ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Number 3, Summer 1978

ON THE UNIVERSAL COMPACTIFICATION OF A CONE

MICHAEL FRIEDBERG

ABSTRACT. Herein we determine the universal (Bohr) compactification of a wide class of semigroups with the discrete topology; this class includes the positive additive rationals, p-adic rationals, reals, as well as the interior of a closed proper cone in R^n. Using the notion of the greatest semilattice homomorphic image, we describe the structure of the universal compactification of a closed proper cone in R^n supplied with the discrete topology.

0. **Introduction.** The universal compactification of the topological semigroup S is a pair (U, u) where U is a compact semigroup, $u : S \to U$ is a continuous homomorphism of S onto a dense subsemigroup of U, and for any other continuous homomorphism $f: S \to T$ with T a compact semigroup there is a continuous homomorphism $\bar{f} : U \to T$ such that $\bar{f} \circ u = f$. The pair (U, u) is known to exist for any topological semigroup (c.f. [13] or [7]) and is unique with respect to the obvious notion of equivalence.

First a comment on terminology: Several authors, including this author, have referred to (U, u) as the Bohr compactification of S. In [18], the Bohr compactification of S is a pair (B, b) where B is a compact commutative semigroup in which the semicharacters (i.e., continuous homomorphisms into the semigroup of complex numbers z with $|z| \leqq 1$) separate points, $b : S \to B$ is a continuous homomorphism of S onto a dense subsemigroup of B, and for any semicharacter γ on S there is a semicharacter $\bar{\gamma}$ with $\bar{\gamma} \circ b = \gamma$. One sees immediately that this definition is much more consistent with the terminology for topological groups; in this sense, the Bohr and universal compactifications may differ (e.g., any non-degenerate compact connected semilattice). We shall henceforth use this terminology.

Our purpose in this work is to make a contribution to the determination of the universal compactification of a closed proper cone in R^n. § 1 sets forth definitions, notation, references, and some general information. In § 2 we develop some techniques for computing certain closed subgroups of the Bohr compactification of dense subgroups of R^n with the discrete topology; at the end of the Section we give examples using the techniques developed. In § 3, we give a description of the universal compactification of a wide class of subsemigroups

Received by the editors on April 5, 1976.
AMS (*MOS*) Primary: 22A15, 22B05, 20M30.

(p-divisible K-semigroups) of R^n with the discrete topology; these include the additive semigroups of positive rationals, positive p-adic rationals, positive reals, and the interior of a closed proper cone in R^n. The description is as a direct product of a certain compact group and the universal compactification of a closed proper cone in R^n with the usual topology. We then give examples, some of which are known and some of which are new. In § 4, we show that the Archimedean components of a closed proper cone in R^n are the interiors of closed proper sub-cones, and that the greatest semilattice homomorphic image of a closed proper cone in R^n satisfies the finite chain condition. Lastly, in § 5 we use techniques developed by A. H. Clifford [5], Hofmann and Mostert [12], and T. T. Bowman [1] to describe (as a special case) the universal compactification of a closed proper cone in R^n with the discrete topology. Essentially it is the disjoint union of universal compactifications of interiors of closed proper cones (as described in § 3). Hopefully, our techniques will aid in the final determination of the universal compactification of a closed proper cone. Our work is a result of an attempt to generalize the work by J. A. Hildebrant in [9] and [10] as well as that of K. H. Hofmann in [11] (Theorem III) and of Hofmann and Mostert in [12] (p. 140).

1. **Preliminaries.** If X is a set and $A \subseteq X$ and $B \subseteq X$ we denote by $A \backslash B$ the set theoretic difference and by $A \dot{\cup} B$ the disjoint union of A and B. If, in addition, X is a topological space we denote by A^* and A^0 the closure and interior respectively of A. A topological semigroup consists of a non-empty set S, an Hausdorff topology on S, and a jointly continuous associative binary operation on S. Since we deal primarily with commutative semigroups, we shall generally use additive notation. In particular, if $A \subseteq S$ and $B \subseteq S$, $A + B$ is the complex sum of A and B and if S is a group, $A - B$ is the complex difference. If S is a semigroup, S_d denotes the topological semigroup consisting of S with the discrete topology. An iseomorphism of topological semigroups is a homomorphism which is a homeomorphism onto its image. If S is a semigroup and σ an equivalence relation on S we use the notation S/σ for the set of equivalence classes modulo σ. In particular, there is a smallest congruence ρ on S such that S/ρ is a semilattice; S/ρ is called the greatest semilattice homomorphic image of S, and the congruence classes are called the Archimedean components of S. Green's $\mathcal{H}$-relation on S is defined by saying $(a, b) \in \mathcal{H}$ only in case $\{a\} \cup (a + S) = \{b\} \cup (b + S)$ and $(S + a) \cup \{a\} = (S + b) \cup \{b\}$; in the commutative case, $\mathcal{H}$ is a congruence relation. If $e^2 = e$ is an idempotent, $H(e)$ denotes the $\mathcal{H}$-class of e and is the largest subgroup of S c[illegible]ing e. A compact holoid is a compact semigroup in which

$\mathcal{H}$ is degenerate (i.e., $a\mathcal{H}b$ only in case $a = b$). We denote by R^n the topological vector space of real n-tuplets with the usual vector operations and the Euclidean topology. A non-empty subset C of R^n is a proper cone if $C + C \subseteq C$, $rC \subseteq C$ for $r \geqq 0$, and $C \cap -C = \{0\}$. A semigroup S is (uniquely) p-divisible where $p \geqq 2$ if for each $x \in S$ there is a (unique) $y \in S$ with $py = x$, and S is (uniquely) divisible if it is (uniquely) p-divisible for each integer $p \geqq 2$. For a discussion of divisible semigroups see [3], or [14]. The following theorem is basic:

KEIMEL'S EXTENSION THEOREM ([14], 1.1). *Let C be a closed proper cone in R^n and T a dense subsemigroup of C^0 such that $(T - T) \cap C^0 \subseteq T$. Any homomorphism $f : T_d \to S$, where S is a compact holoid, is continuous in the relative topology and f may be extended in a unique way to a continuous homomorphism $\bar{f} : C \to S$. Any semigroup T as in Keimel's Theorem we call a K-semigroup.*

The category of locally compact Hausdorff Abelian topological groups and continuous homomorphisms is denoted by LCA. If $G \in LCA$ then $\hat{G}$ denotes the dual group of continuous characters on G to R/Z. $\hat{G} \in LCA$ and if $f \in \operatorname{Hom}(G, H)$ where $H \in LCA$ then $\hat{f} \in \operatorname{Hom}(\hat{H}, \hat{G})$ is defined by $\hat{f}(\gamma) = \gamma \circ f$ for $\gamma \in \hat{H}$. The Bohr compactification of G is the (unique) pair $(B(G), b_G)$ where $B(G) = \widehat{(\hat{G})_d}$ and $b_G : G \to B(G)$ is a continuous isomorphism of G onto a dense subgroup of $B(G)$ where $[b_G(g)](\gamma) = \gamma(g)$ for $g \in G$ and $\gamma \in (\hat{G})_d$. The pair $(B(G), b_G)$ is also the universal compactification of G. Finally we denote by $[0, \infty)((0, \infty))$ the additive semigroup of non-negative (positive) real numbers with the usual topology. Our standard reference for algebraic semigroups is Clifford and Preston [6], for topological semigroups is Hofmann and Mostert [12], and for topological groups is Hewitt and Ross [8].

2. In this section we develop some techniques which will aid in the calculation of certain compact Abelian groups which occur in the universal compactification of a K-semigroup. For this purpose let M be an arbitrary locally compact Abelian topological group, K a dense subgroup of M, and $i : K_d \to M$ the inclusion. Let H denote the subgroup of $B(K_d)$ consisting of those $h \in B(K_d)$ for which there is a net $\tilde{k}$ in K with $h = \lim b_{K_d}(\tilde{k})$ and with $\lim \tilde{k} = 0$ in the relative topology inherited by K as a subset of M.

PROPOSITION 2.1. *H is a closed subgroup of $B(K_d)$ and if $\tau : B(K_d) \to B(M)$ is the continuous homomorphism induced by $i : K_d \to M$, then τ is a surmorphism and has H as its kernel. Hence, the mapping*

$I : B(K_d)/H \to B(M)$ defined by $I(h + H) = \tau(h)$ is an iseomorphism of $B(K_d)/H$ onto $B(M)$.

PROOF. Let $\xi : K \to B(K_d)/H$ be defined by the rule: $\xi(k) = b_{K_d}(k) + H$. If $\bar{k}$ is any net in K converging to 0 in the relative topology on K, then any cluster point of the net $b_{Kd}(\bar{k})$ belongs to H; it follows that ξ is continuous at 0 in the relative topology on K and, hence, is continuous at each member of K. It is known ([8], p. 85) then that ξ may be extended to a continuous homomorphism $\xi_0 : M \to B(K_d)/H$ and ξ_0 may be extended to $\bar{\xi}_0 : B(M) \to B(K_d)/H$ satisfying $\bar{\xi}_0 \circ b_G = \xi_0$. Now since $i : K_d \to M$ is both injective and epic it follows that $\tau : B(K_d) \to B(M)$ is a surmorphism. If $h \in H$ there is a net $\bar{k}$ in K_d converging to 0 in the relative topology on K such that $h = \lim b_{K_d}(k)$. Hence, $\tau(h) = \lim \tau\, b_{K_d}(\bar{k}) = \lim b_G(\bar{k}) = b_G(\lim \bar{k}) = b_G(0) = 0$; this establishes that H is a subgroup of the kernel of τ. It follows immediately that $I : B(K_d)/H \to B(M)$ defined by $I(h + H) = \tau(h)$, is a well-defined continuous homomorphism and is surjective. Now let $k \in K_d$; we have $(\bar{\xi}_0 \circ I)(b_{K_d}(k) + H) = \bar{\xi}_0(\tau(b_{K_d}(k))) = \bar{\xi}_0(b_G(i(k))) = \xi_0(i(k)) = \xi(k) = b_{K_d}(k) + H$. Thus, $\bar{\xi}_0 \circ I$ is the identity function on a dense subset of $B(K_d)/H$ and is, therefore, equal to the identity everywhere. Now let $g \in M$ and $\bar{k}$ a net in K with $g = \lim \bar{k}$; $(I \circ \bar{\xi}_0)(b_G(g)) = I(\xi_0(g)) = I(\lim \xi(\bar{k})) = \lim I(\xi(\bar{k})) = \lim I(b_{K_d}(\bar{k}) + H) = \lim \tau(b_{K_d}(\bar{k})) = \lim b_G(\bar{k}) = b_G(g)$. Thus $I \circ \bar{\xi}_0$ is the identity function on $B(M)$ and, hence, I and $\bar{\xi}_0$ are mutually inverse. That H is the kernel of I is a consequence and the proof is completed.

Now, let $\hat{M} \mid K_d$ denote the subgroup of $(\widehat{K_d})$ consisting of restrictions to K of the characters of the topological group M. Since K is dense in M, $\hat{M} \mid K_d$ consists of those characters in $(\widehat{K_d})$ which are continuous in the relative topology on K as a subset of M.

