $(1, m, n, 0, 0)$, where m and n are integers. If we let C be the compact subset of G^* consisting of the elements each of whose coordinates has absolute value less than or equal to 1, then one sees directly that C is invariant modulo D.

The subgroup I satisfies the hypotheses of Lemma 3.3 and it is a compact extension of the three-dimensional Heisenberg group, which is after all an SCTSN-group.

If we let $G = H'' = G^*/D$, we find that H'' is an extension of R^2 by a three-dimensional torus which agrees with the theorem of course.

4. **Examples and Generalizations.** We present here some examples of how Theorem 3.1 can be used in reverse. Also, we conclude the section by extending Theorem 3.1 to the non-Lie setting of "Almost connected" groups.

EXAMPLE 4.1. *Let K be a compact connected subgroup of a connected Lie group G.* Suppose there exists a representation T of G which vanishes at infinity on G and such that $T|_K$ has a finite spectrum, i.e., there exists a finite set A of irreducible representations of K such that each irreducible representation of K occurring in $T|_K$ is equivalent to some element of A. Then the smallest normal subgroup K' of G containing K is itself compact. Indeed the assumption about $T|_K$ implies that $T|_K$ is uniformly continuous. Then by part (i) of Theorem 3.1 we know that K' contains a compact normal subgroup L for which K'/L is a simply connected nilpotent group. But clearly then K would have to belong to L, whence $K' = L$.

EXAMPLE 4.2. Now let G be a connected semisimple Lie group of the noncompact type, and write $G = KAN$ for the Iwasawa decomposition of G. Suppose that K contains a two-dimensional torus T^2, and suppose that there exists a discrete series representation π of G, i.e., π is an irreducible subrepresentation of the regular representation of G. Now $\pi|_{(T^2)}$ is a direct sum of characters of T^2, and these characters are given by pairs (m, n) of integers. Let A be the set of all pairs occurring in $\pi|_{(T^2)}$. (We are thinking of T^2 as being parametrized by pairs $(e^{i\theta}, e^{i\varphi})$ of complex numbers.)

If α is an irrational number, we denote by H the one-parameter subgroup defined by the pairs $(e^{it}, e^{i\alpha t})$. Now π vanishes at infinity on G, being a subrepresentation of the regular representation, and the smallest closed normal subgroup H'' of G containing H is definitely not a compact extension of a vector group, H'' necessarily being semisimple. According to Theorem 3.1, $\pi|_H$ can have no nonzero absolutely equicontinuous vectors. But $\pi|_H$ is a direct sum of characters of the real line, and these characters are given by the real numbers $m + n\alpha$, where (m, n) belongs to A. One thing then that we can say is that the set of real numbers $M + n\alpha$ for (m, n) belonging to A is not bounded, for otherwise $\pi|_H$ would be uniformly continuous. This unboundedness statement must hold for every irrational number α. Hence we have ruled out certain kinds of subsets of the lattice points which can occur in the restriction to T^2 of π. One can construct infinite sets A such that for some α the numbers $m + n\alpha$ form a bounded set. So we have ruled out a nonempty class of sets.

Finally, we shall extend Theorem 3.1 to a non-Lie context. Other generalizations are undoubtedly possible, but we content ourselves with the following:

THEOREM 4.3. *Let H be a connected Lie group, and let I be a continuous one-to-one homomorphism of H into an almost connected locally compact group G. (A group is almost connected if, modulo its connected component of the identity, it is compact.) Suppose there exists an irreducible multiplier representation T of G which vanishes at infinity on G, and suppose there exists a nonzero vector φ in $H(T)$ which is absolutely equicontinuous for $T|_{I(H)}$, i.e., in the topology of H. Then the smallest closed normal subgroup H'' of G containing $I(H)$ is a compact extension of a vector group.*

PROOF. By a fundamental structure theorem for almost connected groups, (see [**10**]), there exists a compact normal subgroup L of G such that G/L is a connected Lie group. Let π denote the projection of G onto G/L. Then by the Mackey procedure we know that there exists a locally continuous multiplier ω on G/L, an ω-representation S of G/L, and a finite dimensional $\bar{\omega} \cdot \pi$-representation M on G such that T is equivalent to the tensor product $M \otimes (S \cdot \pi)$. It follows immediately that S vanishes at infinity since T does.

We let H^* be the Lie subgroup $\pi(I)H))$ of G/L. We will show that there exists a nonzero vector w in $H(S)$ which is absolutely equicontinuous for $S|_{(H^*)}$. Then the smallest closed normal subgroup of G/L containing H^* will be a compact extension of a vector group, and so then will be H'' since L is compact.

