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NORM-DECREASING ISOMORPHISMS OF THE TRACE-CLASS ALGEBRAS OF H^* ALGEBRAS

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ABSTRACT. Let A_1 and A_2 be H* algebras and let $\tau(A_1)$, $\tau(A_2)$ be their trace classes. We show that an algebra isomorphism T of $\tau(A_1)$ onto $\tau(A_2)$ preserves the trace if any of the following conditions is satisfied:

(i) $A_1 = A_2 = A$, a simple H* algebra

(ii) T is an isometry on the minimal idempotents of A_1

(iii) T is norm-decreasing and $A_1 = A_2 = A$ is the direct sum of a finite number of simple H^* algebras. We also show that if T does preserve the trace, and it is norm-decreasing; then, the induced isomorphism T^m of the multiplier algebra $(\tau(A_1))^m$ onto $(\tau(A_2))^m$ is an isometry.

1. **Introduction.** Wendel in [10] and [11], Rigelhof in [6] and Wood in [12] and [13] have all shown that norm-decreasing isomorphism of some group algebra onto another of the same kind implies an isometry. We shall attempt to show, in this paper, that a norm-decreasing algebra isomorphism T of the trace-class algebra $\tau(A_1)$ onto another $\tau(A_2)$ which preserves the trace, induces an isometric algebra isomorphism T^m of the multiplier algebra $(\tau(A_1))^m$ onto $(\tau(A_2))^m$. The major contribution in this paper, is our investigation of when an algebra isomorphism T preserves the trace and lemma 3.1 is the basis of this investigation. The theory of the trace-class algebra itself was developed in [7] and [8]. [7] was a generalisation of Schatten's work on the trace-class algebra of operators on a Hilbert space in [9].

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2. **Preliminaries.** The trace-class for A, (denoted by $\tau(A)$) is defined to be the set $\{xy : x, y \in A\}$ (see [7]). It is dense in A by lemma 2.7 of [1]. A projection in A is a non-zero member e of A such that $e^2 = e = e^* \neq 0$ (e is a non-zero self-adjoint idempotent). We refer to a mutually orthogonal maximal family $(e_\alpha)_{\alpha \in \Gamma}$ as a projection base.

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When each e_α is irreducible (in the sense of 27B of [4]) (e_α) is called an irreducible projection base.

Let (e_α) be a projection base. The trace, tr, on $\tau(A)$ is defined by $\mathrm{tr}(a) = \sum_\alpha (ae_\alpha, e_\alpha)\ \forall_{a\in\tau}(A)$. In fact,

$$\text{(1)} \qquad \mathrm{tr}(a) = \mathrm{tr}(xy) = (y, x^*) = (x, y^*) = \mathrm{tr}(yx)$$

where $a = xy \quad x, y \in A$.

In particular, $\mathrm{tr}(e_\beta) = \sum_\alpha (e_\beta e_\alpha, e_\alpha) = (e_\beta, e_\beta) = \|e_\beta\|^2\ \forall \beta \in \Gamma$ (see p. 97–98 of [7]).

From 27E of [4], if (e_α) is irreducible in a simple H^* algebra A, there exists an orthogonal basis $(e_{\alpha\beta})_{\alpha,\beta\in\Gamma}$ satisfying the following properties:

$$e_{\alpha\alpha} = e_\alpha; \qquad (e_{\alpha\beta}, e_{\alpha\beta}) = (e_\alpha, e_\alpha)\,\forall_{\alpha\in\Gamma}$$

$$(e_{\alpha\beta}, e_{ks}) = e \qquad (\text{unless } \alpha = k, \beta = s).$$

$$e_{\alpha\beta}e_{ks} = \begin{cases} e_{\alpha s} & \text{when } \beta = k \\ 0 & \text{when } \beta \neq k \end{cases}$$

$$\text{and } e_{\alpha\beta} = e_{\beta\alpha}{}^*.$$

Also, every $y \in A$ can be written as

$$\text{(2)} \qquad y \sim \sum_{\alpha,\beta} C_{\alpha\beta}\, e_{\alpha\beta}.$$