PROPOSITION 2.2. *The subgroup H of $B(K_d)$ is the annihilator of the subgroup $\hat{M} \mid K_d$ of $(\widehat{K_d})$. That is, $r \in H$ only in case $r(\lambda) = 1$ for $\lambda \in \tilde{M} \mid K_d$.*

PROOF. For the moment, let $M^{\perp}$ denote the annihilator of the subgroup $\hat{M} \mid K_d$. If $h \in H$, there is a net $\bar{k}$ in K with $\lim \bar{k} = 0$ and $\lim b_{K_d}(\bar{k}) = h$. Now let $\lambda \in \hat{M} \mid K_d$; because the topology on $B(K_d)$ is the topology of point-wise convergence on members of K_d it follows that $h(\lambda) = \lim b_{K_d}(\bar{k})(\lambda) = \lim \lambda(\bar{k}) = 1$ since λ is continuous. Thus h is identically equal to 1 on $\hat{M} \mid K_d$ and we have shown $H \subseteq M^{\perp}$. For the reverse inclusion we let $\bar{I} : \widehat{B(M)} \to (\widehat{B(K_d)/H})$ be the isomorphism induced by the iseomorphism $I : B(K_d)/H \to B(M)$. Let $G_0 = \{\gamma \in \widehat{B(K_d)} : \gamma \equiv 1 \text{ on } H\}$; G_0 is the annihilator of the subgroup H of $B(K_d)$ and therefore G_0 is isomorphic to the character group of $B(K_d)/H$ under the isomorphism α, where if λ is a character on $B(K_d)/H$ then $\alpha(\lambda)$ is the character on $B(K_d)$ defined by the rule: $\alpha(\lambda)(h) = \lambda(h + H)$ for $h \in B(K_d)$ (c.f. [8], p. 365). By Pontryagin duality, there is an iseomorphism $\phi : (\hat{M})_d \to \widehat{B(M)}$ defined by the rule: $\phi(\mu)(h) = h(\mu)$ for $\mu \in (\hat{M})_d$ and $h \in B(M)$. Thus, $\alpha \circ \bar{I} \circ \phi$: $(\hat{M})_d \to G_0$ is an isomorphism and is surjective. Now let $\gamma \in M^\perp$ and suppose $\gamma \notin H$. Thus, $\gamma(\lambda) = 1$ for all $\lambda \in \hat{M} \mid K_d$ and there is a $F \in \widehat{B(K_d)}$ such that $F(\gamma) \neq 1$ and $F(\mu) = 1$ for all $\mu \in H$. It follows that $F \in G_0$; choose $\nu \in (\hat{M})_d$ such that $(\alpha \circ \bar{I} \circ \phi)(\nu) = F$. Let ν^* be the restriction of ν to K_d so that $\nu^* \in \hat{M} \mid K_d$. We have that $F(\gamma) = [\alpha \circ \bar{I} \circ \phi(\nu)](\gamma) = \bar{I}(\phi(\nu))(\gamma + H) = \phi(\nu)(I(\gamma + H)) = \phi(\nu)(\tau(\gamma)) = \tau(\gamma)(\nu) = \gamma(\nu^*) = 1$ since $\gamma \in M^\perp$. But $F(\gamma) \neq 1$ giving a contradiction to $M^\perp \subsetneq H$ and the proof of Proposition 2.2 is complete.

COROLLARY 2.3. *If M is a locally compact Abelian group, then $B(M)$ is iseomorphic to $B(M_d)/M^\perp$ where $M^\perp$ is the annihilator of the subgroup $\hat{M} \mid M_d$ of $(\widehat{M_d})$ consisting of the continuous characters of M.*

COROLLARY 2.4. *If K is a dense subgroup of R^n and H is the annihilator of $\hat{R}^n \mid K_d$, then $B(K_d)$ is iseomorphic to $H \oplus \Sigma_a{}^c$ where c denotes the cardinality of the continuum and Σ_a is the a-adic solenoid with $a = (2, 3, 4, \cdots)$ (c.f. [8]).*

PROOF. The group $\hat{R}^n \mid K_d$ is a divisible subgroup of $(\widehat{K_d})$ so there is a homomorphic retraction $r : (\widehat{K_d}) \to \hat{R}^n \mid K_d$. Define $\bar{r} : B(K_d) \to B(K_d)$ by the rule: $\bar{r}(\Gamma)(\lambda) = \Gamma(\lambda - r(\lambda))$ where $\Gamma \in B(K_d)$, $\lambda \in (\widehat{K_d})$. It is a simple matter to check that $\bar{r}$ is a continuous homomorphic retraction of $B(K_d)$ onto H. It follows that $B(K_d)$ is iseomorphic to $H \oplus (B(K_d)/H)$. By Propositions 2.1 and 2.2, $B(K_d)/H$ is iseomorphic with $B(R^n)$ and the fact that $B(R^n)$ is iseomorphic to $\Sigma_a{}^c$ is well-known. Thus, $B(K_d)$ is iseomorphic to $H \oplus \Sigma_a{}^c$ and we are done.

We believe the next proposition is probably known, but lacking a reference we give a proof. Let M denote a dense subgroup of R^n which contains Z^n. For $j \in Z^n$ define a character ξ_j on $(M/Z^n)_d$ by the rule: $\xi_j(\exp(2\pi i x_1), \cdots, \exp(2\pi i x_n)) = \exp(2\pi i(x \cdot j))$ where $(x_1, \cdots, x_n) = x \in M$ and $x \cdot j$ denotes the ordinary scalar product in R^n. We denote the group $(M/Z^n)_d$ by G and let $K = \{(j, \xi_j) : j \in Z^n\}$. Then K is a subgroup of $R^n \times \hat{G}$; we let $N = \{\xi_j : j \in Z^n\}$ and we note that N is a subgroup of $\hat{G}$. Finally, for $v \in R^n$ and $\lambda \in \hat{G}$ we define a function $[v, \lambda] \in (\widehat{M_d})$ by the rule:

$$[v, \lambda](x) = \exp(-2\pi i(x \cdot v)) \cdot \lambda(\exp(2\pi i x_1), \cdots, \exp(2\pi i x_n))$$
$$\text{for } x = (x_1, \cdots, x_n) \in M.$$

PROPOSITION 2.5. *The function* $(v, \lambda) + K \to [v, \lambda]$ *is an iseomorphism of* $(R^n \times \hat{G})/K$ *onto* $(\widehat{M_d})$. *Further,* $((\widehat{M_d})/(\widehat{R^n} \mid M_d))_d$ *is isomorphic to* $(\hat{G}/N)_d$.

PROOF. Let $\theta : M_d \to R^n \times (M/Z^n)_d$ be defined by: $\theta(x) = (x, (\exp(2\pi i x_1), \cdots, \exp(2\pi i x_n)))$. Since θ is injective there is induced an epimorphism $\eta : \widehat{R^n} \times \hat{G} \to (\widehat{M_d})$ such that $\eta(\gamma, \lambda)(x) = \gamma(x)\lambda(\exp(2\pi i x_1), \cdots, \exp(2\pi i x_n))$ for $x \in M$. If $\eta(\gamma, \lambda) = 0$ then $\gamma(x) = \lambda(\exp(2\pi i x_1), \cdots, \exp(2\pi i x_n))^{-1}$ so that for $x \in Z^n, \gamma(x) = 1$. There is thus a $j \in Z^n$ such that $\gamma(x) = \exp(-2\pi i(j \cdot x))$ for $x \in R^n$. If we define $\gamma_j \in \widehat{R^n}$ by the rule: $\gamma_j(x) = \exp(-2\pi i(j \cdot x))$ for $x \in R^n$, then it follows that the kernel of η is $\{(\gamma_j, \xi_j) : j \in Z^n\}$.

Now let U be a neighborhood of 0 in R^n such that $U \cap Z^n = \{0\}$ and set $V = U \times \{0\}$, a neighborhood of $(0, 0)$ in $R^n \times (M/Z^n)_d$. Clearly, $V \cap \theta(M_d) = \{(0, 0)\}$, from which we conclude that $\theta(M_d)$ is a discrete, hence closed, subgroup of $R^n \times (M/Z^n)_d$. (We extend our thanks to J. D. Lawson for this observation.) The inclusion $i : \theta(M_d) \to R^n \times (M/Z^n)_d$ induces a surmorphism $\hat{i} : \widehat{R^n} \times \hat{G} \to \widehat{\theta(M_d)}$ since each character on $\theta(M_d)$ can be extended to a character on $R^n \times (M/Z^n)_d$. The corestriction $\theta_0 : M_d \to \theta(M_d)$ of θ is an iseomorphism so the induced morphism $\hat{\theta}_0 : \widehat{\theta(M_d)} \to \widehat{M}_d$ is an iseomorphism. Finally, $\theta = i \circ \theta_0$ so that $\eta = \hat{\theta}_0 \circ \hat{i}$ is a surmorphism. Now $\widehat{R^n} \times \hat{G}$ is σ-compact and locally compact from which it follows that η is an open mapping ([8], p. 42]. Hence, η induces an iseomorphism $\bar{\eta}$ of $\widehat{R^n} \times \hat{G}$ modulo $\{(\gamma_j, \xi_j) : j \in Z^n\}$ onto $(\widehat{M_d})$ and the first conclusion follows by replacing the character $\gamma_v \in \widehat{R^n}$ by the vector $v \in R^n$ in the canonical way.

Now suppose $[v, \lambda] \in \widehat{R^n} \mid M_d$; hence there is a $w \in R^n$ such that $[v, \lambda](x) = \exp(-2\pi i(x \cdot w))$ for all $x \in M$. By definition it follows that $\exp(-2\pi i(v \cdot x))\lambda(\exp(2\pi i x_1), \cdots, \exp(2\pi i x_n)) = \exp(-2\pi i(x \cdot w))$ for all $x \in M$; it follows that $\lambda(\exp(2\pi i x_1), \cdots, \exp(2\pi i x_n)) = \exp(2\pi i(v - w) \cdot x)$ for $x \in M$. Then since $Z^n \subseteq M$ we get $j_0 = v - w \in Z^n$ and so $\lambda = \xi_{j_0}$. We obtain $[v, \lambda] = [v, \xi_{j_0}]$ and it follows that $(\bar{\eta})^{-1}(\widehat{R^n} \mid M_d) = \{(v, \xi_j) + K : v \in R^n, j \in Z^n\}$. There is induced an isomorphism of $(R^n \times \hat{G}/K)/(\bar{\eta})^{-1}(\widehat{R^n} \mid M_d)$ onto $(\widehat{M_d})/\widehat{R^n} \mid M_d$. But $(\bar{\eta})^{-1}(\widehat{R^n} \mid M_d) = (R^n \times N)/K$ so we get $(R^n \times \hat{G})/R^n \times N$ is isomorphic to $(\widehat{M_d})/(\widehat{R^n} \mid M_d)$ and finally that $(\hat{G}/N)_d$ is isomorphic to $((\widehat{M_d})/(\widehat{R^n} \mid M_d))_d$ which is the second assertion of this Proposition and we are finished.

Now let T be a K-semigroup where T is contained and dense in the interior of the closed cone C in R^n with $R^n = C - C$ and $(T - T) \cap C^0 \subseteq T$. We let $G = T - T$, the (dense) subgroup of R^n generated by T. Let H denote the subgroup of $B(G_d)$ consisting of those $h \in B(G_d)$ for which there is a net $\tilde{g} \in G$ with $\lim \tilde{g} = 0$ and $\lim b_{G_d}(\tilde{g}) = h$. Let H_0 denote the subset of H consisting of those $h \in B(G_d)$ for which there is a net $\tilde{t} \in T$ with $\lim \tilde{t} = 0$ and $\lim b_{G_d}(\tilde{t}) = h$.

Proposition 2.6. $H_0 = H$.