Let φ be a nonzero absolutely equicontinuous vector for $T|_{I(H)}$. Then $\varphi = \sum_{j=1}^{n} \sum_i a_{ij}(\varphi_j \otimes \psi_i)$, where $[\varphi_j]$ is an orthonormal basis for the finite dimensional space $H(M)$, and where $[\psi_1]$ is an orthonormal basis for $H(S)$. We know that the set of functions $h \to |(T_{I(h)}(T_g(\varphi)), T_g(\varphi))|$, which equals

$$\Big| \sum_{j} \sum_{j'} \sum_{i} \sum_{i'} a_{ij} \overline{a_{i'j'}} (M_{I(h)}(M_g(\varphi_j)), M_g(\varphi_{j'})) \\ (S_{\pi(I(h))}(S_{\pi(g)}(\psi_i)), S_{\pi(g)}(\psi_{i'})) \Big|$$

is an equicontinuous family of functions on H. Now M, being finite dimensional, is uniformly continuous. (The local continuity of ω is being used here.) Therefore, adding and subtracting

$$\sum_{j}\sum_{j'}\sum_{i}\sum_{i'} \quad a_{ij}\overline{a_{i'j'}}(M_g(\varphi_{j'}), M_g(\varphi_{j'}))$$

$$(S_{\pi(I(h))}(S_{\pi(g)}(\psi_i)), S_{\pi(g)}(\psi_{i'})),$$

and using the triangle inequality in both directions, we find that the functions

$$h \longrightarrow \left| \sum_{j=1}^{n} \sum_{i} \sum_{i'} \quad a_{ij}\, \overline{a_{i'j}} \;\; (S_{\pi(I(h))}(S_{\pi(g)}(\psi_i)), S_{\pi(g)}(\psi_{i'})) \right|$$

which equals

$$\left| \sum_{j=1}^{n} \;\; (S_{\pi(I(h))}(S_{\pi(g)}(w_j)), S_{\pi(g)}(w_j)) \right|$$

are equicontinuous where $w_j = \sum_i a_{ij}\psi_i$.

Now some one of the vectors $[w_j]$ must be nonzero, for otherwise φ would have to be zero. What this proves then is that there exists a nonzero vector in the space of nS which is absolutely equicontinuous for $nS|_{\pi(I(H))}$. Now nS vanishes at infinity, but it is no longer irreducible, unless $n = 1$. However nS is a primary representation, and theorem 3.1 part (ii) is valid for primary representations. Hence the smallest closed normal subgroup of G/L containing H^* is the desired compact extension of a vector group.

5. **An Application to Quantum Mechanics.** We consider here a physical system which for convenience consists of a single particle not necessarily free from external forces. We denote by S the group of "symmetries" of the system, i.e., the group of all transformations of space (R^3) which preserve the "physics" of the system.

ASSUMPTION 1. Each element s of S is assumed to be a continuous transformation of space of the form $s(q) = A_s(q) + q_s$, where A_s is an invertible linear transformation and q_s is a vector in R^3. The transformation A_s is called the *rotational* part of s and the vector q_s is called the *translational* part of s. If we denote elements of S by pairs (A, q), then multiplication in S is given by $(A_1, q_1)(A_2, q_2) = (A_1A_2, q_1 + A_1(q_2))$. We have too that the mapping $s \longrightarrow A_s$ is a continuous homomorphism of S onto a subgroup of $GL(3, R)$.

This first assumption is frequently satisfied because of the presence of other hypotheses on the system. For instance Assumption 1 holds if each element of S is distance-preserving. What we shall do here is to derive conditions on a one-parameter subgroup of S which ensure that it consists entirely of pure translations.

ASSUMPTION 2. We adopt the following quantum mechanical model of our system. There exists a Hilbert space K, a multiplier representation π of S acting in K, and a K-projection-valued measure $\underline{p}$ on the sigma algebra of Borel subsets of R^3 such that:

(i) $\pi_s \underline{p}_E \pi_s^{-1} = \underline{p}_{s(E)}$ for all s in S and all Borel subsets E of R^3.

(ii) The Hilbert space K is irreducible under the joint action of π and $\underline{p}$.

It follows from (ii) that $\underline{p}$ is ergodic with respect to the action of S on R^3. Therefore we can realize K as the Hilbert space $L^2(R^3, K', \mu)$, where K' is some other Hilbert space and μ is a Borel measure on R^3. There exists a unitary representation V of R^3 determined by $\underline{p}$ and defined by $V_q = \int e^{1(q,q')}\, d\underline{p}(q')$. Alternatively, if φ is an element of $L^2(R^3, K', \mu)$, then

$$(V_q(\varphi), \varphi) = \int_{R^3} e^{i(q,q')}\|\varphi(q')\|^2\, d\mu(q').$$

The two assumptions made thus far are both from the realm of axiomatic quantum mechanics. The others we shall make are both more specific, and they should be distinguished from these first two.