By the corollary to theorem 1 of [8], $\tau(A)$ is a Banach* algebra with respect to a $\tau(\,\cdot\,)$ norm. For each $a \in A$, there exists a sequence (λ_n) of positive numbers and (e_n) as above such that $a^*a = \sum \lambda_n e_n$ (see corollary 1 of [7]). $[a]$ is defined by

$$[a] = \sum \mu_n e_n \text{ where } \mu_n = \lambda_n{}^{1/2} \geqq 0.$$

For each $a \in A$, there exists a unique $[a]$ in A such that $[a]^2 = a^*a$ (see lemma 2 of [7]). The $\tau(\,\cdot\,)$ norm with respect to which $\tau(A)$ is complete is defined by

$$\tau(a) = \mathrm{tr}[a] = \sum_\alpha ([a]\,e_\alpha, e_\alpha).$$

It is easy to see that

$$\text{(3)} \qquad \begin{aligned} &\tau(a^*a) = \mathrm{tr}(a^*a) = \|a\|^2\ \forall a \in A \text{ and in particular} \\ &\tau(e_\alpha) = \mathrm{tr}(e_\alpha) = \|e_\alpha\|^2. \end{aligned}$$

A^m is the Banach algebra of all multipliers of A and g is a left multiplier if it is a bounded linear operator of A such that

$$g(xy) = (gx)y, \qquad x, y \in A$$

(see p. 14 of [3]).

3. **Preservation of the trace for a simple A.** Throughout this section, A is a simple H^* algebra.

LEMMA 3.1. *Every minimal idempotent in A has trace equal $\tau(e_k) = \|e_k\|^2$ for any $k \in \Gamma$, where $(e_\alpha)_{\alpha\in\Gamma}$ is an irreducible projection base.*

PROOF. Let f be a minimal idempotent in A. Then $\sum_{\alpha,\beta} (\sum_p C_{\alpha p} C_{p\beta})e_{\alpha\beta} = (\sum_{\alpha,p} C_{\alpha p}e_{\alpha p}) (\sum_{q,\beta} C_{q\beta}e_{q\beta}) = f^2 = f = \sum_{\alpha,\beta} C_{\alpha\beta}e_{\alpha\beta}$.
Since $e_{\alpha\beta}$ are mutually orthogonal, we have

$$\sum_p C_{\alpha p}C_{p\beta} = C_{\alpha\beta} \qquad \forall p, \alpha, \beta \in \Gamma \tag{4}$$

Also,

$$\begin{aligned}\sum_{\alpha,\beta} (C_{\alpha r}C_{r\beta})e_{\alpha\beta} &= \Big(\sum_{\alpha,p} C_{\alpha p}e_{\alpha p}\Big)\, e_{rr}\Big(\sum_{q,\beta} C_{q\beta}e_{q\beta}\Big)\\ &= fe_{rr}f = \lambda f\\ &= \lambda \sum_{\alpha,\beta} C_{\alpha\beta}e_{\alpha\beta}, \lambda \text{ a complex number.}\end{aligned}$$

Hence $C_{\alpha r}C_{r\beta} = \lambda\, C_{\alpha\beta}$ $\forall\, \alpha, \beta \in \Gamma$. When $\beta = r$, we have $C_{\alpha r}C_{rr} = \lambda\, C_{\alpha r}$. Therefore $\lambda = C_{rr}$ if $C_{\alpha r} \neq 0$. Hence $C_{\alpha r}C_{r\beta} = C_{rr}C_{\alpha\beta}$, and when $\alpha = \beta$,

$$C_{\alpha r}C_{r\alpha} = C_{rr}C_{\alpha\alpha}. \tag{5}$$

(4) and (5) now give $\sum_{\alpha,r} C_{\alpha r}C_{r\alpha} = \sum C_{\alpha\alpha} = (\sum C_{\alpha\alpha})^2$. Hence

$$\sum C_{\alpha\alpha} = 1. \tag{6}$$

Therefore

$$\begin{aligned}\mathrm{tr}(f) &= \sum_\alpha (fe_\alpha, e_\alpha)\\ &= \sum_\alpha \Big(\Big[\sum_{r,s} C_{rs}e_{rs}\Big]\, e_\alpha, e_\alpha\Big)\end{aligned}$$

$$= \|e_k\|^2 \sum C_{\alpha\alpha}$$

$$= \tau(e_k) \sum C_{\alpha\alpha} = \tau(e_k) \text{ by (6)}$$

and the proof is complete.