Proof. Clearly, the inclusion $H_0 \subseteq H$ is valid. Let $h \in B(G_d)$ and $\tilde{g}$ a net in G with $\lim \tilde{g} = 0$ and $\lim b_{G_d}(\tilde{g}) = h$; denote the directed set which is the domain of $\tilde{g}$ by $\mathfrak{D}$. For each positive integer n, let $O_n = \{c \in C^0 \mid \|c\| < 1/n\}$ and note that O_n is open. We have $\emptyset \neq O_n \cap T \subseteq O_n \cap (T - T) = (O_n \cap C^0) \cap (T - T) \subseteq O_n \cap T$ and so $O_n \cap T = O_n \cap G$ is open in G with respect to the relativized Euclidean topology on G. With respect to this topology, G is a topological group and hence $(O_n \cap T) - (O_n \cap T)$ is a relative neighborhood of 0 in G. Let $\mathfrak{D}' = \{(\alpha, n) : \alpha \in \mathfrak{D}, n \geqq 1, \text{ and } \tilde{g}_\alpha \in (O_n \cap T) - (O_n \cap T)\}$; since $\lim \tilde{g} = 0$ it follows that $\mathfrak{D}'$ is directed under the partial order; $(\alpha, n) \leqq (\beta, m)$ only in case $\alpha \leqq \beta$ and $n \leqq m$. For $(\alpha, n) \in \mathfrak{D}'$ pick $s(\alpha, n) \in O_n \cap T$ and $t(\alpha, n) \in O_n \cap T$ with $\tilde{g}_\alpha = s(\alpha, n) - t(\alpha, n)$, and finally let $k(\alpha, n) = \tilde{g}_\alpha$. It follows easily that $\tilde{k}$ is a subnet of $\tilde{g}$ so $\lim b_{G_d}(\tilde{k}) = h$. Also evident is the fact that $\lim \tilde{s} = \lim \tilde{t} = 0$. By taking subnets if necessary we get points s_0, $t_0 \in B(G_d)$ with $\lim b_{G_d}(\tilde{s}) = s_0$ and $\lim b_{G_d}(\tilde{t}) = t_0$ and $h = s_0 - t_0$. Since, however s_0, $t_0 \in H_0$ and H_0 is clearly a closed subgroup of $B(G_d)$ it follows that $h = s_0 - t_0 \in H_0$ and the proof is complete.

Corollary 2.7. *H_0 is iseomorphic with the character group of the discrete Abelian group* $(\hat{G}_d/\hat{R}^n \mid G_d)_d$.

Proof. By Proposition 2.2, H is the annihilator of the subgroup $R^n \mid G_d$ of $(\widehat{G_d})$. However, it is known (c.f. [8], p. 365) that the annihilator of $\hat{R}^n \mid G_d$ is (iseomorphic to) the character group of $(\widehat{G_d})/\hat{R}^n \mid G_d$. The conclusion now follows from Proposition 2.6.

We compute now the annihilator H of $\hat{R}^n \mid G_d$ for some specific subgroups G of R^n. These will be useful in § 3 in computing the universal compactification of specific K-semigroups.

Example 1. Let $G = Q$ the additive group of rational numbers. By Proposition 2.2, H is the annihilator of $\hat{R} \mid Q_d$; hence H is the character group of $(\hat{Q}_d/\hat{R} \mid Q_d)$ and, by Proposition 2.5, also the character group of $((\widehat{Q/Z})_d/N)_d$ where $N = \{\xi_j : j \in Z\}$. Since $(\widehat{Q_d})$ is uniquely divisible, $(\widehat{Q_d})/\hat{R} \mid Q_d$ is a rational vector space. Now since Q/Z is countably infinite, the cardinality of $(\widehat{Q/Z})_d$ is c (c.f. [8], p. 396) and since N is countable the cardinality of $(\widehat{QZ})_d/N$ is also c. Hence $(\widehat{Q_d}/\hat{R} \mid Q_d)_d$ is a rational vector space of dimension c, and therefore is the weak direct sum of c copies of Q_d. It follows that H is iseomorphic to $\Sigma_a{}^c$ where Σ_a is the a-adic solenoid (c.f. [8], p. 114).

Example 2. *Let* $G = R^n$; H is the character group of $((\widehat{R_d{}^n})/\hat{R}^n \mid R_d{}^n)_d$. Since $R_d{}^n$ is a rational vector space of dimension c, $(\widehat{R_d{}^n})$ is iseomorphic to $\Sigma_a{}^c$. Each Σ_a is uniquely divisible so that $\Sigma_a{}^c$ is a rational vector space of cardinality, and hence dimension, 2^c. It follows that $\widehat{R_d{}^n}$ is a rational vector space of dimension 2^c. However, $\hat{R}^n \mid R_d{}^n$ is a rational vector space of dimension c; it follows that $((\widehat{R_d{}^n})/\hat{R}^n \mid R_d{}^n)_d$ is a rational vector space of dimension 2^c, and therefore that H is iseomorphic to $\Sigma_a{}^m$ with $m = 2^c$.

Example 3. *Let* $G = Q_p$ the additive group of p-adic rationals with p a prime. We wish to compute $(\widehat{Q_p/Z})_d/N$, where $N = \{\xi_j : J \in Z\}$. Now $Q_p/Z = Z(p\infty)$ and $\xi_j(z) = z^j$ for $x \in Z(p\infty)$. Note that $(\widehat{Q_p})_d$ is iseomorphic to Σ_p the p-adic solenoid and since Σ_p is connected it is divisible (c.f. [8], p. 385); it follows that $(\widehat{Q_p})_d/\hat{R} \mid (Q_p)_d$ and therefore $(\widehat{Z(p\infty)})_d/N$ is divisible. For each prime q, let G_q denote the subgroup of $(\widehat{Z(p\infty)})_d/N$ consisting of the q^n-th roots of unity for $n = 1, 2, \cdots$. For each $\lambda \in (\widehat{Z(p\infty)})_d$ there is a sequence of intergers $\{m_n\}_{n=1}^{\infty}$ such that $m_n \equiv m_{n+1}(p^n)$ and $\lambda(\exp(2\pi i p^{-n})) = \exp(2\pi i m_n p^{-n})$ for $n = 1, 2, 3, \cdots$. If $\{r_n\}_{n=1}^{\infty}$ is another such sequence corresponding to λ, then $r_n \equiv m_n(p^n)$ for $n = 1, 2, \cdots$. Conversely, any such sequence defines an element of $(\widehat{Z(p\infty)})_d$ in the way described. Let $\lambda \in (\widehat{Q_p})_d$ and suppose $p^n\lambda \in \hat{R} \mid (Q_p)_d$. Thus, there is a number $r \in R$ such that $(p^n\lambda)(x) = \exp(2\pi i r x)$ for $x \in Q_p$, or $\lambda(p^n x) = \exp(2\pi i r x)$ for $x \in Q_p$. Since Q_p is p-divisible we get $\lambda(z) = \exp(2\pi i (r p^{-n}) z)$ for $z \in Q_p$. We have shown that the only p^n-th root of unity in $(\widehat{Q_p})_d/\hat{R} \mid (Q_p)_d$ is the identity and thus the same holds for $\widehat{Z(p\infty)}_d/N$.

Let q be a prime, $q \neq p$, and $k_0 \geqq 1$. For each $n \geqq 1$ choose integers k_n and j_n with $k_n q^{k_0} + j_n p^n = 1$. Suppose $\lambda \in \widehat{Z(p\infty)}_d$ with $q^{k_0}\lambda \in N$ and let $\{m_n\}_{n=1}^{\infty}$ be a defining sequence for λ. There is an integer n_0 such that $q^{k_0} = \xi_{n_0}$; then $\exp(2\pi i q^{k_0} m_n p^{-n}) = \exp(2\pi i n_0 p^{-n})$ for $n = 1, 2, \cdots$. Thus, $n_0 \equiv q^{k_0} m_n (p^n)$ for $n = 1, 2, \cdots$; we get $n_0 k_n \equiv k_n q^{k_0} m_n \equiv (1 - j_n p^n) m_n \equiv m_n(p^n)$. Hence, $\lambda(\exp(2\pi i p^{-n})) = \exp(2\pi i n_0 k_n p^{-n})$ for $n = 1, 2, \cdots$. For $n_0 \in Z$, define $\lambda_{n_0} \in \widehat{Z(p\infty)}_d$ as that character corresponding to the sequence

$\{m_n\}_{n=1}^{\infty}$ with $m_n = n_0 k_n$. Note that $m_n - m_{n+1} = (k_n - k_{n+1})n_0$; but $(k_n - k_{n+1})q^{k_0} + (j_n - j_{n+1}p)p^n = 0$ so $(k_n - k_{n+1})q^{k_0} \equiv O(p^n)$ so $k_n \equiv k_{n+1}(p^n)$ and thus $m_n \equiv m_{n+1}(p^n)$ for $n = 1, 2, \cdots$, so we have checked that the sequence $\{m_n\}_{n=1}^{\infty}$ is of the proper kind. The function $n_0 \to \lambda_{n_0} + N : Z \to \widehat{Z(p\infty)}_d/N$ is a homomorphism and, by our preceding remarks, is onto the q^{k_0} roots of unity of $\widehat{Z(p\infty)}_d/N$. Suppose $\lambda_{n_0} \in N$ and choose $j \in Z$ with $\lambda_{n_0} = \xi_j$; then $\exp(2\pi i n_0 k_n p^{-n}) = \exp(2\pi i j p^{-n})$ for $n = 1, 2, \cdots$. We conclude $n_0 k_n \equiv j(p^n)$ for $n = 1, 2, \cdots$; then $n_0 k_n q^{k_0} \equiv j\, q^{k_0}(p^n)$ and $n_0(1 - j_n p^n) \equiv j\, q^{k_0}(p^n)$ so $n_0 \equiv j\, q^{k_0}(p^n)$ for all n. Thus $n_0 = j\, q^{k_0}$ so the kernel of the map $n_0 \to \lambda_{n_0} + N$ is the cyclic subgroup (q^{k_0}) of Z generated by q^{k_0}. Consequently, the set of q^{k_0} roots of unity of $\widehat{Z(p\infty)}_d/N$ is isomorphic to $Z/(q^{k_0})$ and it follows that G_q is isomorphic to $Z(q\infty)$. The torsion subgroup F of $\widehat{Z(p\infty)}_d/N$ is isomorphic to the weak direct sum $\Sigma^*\{Z(q\infty) : q$ a prime, $q \neq p\}$. Note that F is countable so that $(\widehat{Z(p\infty)}_d/N)/F$ is a rational vector space of cardinality, and hence dimension, c. Since F is divisible,$(\widehat{Z(p\infty)}_d/N$ is isomorphic to $(\Sigma^*\{Z(q\infty) : q$ a prime $q \neq p\}) \times Q_d{}^{c*}$ where $Q_d{}^{c*}$ is the weak direct sum of c copies of Q_d. Hence, H is iseomorphic to $\pi\{\Delta_q : q$ a prime, $q \neq p\} \times \Sigma_a{}^c$ where Δ_q is the compact group of q-adic integers (c.f. [8], p. 107).

3. For the purposes of this section, we let T denote a fixed p-divisible K-semigroup where $p \geqq 2$, and C denotes a closed proper cone in R^n such that T is a dense subsemigroup of C^0, $(T - T) \cap C^0 \subseteq T$, and $R^n = C - C$. Denote by (U, u) the universal compactification of T_d and by (A, a) the universal compactification of the semigroup C endowed with the usual topology of R^n. Finally, let $G = T - T$, the subgroup of R^n generated by T; notice that G is p-divisible.

Lemma 3.1. (a) *The semigroup A is uniquely divisible and U is uniquely p-divisible. Both A and U are commutative.*

(b) *The element $a(0)$ is an identity for A and $H(a(0)) = \{a(0)\}$.*

(c) *The mapping $a : C \to A$ is an iseomorphism of C onto $a(C)$.*

(d) *U has an identity θ and $H(\theta) = \cap\{u(\{t \in T : \|t\| < \epsilon\})^* : \epsilon > 0\}$.*

Proof. (a) is fairly well-known and the fact that $a(0)$ is an identity for A is obvious. Let C^∞ denote the one-point compactification of C with addition extended such that $c + \infty = \infty + c = \infty$ for all $c \in C^\infty$; C^∞ is a compact semigroup. Let $i : C \to C^\infty$ denote the inclusion and let $\mu : A \to C^\infty$ be the unique continuous homomorphism satisfying $\mu \circ a = i$. Since $\mu(a(0)) = i(0) = 0$ and $H(0) = \{0\}$ it follows that $\mu(x) = 0$ for all $x \in H(a(0))$. Let $x \in H(a(0))$ and $\tilde{z}$ a net in C with

$x = \lim a(\tilde{z})$; then $0 = \mu(x) = \lim \mu(a(\tilde{z})) = \lim \tilde{z}$ and thus $a(0) = \lim a(\tilde{z}) = x$ proving (b). If $\tilde{z}$ is any net in C with $\lim a(\tilde{z}) = a(z_0)$ then $\lim \tilde{z} = \lim \mu(a(\tilde{z})) = \mu(a(z_0)) = z_0$ and part (c) is done. We do not give a proof for (d) because of the similarity with the proof of Theorem III, [**11**]. Now let $\mathcal{H}$ denote Green's relation on U (i.e., $b_1 \mathcal{H} b_2$ only in case $b_1 + U = b_2 + U$) which in the present case is a closed congruence on U. Let $\rho : U \twoheadrightarrow U/\mathcal{H}$ denote the natural homomorphism. Further, let $U/H(\theta)$ denote the quotient semigroup of U modulo the action of $H(\theta)$ on U and $\tau : U \twoheadrightarrow U/H(\theta)$ the natural homomorphism. Thus $\tau(x) = \tau(y)$ only in case there is a $g \in H(\theta)$ with $x = y + g$. There is a unique continuous homomorphism $\psi : U/H(\theta) \twoheadrightarrow U/\mathcal{H}$ satisfying $\psi \circ \tau = \rho$.