We shall certainly not wish to restrict our attention to symmetry groups all of whose rotational parts belong to a fixed compact subgroup of $GL(3, R)$. However this situation does occur, and it serves to motivate our extra assumptions. We have the following:

THEOREM 5.1. *Suppose that all the rotational parts* $[A_s]$ *for* s *in* S *belong to a fixed compact subgroup* L *of* $GL(3, R)$. *Then*:

(i) *For each element* φ *in* K *the set of functions* $q \rightarrow V_{(A_s^{-1'}(q))}(\varphi)$ *forms an equicontinuous family of functions from* R^3 *into* K, *as* s *varies over* S.

(ii) *If* S_0 *denotes the subgroup of* S *consisting of the pure translations, then there exists an absolutely equicontinuous vector* φ *in* K *for* $\pi|_{S_0}$.

PROOF. To prove (i), we need only examine $\|V_{(A_s^{-1'}(q))}(\varphi) - \varphi\|^2$, which is

$$\int_{R^3} |e^{i((A_s^{-1'}(q)),q')} - 1|^2 \|\varphi(q')\|_K^2\, d\mu(q'),$$

and this clearly is an equicontinuous family of functions of q if the rotational parts all belong to a fixed compact group.

To prove (ii), let t be an element of S and consider the function of s, where s belongs to S_o, given by

$$|(\pi_s\pi_t(\varphi), \pi_t(\varphi))| = |(\pi_{(t^{-1}st)}(\varphi, \varphi)| = |(\pi_{s'}(\varphi), \varphi)|,$$

where s' is another element of S_0. Namely s' is translation by the vector $A_{(t^{-1})}(q_s)$. Now the equicontinuity is again clear if all the matrices $[A_t]$ belong to a fixed compact group.

Our "extra" assumptions can now be stated. They are motivated by the conclusions of the previous theorem, and they are essentially assumptions about the projection-valued measure $\underline{p}$.

Assumption 3. Every vector φ in K has the property that the set of functions $q \longrightarrow V_{(A_s{}^{-1'}(q))}(\varphi)$ forms an equicontinuous family of functions from R^3 in to K as s varies over S.

Assumption 4. The representation V vanishes at infinity.

As remarked, Assumption 3 holds if all the rotational parts of the symmetries belong to a fixed compact group. However, one can see from the proof to part (i) of the last theorem that this assumption can hold in other instances, that assumption depending so much on the measure μ and where it gives its mass.

Assumption 4 is equivalent to assuming that the function $q \longrightarrow \int e^{i(q,q')}\|\varphi(q')\|_{K'}^2\, d\mu(q')$ vanishes at infinity. If μ is absolutely continuous with respect to Lebesgue measure, then this follows from the Riemann-Lebesgue lemma. The assumption holds at other times as well and it appears to depend on whether the Fourier transform $\hat{\mu}$ of μ belongs to some L_p class. (If μ is a point mass, for instance, then this assumption is not valid. The situation is fairly well understood in this case however.)

The following is the main result of this section. The theorem gives sufficient conditions that a one-parameter subgroup of symmetries be a group of pure translations.

Theorem 5.2. *Suppose assumptions* 1–4 *hold. Let* $\exp(tX)$ *be a one-parameter subgroup of* S. *If* π *vanishes at infinity on* S, *and if there exists a nonzero absolutely equicontinuous vector* φ *in* K *for* $\pi|_{\exp(tX)}$, *then every element* s *in the smallest closed normal subgroup* S'' *of* S *containing* $\exp(tX)$ *is a pure translation.*

Proof. Let G be the semidirect product of R^3 with S where multiplication is given by $(q_1, s_1)(q_2, s_2) = (q_1 + (A_s{}^{-1'}(q_2)), s_1s_2)$. Define a mapping T of G into the unitary group on K by $T_{(q,s)} = V_q\pi_s$. We have that

$$\begin{aligned} T_{(q_1,s_1)}T_{(q_2,s_2)} &= V_{q_1}\pi_{s_1}V_{q_2}\pi_{s_2} \\ &= V_{q_1}[\pi_{s_1}V_{q_2}\pi_{s_1}^{-1}]\pi_{s_1s_2}\sigma(s_1, s_2), \end{aligned}$$