LEMMA 3.2. *Let A be a simple H^* algebra and $(f_\gamma)_{\gamma \in \Gamma'}$ be a maximal family of mutually orthogonal minimal idempotents. Then the trace on $\tau(A)$ is characterised as the unique linear functional L that satisfies*

(i) $L(xy) = L(yx), \quad x, y \in A$ *and*

(ii) $L(f_\gamma) = \tau(e_k), \quad (\gamma \in \Gamma^1)$.

PROOF. Suppose $L(x) = \mathrm{tr}(x)$. Then $L(x)$ satisfies (i) by the trace property and (ii) by lemma 3.1. We shall now show that L and tr take the same value at an arbitrary point $x \in \tau(A)$ if L satisfies (i) and (ii). Af_γ is a minimal closed left ideal in A and, in fact, $\tau(A)f_\gamma = Af_\gamma$. $x \in \tau(A)f_\gamma$ implies $x = xf_\gamma = xf_\gamma f_\gamma$. Therefore, using (i) and (ii), $L(x) = L(xf_\gamma f_\gamma) = L(f_\gamma x f_\gamma) = \lambda\, L(f_\gamma) = \lambda\tau(e_k)$, and using lemma 3.1, $\mathrm{tr}(x) = \mathrm{tr}(xf_\gamma f_\gamma) = \mathrm{tr}(f_\gamma x f_\gamma) = \lambda\ \mathrm{tr}(f_\gamma) = \lambda\ \mathrm{tr}(e_k)$. Since $\sum \tau(A)f_\gamma = \tau(A)$, L and tr take the same value at an arbitrary point $x \in \tau(A)$ and the proof is complete.

REMARK 3.3. 3.2 holds if $(f_\gamma)_{\gamma \in \Gamma'}$ is replaced by $(e_\alpha)_{\alpha \in \Gamma}$ and $\tau(e_k)$ by $\tau(e_\alpha)$ (which is not a constant for all α). e_α can however be expressed as a finite sum of elements of an irreducible projection base (e_{α_n}) and $\tau(e_\alpha) = \tau(e_{\alpha_1}) + \cdots + \tau(e_{\alpha_n})$.

THEOREM 3.4. *Let A be a simple H^* algebra. An algebra automorphism T of $\tau(A)$ preserves the trace.*

PROOF. Since T is algebraic, it maps a minimal idempotent, f, to a minimal one Tf. Hence $\mathrm{tr}(f) = \tau(e_k) = \mathrm{tr}(Tf)$ by 3.1. Define $L(x) = \mathrm{tr}(Tx)$; then $L(xy) = L(yx)$ and $L(f) = \mathrm{tr}(Tf) = \tau(e_k)$. Using 3.2, we have $L(x) = \mathrm{tr}(x) = \mathrm{tr}(Tx)$ and the proof is complete.

REMARK 3.5. An algebra isomorphism T of the group algebra $L_2(G)$ onto another $L_2(G^1)$ for compact groups G, G^1 preserves the trace because the minimal two-sided ideals in $L_2(G)$ are finite dimensional simple H^* algebras, T preserves dimension, and $\tau(e) = n$ (the dimension of a minimal ideal). We shall now indicate that for arbitrary simple H^* algebras A_1 and A_2, an isomorphism T of $\tau(A_1)$ onto $\tau(A_2)$ does not preserve the trace.