Lemma 3.2. *There is a continuous homomorphism $\eta : C \to U/H(\theta)$ for which $\eta(t) = \tau(u(t))$ for all $t \in T$.*

Proof. Notice that $U/\mathcal{H}$ is a compact holoid (i.e., all subgroups are trivial). By Keimel's Extension Theorem (see § 1), $\rho \circ u : T \to U/\mathcal{H}$ is continuous with the relative topology of R^n on T, and can be extended to a continuous homomorphism $\bar{\rho} : C \to U/\mathcal{H}$. Hence, $\bar{\rho}(t) = \rho(u(t))$ for $t \in T$. Now let $z \in C$; since ψ is surjective there is an $x \in U/H(\theta)$ such that $\psi(x) = \bar{\rho}(z)$. Suppose also that $y \in U/H(\theta)$ and $\psi(y) = \bar{\rho}(z)$; pick $x', y' \in U$ such that $\tau(x') = x$ and $\tau(y') = y$. It follows that $\rho(x') = \rho(y')$ and thus $x' + U = y' + U$. Choose $b, c \in U$ with $x' = y' + b$, $y' = x' + c$. We have $\bar{\rho}(z) = \psi(x) = \psi(\tau(x')) = \rho(x') = \rho(b) + \rho(y') = \rho(b) + \bar{\rho}(z)$. Inductively we obtain $\bar{\rho}(z) = k\rho(b) + \bar{\rho}(z)$ for all $k \geqq 1$; setting $k = p^m$ it follows $p^m\bar{\rho}(p^{-m}z) = p^m\rho(b) + p^m\bar{\rho}(p^{-m}z)$ for all $m \geqq 1$. As is easily proven, $U/\mathcal{H}$ is uniquely p-divisible, from which it follows that $\bar{\rho}(p^{-m}z) = \rho(b) + \bar{\rho}(p^{-m}z)$. By continuity of $\bar{\rho}$ we get $\bar{\rho}(0) = \rho(b) + \bar{\rho}(0)$. However, since $\rho(u(T))$ is dense in $U/\mathcal{H}$ then so is $\bar{\rho}(C)$. Hence $\bar{\rho}(0)$ is an identity for $U/\mathcal{H}$ and therefore $\bar{\rho}(0) = \rho(\theta)$. Thus, $\rho(b) = \rho(\theta)$ from which it follows that $b \in H(\theta)$ and $\tau(x') = \tau(y')$ and so $x = y$. We have shown that for each $z \in C$ there is a unique $x \in U/H(\theta)$ with $\psi(x) = \bar{\rho}(z)$; we define $\eta(z) = x$. That $\eta : C \to U/H(\theta)$ is a homomorphism is clear. Let $\tilde{z}$ be a net in C converging to $z_0 \in C$ and let x_1 and x_2 be cluster points of the net $\eta(\tilde{z})$. Consequently, $\psi(x_1)$ and $\psi(x_2)$ are cluster points of the net $\psi(\eta(\tilde{z})) = \bar{\rho}(\tilde{z})$. Since $\lim \tilde{z} = z_0$ we get $\lim \bar{\rho}(\tilde{z}) = \bar{\rho}(z_0)$ so that $\psi(x_1) = \psi(x_2) = \bar{\rho}(z_0)$ and therefore $x_1 = x_2 = \eta(z_0)$, giving the continuity of η. Finally, if $t \in T$ then $\bar{\rho}(t) = \rho(u(t)) = \psi(\tau(u(t)))$ so by definition $\eta(t) = \tau(u(t))$ and the proof of Lemma 3.2 is complete.

By the universal properties of (A, a) there is a continuous homomorphism $\bar{\eta} : A \to U/H(\theta)$ satisfying $\bar{\eta} \circ a = \eta$.

LEMMA 3.3. *The mapping* $\bar{\eta}: A \to U/H(\theta)$ *is an iseomorphism of* A *onto* $U/H(\theta)$. *Furthermore,* $\bar{\eta}(a(t)) = \tau(u(t))$ *for all* $t \in T$.

PROOF. Let $j: T_d \to C$ denote the inclusion and set $m = a \circ j: T_d \to A$. There is induced a continuous homomorphism $\bar{m}: U \to A$ for which $\bar{m} \circ u = m$. Since $H(a(0)) = \{a(0)\}$ there is a continuous homomorphism $\phi: U/H(\theta) \to A$ such that $\phi \circ \tau = \bar{m}$. We claim ϕ is a two-sided inverse of $\bar{\eta}$. In fact, let $t \in T$; then $(\phi \circ \bar{\eta})(a(j(t))) = \phi(\eta(j(t))) = \phi(\eta(t)) = \phi(\tau(u(t))) = \bar{m}(u(t)) = m(t) = a(j(t))$. Thus, $\phi \circ \bar{\eta}$ agrees with the identity function on the dense subset $a(j(T))$ of A and hence is the identity function on A. Similarly, $\bar{\eta} \circ \phi$ agrees with the identity function on the dense subset $\tau(u(T))$ of $U/H(\theta)$ and is therefore equal to the identity function everywhere. This completes the proof of Lemma 3.3.

COROLLARY 3.3. *The mapping* $\eta = \bar{\eta} \circ a: C \to U/H(\theta)$ *is an iseomorphism of* C *onto* $\eta(C)$ *and* $\eta(t) = \tau(u(t))$ *for all* $t \in T$.

We now fix a basis $e_1, e_2, \cdots, e_n$ for the real vector space R^n consisting of elements of C. For $1 \leqq i \leqq n$ set $J_i = (\eta(\{\lambda e_i : \lambda \geqq 0\}))^*$ in $U/H(\theta)$. Clearly each J_i is a compact solenoidal semigroup, $H(\theta) \subseteq \tau^{-1}(J_i)$ and $\tau^{-1}(J_i)/H(\theta)$ is iseomorphic to J_i. One may find a one-parameter semigroup $\sigma_i: [0, \infty) \to \tau^{-1}(J_i)$ such that $\sigma(0) = \theta$ and $\sigma(b) \notin H(\theta)$ for $b > 0$ (c.f. [16]). It follows that $\tau(\sigma_i([0, \infty))) = \eta(\{\lambda e_i : \lambda \geqq 0\})$. We set $P = \Sigma\{\sigma_i([0, \infty)) : 1 \leqq i \leqq n\}$; note that P is a subsemigroup of U and $P \cap H(\theta) = \{\theta\}$.

LEMMA 3.4. *The restriction of* τ *to* P *is injective.*

PROOF. Let $s, t \in P$ such that $\tau(s) = \tau(t)$; there is a $g \in H(\theta)$ such that $s = t + g$. Choose $t_i, s_i \in [0, \infty)$, $1 \leqq i \leqq n$, such that $t = \Sigma\{\sigma_i(t_i) : 1 \leqq i \leqq n\}$ and $s = \Sigma\{\sigma_i(s_i) : 1 \leqq i \leqq n\}$. Set $A = \{i : t_i \leqq s_i\}$ and $B = \{j : s_j < t_j\}$, then $A \dot{\cup} B = \{1, 2, \cdots, n\}$. We consider only the case $A \neq \phi \neq B$, the remaining cases being much less difficult (also c.f. [2], Theorem 2.2). For each $i \in A$ set $s_i = t_i + b_i$, $b_i \in [0, \infty)$, and for $j \in B$, set $t_j = s_j + c_j$, $c_j \in (0, \infty)$. Finally, let $b = \Sigma\{\sigma_i(b_i) : i \in A\}$, $c = \Sigma\{\sigma_j(c_j) : j \in B\}$ and $d = (\Sigma\{\sigma_j(s_j) : j \in B\}) + (\Sigma\{\sigma_i(t_i) : i \in A\})$. It follows that $d + b = d + c + g$. Then $d + p^m b = d + p^m(c + g)$ for all $m \geqq 1$. Since U is uniquely p-divisible and the p^m-th roots of d converge to θ we get $b = c + g$. Let $\tau(\sigma_i(b_i)) = \eta(\lambda_i e_i)$, $\lambda_i \geqq 0$, for $i \in A$ and $\tau(\sigma_j(c_j)) = \eta(\mu_j e_j)$, $\mu_j \geqq 0$, for $j \in B$. We obtain $\eta(\Sigma\{\lambda_i e_i : i \in A\}) = \eta(\Sigma\{\mu_j e_j : j \in B\})$ and, since by Corollary 3.3 η is injective, $\Sigma\{\lambda_i e_i : i \in A\} = \Sigma\{\mu_j e_j : j \in B\}$. By the linear independence of $\{e_1, e_2, \cdots, e_n\}$ we get $\lambda_i = \mu_j = 0$ for

$i \in A$, $j \in B$. Then $\tau(\sigma_j(c_j)) = \eta(\mu_j e_j) = \eta(0) = \tau(\theta)$ so $\sigma_j(c_j) \in H(\theta)$, a contradiction since $c_j > 0$. This establishes that the case $A \neq \emptyset \neq B$ is impossible, concluding the proof of Lemma 3.4.

Now, fix $t \in T$; by Corollary 3.3, $\eta(t) = \tau(u(t))$. Write $t = \Sigma\{\lambda_i e_i : 1 \leqq i \leqq n\}$ where $\lambda_i \in R$, and set $A = \{i : \lambda_i \leqq 0\}$ and $B = \{j : \lambda_j > 0\}$. Letting $x = \Sigma\{(-\lambda_i)e_i : i \in A\}$ and $y = \Sigma\{\lambda_j e_j : j \in B\}$ then $x, y \in C$ and $t + x = y$ (note that since $t \in C^0$, $B \neq \emptyset$. However, if $A = \emptyset$ set $x = 0$). There is a w and $z \in P$ such that $\tau(w) = \eta(x)$ and $\tau(z) = \eta(y)$. Since $t + x = y$, $\eta(t) + \eta(x) = \eta(y)$ and thus $\tau(u(t) + w) = \tau(z)$. We may therefore choose $g \in H(\theta)$ such that $u(t) + w = z + g$. Suppose $w', z' \in P$, $g' \in H(\theta)$ and $u(t) + w' = z' + g'$; then $w' + z = w + z' + (g' - g)$ so that $\tau(w' + z) = \tau(w + z')$ and by Lemma 3.4, $w' + z = w + z'$. Again using unique p-divisibility we conclude that $g' - g = \theta$ or $g' = g$. This observation allows us to define a function $\beta : T_d \to H(\theta)$ by the rule: $\beta(t) = g$ exactly when there exist $w, z \in P$ with $u(t) + w = z + g$.

Lemma 3.5. *The function $\beta : T_d \to H(\theta)$ is a homomorphism.*

Proof. Clear (or see [2], Theorem 2.2).

By the universal properties of (U, u) there is a continuous homomorphism $\bar{\beta} : U \to H(\theta)$ which satisfies $\bar{\beta} \circ u = \beta$.