where σ is the multiplier associated with π. Also we have

$$\begin{aligned}
\pi_{s_1} V_{q_2} \pi_{s_1}{}^{-1} &= \pi_{s_1} \int e^{i(q_2, q')} d\underline{p}(q') \pi_{s_1}{}^{-1} \\
&= \int e^{i(q_2, s_1{}^{-1}(q'))} d\underline{p}(q') \\
&= \int e^{i(q_2, [A_{s_1}{}^{-1}(q') - (A_{s_1}{}^{-1}(q_{s_1}))])} d\underline{p}(q') \\
&= e^{-i(q_2, [A_{s_1}{}^{-1}(q_{s_1})])} \int e^{i(q_2, [A_{s_1}{}^{-1}(q')])} \, d\underline{p}(q') \\
&= \delta(q_2, s_1) V_{(A_{s_1}{}^{-1}(q_2))},
\end{aligned}$$

where $\delta(q_2, s_1) = e^{-i(q_2, [A_{s_1}^{-1}(q_{s_1})])}$.

Therefore $T_{(q_1, s_1)} T_{(q_2, s_2)} = \sigma(s_1, s_2)\delta(q_2, s_1) T_{((q_1, s_1)(q_2, s_2))}$ which shows that T is a multiplier representation of G.

Since V is defined by the integral formula involving $\underline{p}$ it follows too that T is irreducible. Because π vanishes at infinity by hypothesis, and because of Assumptions 3 and 4, it follows also that T vanishes at infinity.

Define H to be the subgroup of G consisting of the pairs $(q, \exp(tX))$. Again by Assumption 3 and the hypotheses of the theorem, there exists a nonzero absolutely equicontinuous vector φ for $T|_H$. Now the smallest closed normal subgroup H'' of G containing H is obviously $R^3 S''$. Since every element of S which is conjugate to an element of $\exp(tX)$ lies on a one-parameter subgroup of S, we have that S'' is connected, and therefore H'' belongs to the connected component of the identity in G. By an obvious extension of Theorem 3.1 part (ii) to disconnected Lie groups G, it follows that H'' is a compact extension of a vector group. Hence so is S''. Let Γ be a compact normal subgroup of S'' for which S'/Γ is a vector group. Now the group $R^3\Gamma$ is a closed subgroup of H'' and so is itself a compact extension of a vector group. But this implies, as usual, that Γ commutes with everything in R^3. Because of the multiplication formula in G we see that every element of Γ is a pure translation. That implies that Γ is trivial since no group of pure translations can be compact except the trivial group. For the same kind of reasoning, H'' can contain no compact normal subgroups. Hence H'' is a vector group and therefore is commutative. But then S'' commutes with everything in R^3, and as before this implies that each element of S'' is a pure translation.

The peculiar connection between the physics and the mathematical model is this: If there exists a one-parameter subgroup $\exp(tX)$ satisfying the hypotheses of the theorem, and if the smallest closed normal subgroup S'' of S containing $\exp(tX)$ is three dimensional, then there can be no external forces on this system. Indeed translation in any direction would be a symmetry of the system, and therefore no force can be acting. (Translation in almost any direction within a nonzero force field will change the energy of the system and so will not be a symmetry.)

It is obvious that these same kinds of considerations can be applied to systems with more degrees of freedom. The basic fact is a relationship between absolutely equicontinuous vectors and translational symmetry.

Bibliography

1. L. Baggett, *A Weak Containment Theorem for Groups with a Quotient R-Group,* TAMS **128** (1967), 277–290.

2. L. Baggett and A. Kleppner, *Multiplier Representations of Abelian Groups,* Jour. Func. Anal. **14** (1973), 299–324.

3. J. Dixmier, *Les C*-Algèbres et leurs représentations,* Gauthier-Villars, Paris (1964).

4. S. Helgason, *Differential Geometry and Symmetric Spaces,* Academic Press (1962).

5. R. Kadison and I. Singer, *Some Remarks on Representations of Connected Groups,* Proc. Nat. Acad. Sc. USA **38** (1952), 419–423.

6. A. Kirillov, *Unitary Representations of Nilpotent Lie Groups,* (in Russian) Usp. Mat. Nauk. **17** (1962), 57–110.

7. G. Mackey, *Induced Representations of Locally Compact Groups* I, Annals Math. **55** (1952), 101–139.

8. ———, *The Mathematical Foundations of Quantum Mechanics,* W. J. Benjamin, Amsterdam (1963).

9. ———, *Unitary Representations of Group Extensions* I, Acta Math. **99** (1958), 265–311.

10. D. Montgomery and L. Zippin, *Topological Transformation Groups,* Interscience, N.Y. (1955).

11. Seminaire "Sophus Lie", Théorie des Algèbres de Lie, Topologie des Groupes de Lie, Paris (1955).

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