By redefining the inner product in a simple H^* algebra, we shall indicate that the norm of a minimal idempotent can either be increased from 1 to a $c > 1$ or decreased from $c > 1$ to 1. The only nontrivial part is to show that the norm defined by the new inner-product, in each case, satisfies the multiplicative property.

Let A_1 be a simple H^* algebra and f, a minimal idempotent in A_1 such that $\|f\| = 1$. We define a new inner product $[\,,]$ in A_1 by

$$[x, y] = (1/\alpha^2 x, y), \text{ where } e < \alpha \leqq 1.$$

Let A_2 be the algebra formed with this inner-product. Then $\|f\|_2 > 1$, $\mathrm{tr}_2(f) \geqq \mathrm{tr}_1(f)$ and

$$\begin{aligned}\|xy\|_2^2 &\leqq 1/\alpha^2\, \|xy\|_1^2 \leqq 1/\alpha^2\, \|x\|_1^2\, \|y\|_1^2 \\ &\leqq (1/\alpha^2\, \|x\|_1^2)\,(1/\alpha^2\, \|y\|_1^2) \\ &= \|x\|_2^2\, \|y\|_2^2.\end{aligned}$$

Conversely, let A_1 and f be as above but such that $\|f\|_1 = \beta > 1$. Suppose a new inner-product $[\,,]$ is defined by $[x, y] = ((1/\beta^2)\, x, y)$. Then $\|f\|_2 = (1/\beta)\ \|f\|_1 = 1$ and $\mathrm{tr}_2(f) < \mathrm{tr}_2(f)$. To show that $\|xy\|_2 \leqq \|x\|_2\, \|y\|_2$, it suffices to prove that

$$\|xy\|_1 \leqq (1/\beta)\, \|x\|_1\, \|y\|_1 \tag{7}$$

For then,

$$\begin{aligned}\|xy\|_2^2 &= 1/\beta^2 \|xy\|_1^2 \leqq 1/\beta^2 \|x\|_1^2 1/\beta^2 \|y\|_1^2 \\ &= \|x\|_2^2\, \|y\|_2^2.\end{aligned}$$

Let $x, y \in A_1$. Then

$$\begin{aligned}\|xy\|_1^2 &= \left\| \left(\sum_{\alpha,\beta} \lambda_{\alpha\beta} e_{\alpha\beta} \right) \left(\sum_{i,j} c_{ij} e_{ij} \right) \right\|_1^2 \\ &= \left\| \sum_{\alpha,\beta,j} \lambda_{\alpha\beta}\, c_{\beta j}\, e_{\alpha j} \right\|_1^2 \\ &= \sum_{\alpha,j} \left| \sum_{\beta} \lambda_{\alpha\beta}\, c_{\beta j} \right|^2 \|e_k\|_1^2 \\ &\leqq \frac{1}{\|e_k\|^2} \left(\sum_{\alpha,\beta} |\lambda_{\alpha\beta}|^2\, \|e_k\|_1^2 \right) \left(\sum_{\beta,j} |c_{\beta j}|^2 \|e_k\|_1^2 \right) \\ &= \frac{1}{\beta^2} \|x\|_1^2\, \|y\|_1^2.\end{aligned}$$

(note: $\|e_k\|_1 = \beta$ since e_k irreducible implies e_k minimal by lemmas 27B and 27D of [4]). (7) now follows. Therefore, given two simple H^* algebras A_1 and A_2, an isomorphism T of $\tau(A_1)$ onto $\tau(A_2)$ does not in general preserve the trace. But the following result holds.

THEOREM 3.7. *Let A_1 $(i = 1, 2)$ be a simple H^* algebra. An algebra isomorphism T of $\tau(A_1)$ onto $\tau(A_2)$ is such that* $\operatorname{tr}(Tx) = k \operatorname{tr}(x)$ $\forall x \in \tau(A_1)$ *where k is a constant greater than zero.*

PROOF. Let f be a minimal idempotent in $\tau(A_1)$. Tf is also minimal in $\tau(A_2)$. Suppose $\operatorname{tr}(f) = d_1$ and $\operatorname{tr}(Tf) = d_2$. Define $L(x) = \operatorname{tr}(Tx)/k$ where $k = d_2/d_1$. Then $L(xy) = L(yx)$ and $L(f) = \operatorname{tr}((Tf)/k) = d_1 = \operatorname{tr}(f)$. Using 3.2, we have $L(x) = \operatorname{tr}(x) = \operatorname{tr}(Tr)/k)$ $\forall x \in \tau(A_1)$, i.e., $\operatorname{tr}(Tx) = k \operatorname{tr}(x)$.