Lemma 3.6. *The function $\bar{\beta} : U \to H(\theta)$ is a continuous homomorphic retraction of U onto $H(\theta)$.*

Proof. Let $g \in H(\theta)$ and t a net in T with $g = \lim u(\tilde{t})$. For each α in the domain of $\tilde{t}$ there is a decomposition $A_\alpha \cup B_\alpha = \{1, 2, \cdots, n\}$ and $t_i^\alpha \in [0, \infty)$ such that $u(\tilde{t}_\alpha) + (\Sigma\{\sigma_i(t_i^\alpha) : i \in A_\alpha\}) = (\Sigma\{\sigma_j(t_j^\alpha) : j \in B_\alpha\}) + \beta(\tilde{t}_\alpha)$. By taking a subnet if necessary we may assume $A_\alpha = A_0$ and $B_\alpha = B_0$ for all α. Thus

$$(*)\qquad u(\tilde{t}) + (\Sigma\{\sigma_i(\tilde{t}_i) : i \in A_0\}) = (\Sigma\{\sigma_j(\tilde{t}_j) : j \in B_0\}) + \beta(\tilde{t}).$$

For $i \in A_0$ choose a net $\tilde{\lambda}_i \in [0, \infty)$ such that $\eta(\tilde{\lambda}_i e_i) = \tau(\sigma_i(\tilde{t}_i))$ and for $j \in B_0$ a net $\tilde{\mu}_j \in [0, \infty)$ such that $\eta(\tilde{\mu}_j e_j) = \tau(\sigma_j(\tilde{t}_j))$. Since $\eta(\tilde{t}) = \tau(u(\tilde{t}))$ it follows that $\eta(\tilde{t} + \Sigma\{\tilde{\lambda}_i e_i : i \in A_0\}) = \eta(\Sigma\{\tilde{\mu}_j e_j : j \in B_0\})$ and by Corollary 3.3, $\tilde{t} + \Sigma\{\tilde{\lambda}_i e_i : i \in A_0\} = \Sigma\{\tilde{\mu}_j e_j : j \in B_0\}$. Now since $g = \lim u(\tilde{t})$ we get $\eta(0) = \tau(\theta) = \tau(g) = \lim \tau u(\tilde{t}) = \lim \eta(\tilde{t})$; again by Corollary 3.3, $\lim \tilde{t} = 0$. This implies that $\lim \tilde{\lambda}_i = \lim \tilde{\mu}_j = 0$ for all $i \in A_0$ and $j \in B_0$. It follows that $\lim \tau(\sigma_i(\tilde{t})) = \tau(\theta)$ for all i, $1 \leqq i \leqq n$, and thus $\lim \sigma_i(\tilde{t}_i) = \theta$ for each i. Recalling that $\beta(\tilde{t}) = \bar{\beta}(u(\tilde{t}))$ and using $(*)$ it follows that $g = \lim u(\tilde{t}) = \lim[u(\tilde{t}) + \Sigma\{\sigma_i(\tilde{t}_i) : i \in A_0\}] = \lim[\Sigma\{\sigma_j(\tilde{t}_j) : j \in B_0\} + \bar{\beta}(u(\tilde{t}))] = \lim \bar{\beta}(u(\tilde{t})) = \bar{\beta}(g)$ and we are done with the proof.

LEMMA 3.7. *If $\bar{\beta} : U \to H(\theta)$ is any continuous homomorphic retraction, then $\Delta = \bar{\beta} \times \tau$ is an iseomorphism of U onto $H(\theta) \times (U/H(\theta))$.*

PROOF. Clearly, Δ is continuous; suppose $\Delta(x) = \Delta(y)$. Then $\tau(x) = \tau(y)$, which implies there is a $g \in H(\theta)$ with $x = y + g$. However, since $\bar{\beta}(x) = \bar{\beta}(y)$ we get $\bar{\beta}(x) = \bar{\beta}(y) + \bar{\beta}(g) = \bar{\beta}(x) + g$ and $g = \theta$ implying $x = y$. Let $(g, \tau(x)) \in H(\theta) \times U/H(\theta)$ where $x \in U$; set $y = x + (g - \bar{\beta}(x))$. Clearly $\tau(y) = \tau(x)$; further $\bar{\beta}(y) = \bar{\beta}(x) + \bar{\beta}(g - \bar{\beta}(x)) = \bar{\beta}(x) + (g - \bar{\beta}(x)) = g$ and $\Delta(y) = (g, \tau(x))$. This establishes Lemma 2.7.

We are now in a position to state and prove our first main result:

THEOREM 1. *Let T be a p-divisible K-semigroup, $p \geqq 2$, C a closed proper cone in R^n with $R^n = C - C$, T a dense subsemigroup of C^0, with $(T - T) \cap C^0 \subseteq T$. Let (A, a) denote the universal compactification of C in the usual topology of R^n and let (B, b) denote the Bohr compactification of the discrete Abelian group $G_d = (T - T)_d$. Finally, let $i : T_d \to G_d$ and $j : T_d \to C$ denote the inclusions, then*:

(i) *The universal compactification of T_d is $(\mathcal{B}, (b \circ i) \times (a \circ j))$ where $\mathcal{B} = [(b \circ i) \times (a \circ j)(T_d)]^*$ in $B \times A$.*

(ii) *There is a continuous homomorphic retraction $\mu : B \times A \to \mathcal{B}$.*

(iii) *$\mathcal{B}$ is iseomorphic to $H_0 \times A$ where H_0 is the closed subgroup of B to which an element g belongs only in case there is a net $\tilde{t}$ in T with* $\lim \tilde{t} = 0$ *and* $\lim b(\tilde{t}) = g$.

PROOF. Consider the homomorphism $\beta : T \to H(\theta)$ of Lemma 3.5. Extend to a homomorphism, which we also denote by β, from G_d to $H(\theta)$ by defining $\beta(t_1 - t_2) = \beta(t_1) - \beta(t_2)$. Since $\beta(t) = \bar{\beta}(u(t))$ it follows that $\beta(T)$ and, therefore, $\beta(G)$ is dense in $H(\theta)$. By the universal properties of (B, b) there is a continuous homomorphism $\beta^* : B \to H(\theta)$ satisfying $\beta^* \circ b = \beta$. By Lemma 3.6, $\bar{\beta} : U \to H(\theta)$ is a continuous homomorphic retraction and by Lemma 3.7, $\Delta = \bar{\beta} \times \tau$ is an iseomorphism of U onto $H(\theta) \times (U/H(\theta))$. Set $\mu = \Delta^{-1} \circ (\beta^* \times \bar{\eta}) : B \times A \to U$ where $\bar{\eta} : A \to U/H(\theta)$ is the iseomorphism of Lemma 3.3. Let $t \in T$ and recall that $\bar{\eta}(a(t)) = \tau(u(t))$; then $(\beta^* \times \bar{\eta})\ (b \circ i \times a \circ j)(t) = (\beta^* \circ b(i(t)),\ \bar{\eta} \circ a(j(t))) = (\beta(i(t)), \tau(u(t)) = (\beta(t),\ \tau(u(t))$. But $\Delta^{-1}(\beta(t), \tau(u(t))) = u(t)$ so we have $[\mu \circ (b \circ i \times a \circ j)]\,(t) = u(t)$. We have shown that if (U, u) is the universal compactification of T_d, there is a continuous homomorphism $\mu : B \times A \to U$ such that $\mu \circ (b \circ i \times a \circ j) = u$. Thus, if (i) is established, (ii) follows immediately. Let S be a compact semigroup and $f: T_d \to S$ a homomorphism; there is a continuous homomorphism $f^*: U \to S$ with $f^* \circ u = f$. Let f' denote the restriction of $f^* \circ \mu$ to $\mathcal{B}$; if $t \in T$ then $[f' \circ (b \circ i \times a \circ j)](t) = f^*(\mu \circ (b \circ i \times a \circ j)(t)) = f^*(u(t)) = f(t)$. Thus, $(\mathcal{B},\ b \circ i \times a \circ j)$ satisfies the requirements in order to be the universal compactification of T_d, hence (i) is proved. By Lemma 3.7, $\mathcal{B}$ is iseomorphic to $H \times (\mathcal{B}/H)$ where H is the maximal subgroup of $\mathcal{B}$ containing the identity of $\mathcal{B}$. Further, by Lemma 3.3, $\mathcal{B}/H$ is iseomorphic to A. If $(g, x) \in H$ by Lemma 3.1 (c), there is a net $\tilde{t}$ in T such that $\lim \tilde{t} = 0$ and $\lim((b \circ i)(\tilde{t}), (a \circ j)(t)) = (g, x)$. Thus, $x = a(0)$ and $g = \lim b(i(\tilde{t}))$, so $g \in H_0$. Conversely, if $g \in H_0$ and $\lim \tilde{t} = 0$ where $\lim b(\tilde{t}) = g$ then $(g,\ a(0)) = \lim[(b \circ i) \times (a \circ j)](\tilde{t})$ so $(g,\ a(0)) \in \mathcal{B}$ and is clearly a unit. Hence, $H = H_0 \times \{a(0)\}$ and we have completed the proof of (iii) and of Theorem 1.

Remark. Since $H = H_0 \times \{a(0)\}$ the identity of $\mathcal{B}$ is $(b(0), a(0))$.

We now combine the results of § 2 with Theorem 1 to compute the universal compactification of some specific K-semigroups.

Example 4. We take $T = Q^+$, the additive semigroup of positive rational numbers. In this case, $C = [0, \infty)$ and $G = Q$. The universal compactification of C is the universal compact solenoidal semigroup Φ (c.f. [12], Theorem II). The group H_0 is, by Corollary 2.7, the character group of $((\widehat{Q_d})/\hat{R} \mid Q_d)$ which, as computed in Example 1, § 2, is iseomorphic to $\Sigma_a{}^c$. By Theorem 1 (iii), the universal compactification of $Q_d{}^+$ is $\Sigma_a{}^c \times \Phi$. This result was first obtained by J. Hildebrant [9].

Example 5. We take $T = Q_p{}^+$ the positive p-adic rationals where p is a prime. Again $C = [0, \infty)$ and by Example 3, § 2 and Theorem 1 (iii), the universal compactification of the positive p-adic rationals $(Q_p{}^+)_d$ is $\prod \{\Delta_q : q \neq p,\ q \text{ a prime}\} \times \Sigma_a{}^c \times \Phi$. Again this result first appears in [10] by J. Hildebrant.

Example 6. Here we take $T = C^0$ where C is a closed cone in R^n and $R^n = C - C$. Here H_0 is iseomorphic to $\Sigma_a{}^m$, where $m = 2^c$, by Example 2, § 2. Hence the universal compactification of C^0 is $\Sigma_a{}^m \times A$, $m = 2^c$, and A is the universal compactification of C. Thus, if $T = (0, \infty)$, the additive positive reals, the universal compactification of $(0, \infty)_d$ is $\Sigma_a{}^m \times \Phi$ with $m = 2^c$. Slightly more generally, if $T = (0, \infty)^n$ the universal compactification of $((0, \infty)^n)_d$ is $\Sigma_a{}^m \times \Phi^n$ where $m = 2^c$.

4. Let C be a closed proper cone in R^n with $R^n = C - C$ and let G denote a dense p-divisible subgroup of R^n. We set $S_0 = G \cap C$ and

$S = S_0 \backslash \{0\}$. In this section we give a concrete realization of the greatest semilattice homomorphic image $\mathcal{E}$ (c.f. [6], p. 131) of S and we then show that $\mathcal{E}$ satisfies the finite chain condition and that S is a semilattice $\mathcal{E}$ of p-divisible K-semigroups.

LEMMA 4.1. *S is a p-divisible subsemigroup of C. If T denotes the part of S in the interior of C relative to* $C - C = R^n$ *then T is a p-divisible K-semigroup.*

DEFINITION A. A p-divisible subsemigroup E of S is a p-divisible filtre on S if $E = S$ or $S\backslash E$ is an ideal in S. An S-cone is a p-divisible filtre on S which is closed in S with respect to the relative Euclidean topology.