REMARK 3.8. In 3.7, the trace is preserved if $k_1 = 1$; and this is the ease when T is a $*$ isomorphism and $\|T\| = 1$:

$$
\begin{aligned}
1 = \|T\| &= \sup_{x \neq e} \frac{\tau(Tx)}{\tau(x)} \\
&= \sup_{x \neq e} \frac{\operatorname{tr}[Tx]}{\operatorname{tr}[x]} \\
&= \sup_{x \neq e} \frac{\operatorname{tr}(Tx * Tx)^{1/2}}{\operatorname{tr}(x * x)^{1/2}} \\
&= \sup_{x \neq e} \frac{\sum \mu_n \operatorname{tr}(Te_n)}{\sum \mu_n \operatorname{tr}(e_n)} \\
&= \frac{\tau(Te_k)}{\tau(e_k)} = \frac{\operatorname{tr}(Tf)}{\operatorname{tr}(f)} = k_1.
\end{aligned}
$$

It is clear from above that if T is just a $*$ isomorphism, then $\tau(Tx) = \|T\|\tau(x)$ and $\operatorname{tr}(Tx) = \|T\|^2 \operatorname{tr}(x)$.

THEOREM 3.9. *Let A_i $(i = 1, 2)$ be a simple H^* algebra. An isometric algebra isomorphism T of $\tau(A_1)$ onto $\tau(A_2)$ preserves the trace.*

PROOF. Let e_1 be a minimal self adjoint idempotent in $\tau(A_1)$. Te_1 is minimal but not necessarily self adjoint and $\tau(Te_1) = \tau(e_1)$ by hypothesis. If e_2 is a minimal self adjoint idempotent in $\tau(A_2)$, then $\operatorname{tr}(Te_1) = \tau(e_2)$ by 3.1. Therefore

$$\begin{aligned}\tau(e_1) &= \mathrm{tr}(f) \leqq \tau(f) \text{ (by corollary 2 of [7])}\\ &= \tau(Tf) = \tau(e_2) = \mathrm{tr}(Te_1)\\ &= \tau(Te_1) = \tau(e_1).\end{aligned}$$

Hence $\mathrm{tr}(Tf) = \mathrm{tr}(e_2) = \tau(e_2) = \tau(e_1) = \mathrm{tr}(f)$. Since the trace is preserved on minimal idempotents, we conclude the proof as in 3.4.

4. **Preservation of the trace for an arbitrary proper semi simple H^* algebra A.** We shall assume that A is proper in the sense of [1].

THEOREM 4.1. *Let $(e_\alpha)_{\alpha\in\Gamma}$ be an irreducible projection base for A. Then the trace on $\tau(A)$ is characterised as the unique linear functional L on $\tau(A)$ that satisfies*
(i) $L(xy) = L(yx)$, $x, y \in A$ *and*
(ii) $L(e_\alpha) = \tau(e_\alpha)$, $\alpha \in \Gamma$.

PROOF. As in 3.2, we note however that in this case $\tau(e_\alpha)$ is not a constant for all α.

REMARK 4.2. 4.1 is also true if (ii) is replaced by (1)

$$L(f_\alpha) = \mathrm{tr}(f_\alpha) \qquad \alpha \in \Gamma$$

or (2)

$$L(e_\alpha) = \tau(e_\alpha), \qquad \text{where}$$

$(e_\alpha)_{\alpha\in\Gamma}$ is just a projection base.

THEOREM 4.3. *An algebra isomorphism T of $\tau(A_1)$ onto $\tau(A_2)$ preserves the trace if T is an isometry on minimal idempotents.*

PROOF. Since T is algebraic, it maps minimal ideals onto minimal ideals and so preserves the trace on minimal idempotents by 3.9. Using the proof of 4.1 instead of 3.2, the result follows.

THEOREM 4.4. *Let A be an arbitrary H^* algebra which is the direct sum of a finite number of simple H^* algebras A_n. A norm-decreasing automorphism T of $\tau(A)$ preserves the trace.*

PROOF. In view of 4.3, it suffices to show that the trace is preserved on minimal idempotents. Since T is norm-decreasing and A is the direct sum of a finite number of simple H^* algebras A_n, then T either maps $\tau(A_i)$ onto itself ($i = 1, \cdots, n$) and so preserves the trace by 3.4 or T maps each minimal idempotent to a minimal idempotent of equal norm (since there are only a finite number of possible values). The trace is also preserved by 3.9.

We shall now indicate that, in general, if T is a norm-decreasing isomorphism of $\tau(A_1)$ onto $\tau(A_2)$, then there is a constant k such that $\operatorname{tr}(Tr) = k \operatorname{tr}(x)$.

THEOREM 4.5. *If T is an algebra isomorphism of $\tau(A_1)$ onto $\tau(A_2)$, then for each minimal two-sided ideal N of $\tau(A_1)$ there exists k_N $(0 < k_N \leqq \|T\|^2)$ such that* $\operatorname{tr}(Tr) = k_N \operatorname{tr}(x)$ $\forall\, x \in N$.

PROOF. Since T maps minimal ideals onto minimal ideals, the proof follows from 3.7 and 3.8.

In fact, 4.7 is the best possible as the following example will show:

EXAMPLE 4.6. Let $A = \cdots \oplus \underset{\sim}{C}_{-n} \oplus \underset{\sim}{C}_{-(n-1)} \oplus \cdots \oplus \underset{\sim}{C}_{-1} \oplus \underset{\sim}{C}_0 \oplus \underset{\sim}{C}_1 \oplus \cdots \oplus \underset{\sim}{C}_n \oplus \cdots$, where each $\underset{\sim}{C}_n(-\infty \leqq n \leqq \infty)$ is a complex number field. Let the norm of the idempotent e_n in each $\underset{\sim}{C}_n$ (which is the unit element) be defined as follows:

$$\text{For } n < o, \quad \tau(e_n) = 3 + \frac{1}{n}$$

$$\text{For } n = o \quad \tau(e_0) = 2$$

$$\text{For } n > o, \quad \tau(e_n) = 1 + \frac{1}{n}.$$

Let T be an algebraic shift automorphism on $\tau(A)$ which is norm-decreasing. Then we have $Te_n = e_{n+1}$ $\forall_n$ $(-\infty \leqq n \leqq \infty)$. Therefore, for $n <$ e,

$$\begin{aligned}\operatorname{tr}(Te_n) &= \frac{(3 + (1/(n+1)))}{(3 + (1/n))} \operatorname{tr}(e_n)\\ &= k_n \operatorname{tr}(e_n), \quad 0 < k_n \leqq 1.\end{aligned}$$

For $n > 0$,

$$\begin{aligned}\operatorname{tr}(Te_n) &= \frac{(1 + (1/(n+1)))}{(1 + (1/n))} \operatorname{tr}(e_n)\\ &= k_n \operatorname{tr}(e_n), \quad 0 < k_n \leqq 1\end{aligned}$$

and $\operatorname{tr}(Te_1) = \operatorname{tr}(e_0) = \operatorname{tr}(Te_0) = \operatorname{tr}(e_1) = 2$. Hence $\operatorname{tr}(Te_n) = k_n \operatorname{tr}(e_n)$ $0 < k_n \leqq 1, -\infty \leqq n \leqq \infty$.