LEMMA 4.2. *Every proper p-divisible filtre on S is contained in the boundary of C relative to* $C - C = R^n$. *Every p-divisible subsemigroup of S contained in the boundary of C is contained in a proper S-cone.*

PROOF. Let E be a proper p-divisible filtre on S and suppose $y \in E$ and $y \in C^0$ relative to R^n. Thus, $y - C^0$ is a neighborhood of 0 in R^n; if $x \in S$ there is a $n \geqq 1$ such that $p^{-n}x \in y - C^0$. But then $s = y - p^{-n} \cdot x \in G - G \cap C^0 \subseteq G \cap C^0 = T$; hence $p^n s \in S$ and $p^n s + x = p^n y \in E$. However, since $S\backslash E$ is an ideal in S we must have $x \in E$. Consequently, $S \subseteq E$ and E is not proper. For the second assertion we let E_0 be a p-divisible subsemigroup of S contained in the boundary of C relative to R^n. Denote by $\mathcal{M}$ the collection of all p-divisible subsemigroups of S contained in the boundary of C and which contain E_0. Let $\mathcal{A}$ be a maximal tower in $\mathcal{M}$ and let $E = \bigcup \mathcal{A}$; clearly $E \in \mathcal{M}$. Furthermore if E^* is the closure of E in R^n then $E^* \cap S \in \mathcal{M}$ so by the maximality of $\mathcal{A}$, $E = E^* \cap S$ is closed in S. Now, suppose $x, y \in S$ with $x \notin E$ but $x + y \in E$. Denote by $P = \{z \in S \mid z \in E$ or $z + ry \in E$ for some positive p-adic rational $r\}$ and notice that $E \subsetneqq P$ since $x \in P\backslash E$. That P is a p-divisible subsemigroup of S containing E_0 is clear. Suppose $z \in P \cap C^0$; there must be a positive p-adic rational r such that $z + ry \in E$. Notice that T itself is an ideal in S and $z \in T$. It follows that $z + ry \in E \cap C^0$, a contradiction. We have shown that P is a subset of the boundary of C and consequently $P \in \mathcal{M}$. But $E \subsetneqq P$ contradicting the maximality of $\mathcal{A}$. It now follows that $S\backslash E$ is an ideal of S and finally that E is a proper S-cone containing E_0.

DEFINITION B. If E is an S-cone, denote by $T(E)$ the part of S in the interior of E^* relative to the closed subspace $E^* - E^*$ of R^n.

Remark. If E is an S-cone and $G' = E - E$ then G' is a dense p-divisible subgroup of the closed subspace $E^* - E^*$ of R^n. Setting $C' = E^*$, $S_0' = G' \cap C'$, $S' = S_0' \backslash \{0\}$ and $T' = T(E)$ then Lemma 4.1 and Lemma 4.2 remain valid for G', S', T', C', and $E^* - E^*$. Further, $S' = M$; in fact, $S_0' = G' \cap C' \subseteq G \cap C' = G \cap E^* = G \cap C \cap E^* = S_0 \cap E^* = \{0\} \cup (S \cap E^*) = \{0\} \cup E$ so that $S' \subseteq E$. The reverse inclusion is obvious. We also observe that by Lemma 4.1, $T(E)$ is a p-divisible K-semigroup dense in the interior (relative to $E^* - E^*$) of the cone E^*.

Lemma 4.3. *If E_1 and E_2 are distinct S-cones, then $T(E_1) \cap T(E_2) = \phi$.*

Proof. If E_1 and E_2 are disjoint we are through. On the other hand, if E_1 meets E_2 then $E_1 \cap E_2$ is a proper E_2-cone and, hence, by Lemma 4.2 is contained in the boundary of $E_2{}^*$ relative to $E_2{}^* - E^*_2$. Hence no point of $T(E_2)$ may belong to E_1, so $T(E_1) \cap T(E_2) = \phi$.

Now we let $\mathcal{E}$ denote the collection of S-cones and for $E_1, E_2 \in \mathcal{E}$, $E_1 \vee E_2$ denotes the intersection of all S-cones containing $E_1 \cup E_2$.

Lemma 4.4. *$(\mathcal{E}, \vee)$ is a semilattice satisfying the finite chain condition.*

Proof. That $(\mathcal{E}, \vee)$ is a semilattice is obvious. Suppose $E_1, E_2 \in \mathcal{E}$ and $E_1 \subseteq E_2$. Since E_1 is a proper E_2-cone, E_1 is contained in the boundary of $E_2{}^*$ relative to $E_2{}^* - E_2{}^*$. Thus, the closed cone $E_1{}^*$ is a subset of the boundary of $E_2{}^*$ relative to $E_2{}^* - E_2{}^*$. Hence the inductive dimension of $E_1{}^*$ is strictly less than the inductive dimension of $E_2{}^*$. Thus, there cannot be any infinite chains in $\mathcal{E}$, since the inductive dimension of E^*, $E \in \mathcal{E}$, is bounded by n, where again $C - C = R^n$.

Lemma 4.5. *If $E_1, E_2 \in \mathcal{E}$, then $T(E_1) + T(E_2) \subseteq E_1 \vee E_2$).*

Proof. Let $E_3 = E_1 \vee E_2$; if $E_3 = E_1$ then $E_2 \subseteq E_1$ and since $T(E_1)$ is an ideal in E_1 we get $T(E_1) + T(E_2) \subseteq T(E_1) = T(E_1 \vee E_2)$. The conclusion follows likewise if $E_3 = E_2$. Hence we may assume E_1 and E_2 are proper E_3-cones. By Lemma 4.2, $E_1 \cup E_2$ is contained in the boundary F of $E_3{}^*$ relative to $E_3{}^* - E_3{}^*$. Suppose $E_1 + E_2 \subseteq F$; then $E_1 \cup E_2 \cup E_1 + E_2$ is a p-divisible subsemigroup of E_3 contained in F. By Lemma 4.2 there is an E_3-cone E containing $E_1 \cup E_2 \cup E_1 + E_2$ and contained in F. It is easily verified that E is an S-cone primarily because $S \backslash E_3$ is an ideal in S and $E_3 \backslash E$ is an ideal in E_3. However, $E \subsetneq E_3$ which contradicts the definition of $E_1 \vee E_2 = E_3$. Hence, we may assume there exist $x \in E_1$, $y \in E_2$ with $x + y \notin F$; since $E_1 \cup E_2 \subseteq E_3$ it follows that $x + y \in T(E_3)$. Let $z_1 \in T(E_1)$ and $z_2 \in T(E_2)$, let U and V denote respectively the interior of $E_1{}^*$ relative to $E_1{}^* - E_1{}^*$ and the interior of $E_2{}^*$ relative to $E_2{}^* - E_2{}^*$. Then $z_1 - U$ and $z_2 - V$ are neighborhoods of 0 in $E_1{}^* - E_1{}^*$ and $E_2{}^* - E_2{}^*$ respectively. There is an integer $n_0 \geqq 1$ such that for $m \geqq n_0$, $p^{-m}x \in z_1 - U$ and $p^{-m}y \in z_2 - V$. Then $z_1 - p^{-m}x \in (E_1 - E_1) \cap U \subseteq G \cap E_1{}^* = G \cap C \cap E_1{}^* = S_0 \cap E_1{}^* = \{0\} \cup (S \cap E_1{}^*) = \{0\} \cup E_1$; hence we may assume that for $m \geqq n_0$, $z_1 - p^{-m}x \in E_1$ and similarly that $z_2 - p^{-m}y \in E_2$. Then $(z_1 + z_2) - p^{-m}(x + y) \in E_1 + E_2 \subseteq E_3$ for $m \geqq n_0$. Hence, $z_1 + z_2 = p^{-m}(x + y) + z_3$ where $z_3 \in E_3$. Since $x + y \in T(\mathrm{E}_3)$ then $p^{-m}(x + y) \in T(E_3)$ for $m \geqq n_0$ and since $T(E_3)$ is an ideal in E_3, $z_1 + z_2 = p^{-m}(x + y) + z_3 \in T(E_3)$ and we are done.

We are now prepared to state the major result of this section. The result is in the same vein as the result by Brown and La Torre ([4], Theorem 1) where it is shown that a uniquely divisible commutative semigroup is a semilattice of semigroups each of which is the direct sum of a rational vector space and a cone of a rational vector space.

THEOREM 2. *Let G be a p-divisible dense subgroup of R^n and C a closed cone in R^n with $R^n = C - C$. Let $S_0 = G \cap C$ and $S = S_0 \backslash \{0\}$ and denote by $(\mathcal{E}, \vee)$ the semilattice of S-cones. Let $\mathcal{E}_0$ denote the semilattice obtained by adjoining an identity 0 to $\mathcal{E}$. Define $T(0) = \{0\}$ and for $E \in \mathcal{E}$ define $T(E)$ as in Definition B.*

(i) *$S_0 = \cup\{T(E) : E \in \mathcal{E}_0\}$ is a semilattice decomposition of S_0 into subsemigroups of S_0.*

(ii) *For $E \in \mathcal{E}$, $T(E)$ is a p-divisible K-semigroup.*

(iii) *$(\mathcal{E}_0, \vee)$ is a semilattice satisfying the finite chain condition.*

(iv) *$(\mathcal{E}_0, \vee)$ is the greatest semilattice homomorphic image of S_0.*

PROOF. In view of Lemmas 4.3 and 4.5 it suffices for (i) to show that each element of S_0 belongs to $T(E)$ for some $E \in \mathcal{E}$. Hence, let $x \in S_0$, $x \neq 0$, and denote by E the intersection of all S-cones containing x; clearly E is an S-cone. If $x \notin T(E)$ then x belongs to the boundary of E^* relative to $E^* - E^*$. Then $M = \{rx \mid r$ a positive p-adic rational$\}$ is a p-divisible subsemigroup of E contained in the boundary of E^* relative to $E^* - E^*$. By Lemma 4.2, there is an E-cone E_0 containing M (and therefore x) which is contained in the boundary of E^* relative to $E^* - E^*$. Since E is an S-cone it follows that E_0 is an S-cone containing x. However $E_0 \subsetneqq E$ contradicting the choice of E, and $x \in T(E)$. Part (ii) is simply a restatement of the remark preceding Lemma 4.3. Part (iii) is Lemma 4.4. For (iv), note that the greatest semilattice homomorphic image of S_0 is S_0/ρ where ρ is the intersection of all congruences σ on S_0 with S_0/σ a semilattice. Clearly, if σ denotes the relation on S_0 with $x\sigma y$ only in case $x, y \in T(E)$ for some $E \in \mathcal{E}_0$ then σ is a congruence and S_0/σ is isomorphic to $\mathcal{E}_0$ so that $\rho \subseteq \sigma$. Suppose $x\sigma y$ where $x, y \in T(E)$ for $E \in \mathcal{E}$. Let U denote the interior of the cone E^* relative to $E^* - E^*$; since $x \in U$, $x - U$ is a neighborhood of 0 in $E^* - E^*$. There is an integer m with $p^{-m}y \in x - U$. Consequently $x - p^{-m}y \in (E - E) \cap U \subseteq E$ (see the proof of Lemma 4.5). Hence, x divides a power of y in S_0 and similarly y divides a power of x in S_0. This is exactly the statement that $x\rho y$ ([6], Theorem 4.12) and $\sigma \subseteq \rho$; thus $\sigma = \rho$ and (iv) follows.

EXAMPLE 7. Let $G = R^3$ and C the cone with lateral cross-section given by Figure 1.