REMARK 4.7. In 4.5, C_n could be replaced by any simple H^* algebra A_n and (e_n) by an irreducible projection base (e_{α_n}). If (f_{α_n}) is a maximal family of mutually orthogonal minimal idempotents in A such that $f_{\alpha_n} \in A_n$ for each n, then $Tf_{\alpha_n} = f_{\alpha_{n-1}}$. Since $e_{\alpha_n} \in A_n$ for each n and $\operatorname{tr}(f_{\alpha_n}) = \tau(e_{\alpha_n})$, we have $\operatorname{tr}(Tf_{\alpha_n}) = \operatorname{tr}(f_{\alpha_{n+1}}) = \tau(e_{\alpha_{n+1}})$ and $\operatorname{tr}(Tf_{\alpha_n}) = k_n \operatorname{tr}(f_{\alpha_n})$ as above.

5. The main theorem. It is well known that if T is an algebra isomorphism of $\tau(A_1)$ onto $\tau(A_2)$, the induced algebra isomorphism T^m of $(\tau(A_1))^m$ onto $(\tau(A_2))^m$ is given by $T^m g = TgT^{-1}$, $g \in (\tau(A_1))^m$. We shall need the following lemma first.

LEMMA 5.1. $A^m \equiv (\tau(A))^m$.

PROOF. $A^m \subset (\tau(A))^m$. If g is a multiplier on A, the restriction to $\tau(A)$ is a multiplier and maps $\tau(A)$ into $\tau(A)$ (since $g(xy) = (gx)y$). Since multipliers are continuous anyway, we have the above assertion. $(\tau(A))^m \subset A^m$. Since the norm in A satisfies $\|x\| = \sup_{\|y\| \leq 1} \tau(xy)$ (see corollary 4 of [7]), we have that if g is a multiplier on $\tau(A)$, it is continuous with respect to the A norm. For

$$\begin{aligned} \|g(xy)\| &= \sup_{\|z\| \leq 1} \tau(g(xy)z) \\ &= \sup_{\|z\| \leq 1} \tau(g(xyz))) \\ &\leq \|g\| \sup_{\|z\| \leq 1} \tau(xyz) \\ &= \|g\| \, \|xy\|. \end{aligned}$$

Thus it will extend to a multiplier on A.

THEOREM 5.2. *A norm decreasing algebra isomorphism T of $\tau(A_1)$ onto $\tau(A_2)$ which preserves the trace, induces an algebra isomorphism T^m of $(\tau(A_1))^m$ onto $(\tau(A_2))^m$ which is an isometry.*

PROOF. We only need to show that T norm-decreasing implies T^m is an isometry.

$$\begin{aligned} \|(T^m)^{-1}g\| &= \|f_{(T^m)^{-1}g}\| && \text{(by theorem 2 of [8])} \\ &= \sup_{\tau(x) \leq 1} |f_{(T^m)^{-1}g}(x)| && (x \in \tau(A_1)) \end{aligned}$$

(where $f_{(T^m)^{-1}g}$ denotes the linear functional identified with the multiplier $(T^m)^{-1}g \in (\tau(A_1))^m$)

$$\begin{aligned} &= \sup |\mathrm{tr}((T^m)^{-1}gx)| && \text{(by definition)} \\ &= \sup_{\tau(x) \leq 1} |\mathrm{tr}(T^{-1}gTx)| && \text{(since } (T^m)^{-1}g = T^{-1}gT) \\ &= \sup_{\tau(x) \leq 1} |\mathrm{tr}\, g(Tx)| && \text{(since } T \text{ preserves the trace)} \\ &\leq \|g\| \sup_{\tau(x) \leq 1} \tau(Tx) && \text{(by corollary 2 and lemma 5 of ([7])} \\ &\leq \|g\| && \text{(since } T \text{ is norm-decreasing).} \end{aligned}$$

Since $(\tau(A))^m$ is a B^* algebra (see cor. 3.3 of [2]) then by theorem 2.1.1 of [5], T^m is an isometry. The proof is complete.

REMARK 5.3. It is not clear yet whether the assumption of trace preservation can be dropped in 5.2, and no counter example is known.

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