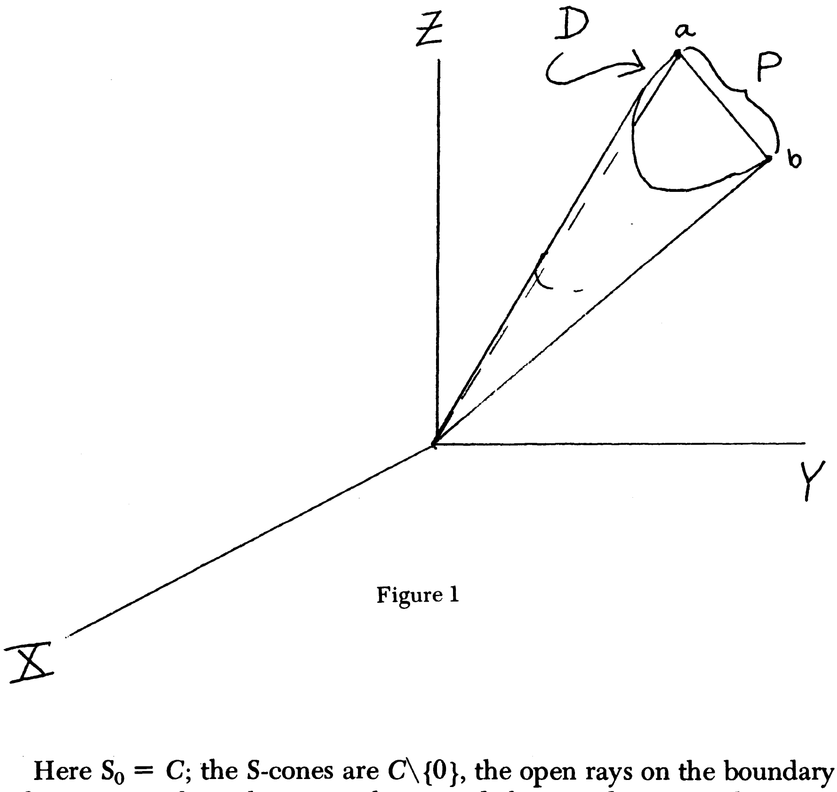

Figure 1

Here $S_0 = C$; the S-cones are $C\backslash\{0\}$, the open rays on the boundary of C passing through semicircle D, and the two-dimensional cone P on the boundary. The semilattice $\mathcal{E}_0$ is described by Figure 2.

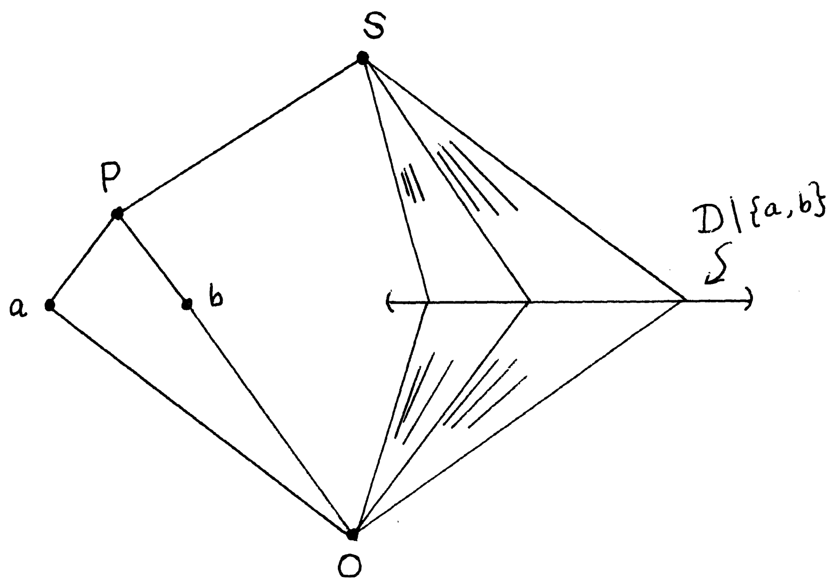

Figure 2

The Archimedean components of C are $\{0\}$, the open rays passing through D, the interior of the two-dimensional cone P and C^0.

5. Let G denote a fixed p-divisible dense subgroup of R^n, C a closed proper cone in R^n, and $S_0 = G \cap C$. In this section, we exploit the results of the previous sections to obtain a concrete realization of the universal compactification of $(S_0)_d$.

Let $S = S_0 \setminus \{0\}$ and let $(\mathcal{E}, \vee)$ be the semilattice of S-cones. Setting $T(0) = \{0\}$, Theorem 2 asserts that S_0 is a semilattice $\mathcal{E}_0$ of semigroups $T(E)$ where, for $E \in \mathcal{E}$, $T(E)$ is a p-divisible K-semigroup.

We shall need a representation for the universal compactification of $(\mathcal{E}_0)_d$; let $(\Omega, \vee)$ denote an arbitrary (upper) semilattice satisfying the finite chain condition. Denote by $\mathcal{P}$ the collection of prime ideals of Ω (i.e., $P \in \mathcal{P}$ only in case P is a semilattice ideal and $\Omega \setminus P$ is a subsemilattice of Ω or $P = \Omega$). For $e \in \Omega$ let $e\!\downarrow\, = \{f \in \Omega : e \vee f = e\}$ and we note that $\Omega \setminus e\!\downarrow\, \in \mathcal{P}$. For $e \in \Omega$ and $P_1, P_2, \cdots, P_n \in \mathcal{P}$ with $e \in \bigcap_{i=1}^n P_i$ let $V(e, P_1, \cdots, P_n) = e\!\downarrow \cap \left(\bigcap_{i=1}^n P_i\right)$ and denote by $\mathcal{A}$ the collection of all possible subsets of Ω of the form $V(e, P_1, \cdots, P_n)$. It is easy to see that $\mathcal{A}$ is a basis for an Hausdorff topology $\mathcal{T}$ on Ω.

Proposition 5.1. *The universal compactification of* $(\Omega_d, \vee)$ *is* $((\Omega, \vee, \mathcal{T}), i)$ *where* $i : \Omega_d \to \Omega$ *is the inclusion. Further,* $(\Omega, \mathcal{T})$ *is* 0-*dimensional.*

Proof. Let 2 denote the upper semilattice $(\{0,1\},\vee)$ and $X = \mathrm{Hom}(\Omega_d, 2)$ the collection of semilattice homomorphisms of Ω_d into 2. If $A \subseteq \Omega$, we denote by 1_A the characteristic function of A. We have that $\chi \in \mathrm{Hom}(\Omega_d, 2)$ only in case $\chi = 1_P$ for some $P \in \mathcal{P}$. The evaluation map $\theta : \Omega \to 2^X$ is an isomorphism of Ω into 2^X. Hence, $\theta(\Omega)$ is a subsemilattice of 2^X satisfying the finite chain condition. By a theorem due to J. W. Stepp ([17], Lemma 8), $\theta(\Omega)$ is a closed and, hence, compact subset of 2^X in the product topology. It follows that $(\Omega, \vee)$ is a compact 0-dimensional topological semilattice with respect to the weak topology $\mathcal{T}'$ generated by the functions in $X = \mathrm{Hom}(\Omega_d, 2)$. The identity function $j : (\Omega, \mathcal{T}') \to (\Omega, \mathcal{T})$ is easily seen to be continuous and is thus a homeomorphism and an isomorphism. Hence, $(\Omega, \vee, \mathcal{T})$ is a compact 0-dimensional semilattice. If $(U(\Omega_d), u)$ is the universal compactification of $(\Omega_d, \vee)$ then since u is an isomorphism (in this case) and Ω satisfies the finite chain condition we get $u : \Omega_d \to U(\Omega_d)$ is an isomorphism onto. The identity function $i : \Omega_d \to (\Omega, \vee, \mathcal{T})$ induces $i^* : U(\Omega_d) \to (\Omega, \vee, \mathcal{T})$ and by our previous observations, i^* is an iseomorphism onto. It now follows that $((\Omega, \vee, \mathcal{T}), i)$ is the universal compactification of Ω_d.

For each $E \in \mathcal{E}$, $T(E)$ is a p-divisible K-semigroup. Let $G_E = T(E) - T(E)$, the subgroup of R^n generated by $T(E)$. Obviously $G_E \subseteq E - E$; choose arbitrarily $t \in T(E)$, then since $T(E)$ is an ideal in E, $E - E = (E + t) - (E + t) \subseteq T(E) - T(E) = G_E$ so that $G_E = E - E$. Denote by (B_E, b_E) the Bohr compactification of $(G_E)_d$ and by (A_E, a_E) the universal compactification of the closed proper cone $C_E = E^*$ with the relative Euclidean topology. Let $i_E : T(E) \to G_E$ and $j_E : T(E) \to C_E$ denote the inclusions and $\beta_E = (b_E \circ i_E) \times (a_E \circ j_E)$; by Theorem 1(i), $(\mathcal{B}_E, \beta_E)$ is the universal compactification of $T(E)_d$ where $\mathcal{B}_E$ is the closure of $\beta_E(T(E))$ in $B_E \times A_E$. In $\mathcal{E} \times (\bigcup\{B_E \times A_E : E \in \mathcal{E}\})$ let D_E denote the subset consisting of those pairs (E, z) where $z \in B_E \times A_E$ and let $D = \dot{\bigcup}\{D_E : E \in \mathcal{E}\}$. We shall use a technique developed by A. H. Clifford [5] to introduce an associative operation on D. We will then use a construction principle to introduce a topology on D which was first formulated by Hofmann and Mostert in their work on hormoi (c.f. [12], p. 140, 5.3). The combination of the two techniques was used previously by T. T. Bowman ([1], Theorem 1.3) to determine the structure of compact semigroups S in which Green's $\mathcal{H}$-relation is a congruence and $S/\mathcal{H}$ is a Lawson semilattice, which is the topological version of Clifford's work previously cited.

If $E, F \in \mathcal{E}$ and $E \vee F = F$ then $G_E \subseteq G_F$ and $C_E \subseteq C_F$. Denote by $[E, F] : B_E \times A_E \to B_F \times A_F$ the continuous homomorphism induced by the inclusion $G_E \times C_E \to G_F \times C_F$ (recall that $B_E \times A_E$ is the universal compactification of $G_E \times C_E$). In particular, for $x \in G_E$, $y \in C_F$, $[E, F](b_E(x), a_E(y)) = (b_F(x), a_F(y))$. If $E_1, E_2, E_3 \in \mathcal{E}$ with $E_1 \subseteq E_2 \subseteq E_3$ then $[E_2, E_3] \circ [E_1, E_2] = [E_1, E_3]$. For $\tilde{x} = (E, x) \in D_E$ and $\tilde{y} = (F, y) \in D_F$ define:

$$(*) \qquad \tilde{x} + \tilde{y} = (E \vee F, [E, E \vee F](x) + [F, E \vee F](y)).$$

This is essentially Clifford's technique in [5], and $(\tilde{x}, \tilde{y}) \to \tilde{x} + \tilde{y}$ is easily seen to be an associative commutative binary operation on D.

In order to introduce a topology in D (via the technique of Hofmann-and Mostert) we let $F \in \mathcal{E}$ and $V = V(F, P_1, \cdots, P_n) \in \mathcal{A}$ as defined preceding Proposition 5.1. Choose an open set U in the product topology of $B_F \times A_F$ and define:

$$(**) \qquad W(U, V) = \{(E, x) \in D : E \in V \text{ and } [E, F](x) \in U\}.$$

The collection of all subsets of D of the form $(**)$ constitutes a basis for a topology on D, which we refer to as the $W(U, V)$-topology. The net $(\tilde{E}, \tilde{x})$ in D converges to (E, x) in D with respect to the $W(U, V)$-topology only in case $\lim \tilde{E} = E$ in the $\mathcal{T}$ topology of Proposition 5.1, and $\lim [\tilde{E}, E](\tilde{x}) = x$ in the product topology on $B_E \times A_E$.

PROPOSITION 5.2. *The set D together with the binary operation $(*)$ and the $W(U, V)$-topology defined by $(**)$ is a compact commutative topological semigroup. The function $x \to (E, x) : B_E \times A_E \to D_E$ is an iseomorphism of $B_E \times A_E$ onto the compact subsemigroup D_E of D.*

PROOF. The proof is almost identical with that given by Bowman in proving Theorem 1.3 [1], and, in any case is straightforward, so we omit it.

Let D_0 denote the compact semigroup obtained by adjoining an identity 0 to D as an isolated point. Define $\beta : S_0 \to D_0$ by the rule: $\beta(0) = 0$ and $\beta(x) = (E, \beta_E(x))$ for $x \in T(E)$. For $E \in \mathcal{E}$, let $\tilde{\mathcal{B}}_E = \{(E, z) : z \in \mathcal{B}_E\}$ and let $\mathcal{B} = (\dot{\bigcup}\{\tilde{\mathcal{B}}_E : E \in \mathcal{E}\}) \dot{\cup} \{0\}$; hence $\beta(S_0) \subseteq \mathcal{B}$.

THEOREM 3. *Let G be a dense p-divisible subsemigroup of R^n and C a closed proper cone in R^n with $R^n = C - C$. The universal compactification of $(S_0)_d = (G \cap C)_d$ is the pair $(\mathcal{B}, \beta)$, where $\mathcal{B}$ has the operation defined by $(*)$, the relative $W(U, V)$-topology defined by $(**)$, and $\mathcal{B}\backslash\{0\}$ is the disjoint union of universal compactifications of p-divisible K-semigroups.*

Proof. Let $x, y \in S$ with $x \in T(E)$ and $y \in T(F)$; since $x + y \in T(E \vee F)$, $\beta(x + y) = (E \vee F, \beta_{E\vee F}(x + y)) = (E \vee F, (b_{E\vee F}(x + y), a_{E\vee F}(x + y))) = (E \vee F, (b_{E\vee F}(x), a_{E\vee F}(x)) + (b_{E\vee F}(y), a_{E\vee F}(y))) = (E \vee F, [E, E \vee F](b_E(x), a_E(x))) + [F, E \vee F](b_F(y), a_F(y))) = (E, (b_E(x), a_E(x))) + (F, (b_F(y), a_F(y))) = \beta(x) + \beta(y)$. Thus β is a homomorphism (in fact, an isomorphism into). Since $\beta_E(T(E))$ is dense in $\mathcal{B}_E$, it follows that $\tilde{\mathcal{B}}_E \subseteq \beta(S_0)^*$ and thus $\mathcal{B} \subseteq \beta(S_0)^*$. Now let $x \in D_E$ and suppose $x \in \beta(S_0)^*$; there is a net $\tilde{z}$ in S_0 such that $\lim \beta(\tilde{z}) = x$ in the $W(U, V)$ topology. Since $\mathcal{E}_0$ satisfies the finite chain condition, there is an $E_0 \in \mathcal{E}_0$ such that $E_0 \vee E = E$, $E_0 \neq E$ and if $F \in \mathcal{E}$ with $E_0 \vee F = F$, $F \vee E = E$ and $E_0 \neq F$ then $F = E$. There is a net $\tilde{F}$ in $\mathcal{E}$ such that $\tilde{z} \in T(\tilde{F})$ and since $\lim \beta(\tilde{z}) = x$ it follows that $\lim \tilde{F} = E$ in the $\mathcal{T}$ topology. Eventually, $\tilde{F} \in V(E, \mathcal{E}\backslash E_0\!\downarrow)$ so that $E_0 \vee \tilde{F} \neq E_0$ and, $E_0 \vee (E_0 \vee F) = E_0 \vee F$, and $(E_0 \vee \tilde{F}) \vee E = E_0 \vee (\tilde{F} \vee E) = E_0 \vee E = E$. Hence by the choice of E_0 we may assume $E_0 \vee \tilde{F} = E$. Now if $E_0 = \{0\}$ then $\tilde{F} = E$ and it follows that $x \in \mathcal{B}_E$. Hence, we assume $E_0 \in \mathcal{E}$; choose arbitrarily $t \in T(E_0)$. Then $\tilde{z} \in T(\tilde{F})$ implies $\tilde{z} + t \in T(\tilde{F} \vee E_0) = T(E)$ and $\beta(\tilde{z}) + \beta(t) = \beta(\tilde{z} + t) = (E, \beta_E(\tilde{z} + t)) \in \tilde{\mathcal{B}}_E$; by continuity of addition in D and the fact that $\tilde{\mathcal{B}}_E$ is closed in D we get $x + \beta(t) \in \tilde{\mathcal{B}}_E$ for all $t \in T(E_0)$. Since $\beta(T(E_0))$ is dense in $\tilde{\mathcal{B}}_{E_0}$ we get $x + y \in \tilde{\mathcal{B}}_E$ for all $y \in \tilde{\mathcal{B}}_{E_0}$. In particular, by the remark following Theorem 1, $(E_0, (b_{E_0}(0), a_{E_0}(0))) \in \tilde{\mathcal{B}}_{E_0}$ and thus $x + (E_0, (b_{E_0}(0), a_{E_0}(0))) \in \tilde{\mathcal{B}}_E$. Let $x = (E, (g, k))$ where $g \in B_E$ and $k \in A_E$; then $x + (E_0, (b_{E_0}(0), a_{E_0}(0)) = (E, (g, k) + [E_0, E](b_{E_0}(0), a_{E_0}(0))) = (E, (g, k) + (b_E(0), a_E(0))) = (E, (g, k)) = x$. Hence $x \in \tilde{\mathcal{B}}_E$ and we have established that $\mathcal{B} = \beta(S_0)^*$; this also establishes the fact that $\mathcal{B}$ is a compact subsemigroup of D_0, since β is a homomorphism.

Now suppose $f : (S_0)_d \to W$ is a homomorphism of S_0 into a dense subsemigroup of the compact semigroup W. Clearly, W has an identity 1 and $f(0) = 1$. For $E \in \mathcal{E}$, let $f_E : T(E) \to W$ denote the restriction of f to $T(E)$. There is a continuous homomorphism $f_E{}^* : \mathcal{B}_E \to W$ for which $f_E{}^* \circ \beta_E = f_E$. We now define $f^* : \mathcal{B} \to W$ by the rule: $f^*(0) = 1$ and $f^*(E, x) = f_E{}^*(x)$ where $(E, x) \in \tilde{\mathcal{B}}_E$. This clearly defines a function on $\mathcal{B}$ to W. Further for $x \in T(E)$, $(f^* \circ \beta)(x) = f^*(E, \beta_E(x)) = f_E{}^*(\beta_E(x)) = f_E(x)$ and consequently, $f^* \circ \beta = f$. Suppose $\beta(\tilde{t})$ is a net in $\beta(S_0)$ converging to $(E, x) \in \tilde{\mathcal{B}}_E$ where $\tilde{t} \in T(\tilde{E})$. Then $\lim \tilde{E} = E$ and $\lim [\tilde{E}, E](\beta_E(\tilde{t})) = x$; note that $[\tilde{E}, E](\beta_E(\tilde{t})) = [\tilde{E}, E](b_E(\tilde{t}), a_E(\tilde{t})) = (b_E(\tilde{t}), a_E(\tilde{t}))$. Now $f^*(E, x) = f_E{}^*(x) = f_E{}^*\ (\lim(b_E(\tilde{t}), a_E(\tilde{t}))) = \lim f_E{}^*(b_E(\tilde{t}), e_E(\tilde{t})) = \lim f_E{}^*(\beta_E(\tilde{t})) = \lim f_E(\tilde{t}) = \lim f(\tilde{t}) = \mathit{lim}\, f_{\tilde{E}}(\tilde{t}) = \lim f_{\tilde{E}}{}^*(\beta_{\tilde{E}}(\tilde{t})) = \lim f^*(\tilde{E}, \beta_{\tilde{E}}(\tilde{t})) = \lim f^*(\beta(\tilde{t}))$. We have shown that f^* is a function on $\mathcal{B}$ to W and there is a dense set (namely $\beta(S_0)$) such that if $\beta(\tilde{t})$ is a net in $\beta(S_0)$ converging to $(E, x) \in \tilde{\mathcal{B}}_E$ then $f^*(\beta(\tilde{t}))$ converges to $f^*(E, x)$. By ([15], Lemma 1), f^* is continuous. Since $f^* \circ \beta = f$ it follows that f^* is a homomorphism on $\beta(S_0)$ and thus, by continuity, on $\beta(S_0)^* = \mathcal{B}$. We may now conclude that $(\mathcal{B}, \beta)$ is the universal compactification of $(S_0)_d$. Finally, the function $x \to (E, x)$: $\mathcal{B}_E \to \tilde{\mathcal{B}}_E$ is clearly an iseomorphism onto.

Corollary. *Let C be a closed proper cone in R^n with $R^n = C - C$, and let $(\mathcal{B}, \beta)$ be the universal compactification of C_d as described above. The connected components of $\mathcal{B}$ are the subsemigroups $\tilde{\mathcal{B}}_E$, $E \in \mathcal{E}$, and $\{0\}$. Each $\tilde{\mathcal{B}}_E$ is iseomorphic to $\Sigma_a{}^m \times A_E$ where $m = 2^C$ and A_E is the universal compactification of the closed proper cone E^*.*

Proof. Each $\tilde{\mathcal{B}}_E$ is iseomorphic to $\mathcal{B}_E$; recall that E is a $(C\backslash\{0\})$-cone and is therefore closed in $C\backslash\{0\}$. Hence $E^* = E \cup \{0\}$ is a closed proper cone; since $T(E)$ is the part of $C\backslash\{0\}$ in the interior of E^* relative to $E^* - E^*$, it follows that $T(E)$ is equal to the interior of the cone E^*. By example 6, the universal compactification of $T(E)$ is $\Sigma_a{}^m \times A_E$ where $m = 2^C$; hence $\tilde{\mathcal{B}}_E$ is iseomorphic to $\Sigma_a{}^m \times A_E$ and is connected. The function $(E, x) \to E : \mathcal{B} \to \mathcal{E}_0$ is continuous; since $\mathcal{E}_0$ is 0-dimensional by Proposition 5.1, it follows that $\tilde{\mathcal{B}}_E$ is a maximal connected set and we are done.

References

1. T. T. Bowman, *A construction principle and compact Clifford semigroups*, Semigroup Forum, **2** (1971), 343–353.

2. D. R. Brown and M. Friedberg, *Representation Theorems for uniquely divisible semigroups*, Duke Math. J. **35** (1968), 341–352.

3. ——, *A survey of compact divisible commutative semigroups*, Semigroup Forum, **1** (1970), 143–161.

4. D. R. Brown and J. G. La Torre, *A characterization of uniquely divisible commutative semigroups*, Pac. J. Math. **18**, No. 1 (1966), 57–60.

5. A. H. Clifford, *Semigroups admitting relative inverses*, Ann. of Math. **42** (1941), 1037–1049.

6. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Vol. I, 1961.

7. M. Friedberg and J. W. Stepp, *A note on the Bohr compactification*, Semigroup Forum **6** (1973), 362–364.

8. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis* I, Springer-Verlag, 1963, Berlin.

9. J. A. Hildebrant, *The universal compact subunithetic semigroup*, Proc. Amer. Math. Soc. **23**, No. 2, (1969), 220–224.

10. ——, *Compact p-adic semigroups*, Duke Math. J. **39**, No. 1, (1972), 39–44.

11. K. H. Hofmann, *Topologische Halbgruppen mit dichter submonogene Unterhalbgruppe*, Math. Z. **74** (1960), 232–276.

12. K. H. Hofmann and P. S. Mostert, *Elements of Compact Semigroups*, Charles E. Merrill, Columbus, Ohio, 1966.

13. Per Holm, *On the Bohr compactification*, Math. Annalen **156** (1964), 34–46.

14. Klaus Keimel, *Eine Exponential funktion für kompakte abelsche Halbgruppen*, Math. Z. **96** (1967), 7–25.

15. J. D. Lawson and J. A. Hildebrant, *Embedding in compact uniquely divisible semigroups*, Semigroup Forum 4, No. 4, (1972), 295–300.

16. P. S. Mostert and A. L. Shields, *One parameter semigroups in a semigroup*, Trans. Amer. Math. Soc. **96** (1960), 510–517.

17. J. W. Stepp, *Algebraic maximal semilattices*, Pac. J. Math. **58**, No. 1, (1975), 243–248.

18. J. L. Taylor, *Measure Algebras*, Conference Board of the Math. Sciences, AMS, No. 16.

UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77004