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A NOTE ON EXCHANGEABLE SEQUENCES OF EVENTS

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ABSTRACT. Bruno de Finetti's (1931) representation for the law of an exchangeable sequence of 0's and 1's is exhibited as an invariant limit in the ergodic theorem for a transformation first defined by L. K. Arnold (1968) and studied by Hajian, Ito, and Kakutani (1972) in the context of σ-finite invariant measures. A known result on almost-sure convergence of normalized sums for such a sequence emerges as a corollary.

0. **Background.** A random sequence $\{X_k(\omega)\}_{k=1}^{\infty}$ on a probability space $(\Omega, \mathcal{F}, \mu)$ is said to be *exchangeable* if for any permutation σ of a finite set $\{k_1, \cdots, k_n\}$ of indices and for any $A_1, \cdots, A_n \in \mathcal{F}$ measurable events in Ω,

$$\mu(\{\omega \in \Omega : X_{\sigma(k_1)}(\omega) \in A_1, \cdots, X_{\sigma(k_n)}(\omega) \in A_n\}) = \mu(X_{k_1} \in A_1, \cdots, X_{k_n} \in A_n),$$

where we suppress explicit mention of ω on the right-hand side. A sequence of measurable events $\{A_k\}_{k=1}^{\infty}$ is called exchangeable whenever the sequence $\{I_{A_k}(\cdot)\}$ of its indicator functions is.

The simplest example of exchangeable events is that of an independent sequence $\{A_k\}_{k=1}^{\infty}$ with all $\mu(A_k)$ equal. This case occurs when the indicator $I_{A_k}(\omega) = \omega_k$ is the k^{th} coordinate of a point $\omega \in \{0, 1\}^{\infty} \cong \Omega$, where $\mathcal{F}$ is the product σ-algebra and μ the infinite-product probability measure on $\{0, 1\}^{\infty}$ assigning probability $\mu(A_1) = \mu(A_k)$ to $\{1\} \times \{0, 1\}^{\infty}$.

Given any sequence $\{A_k\}_{k=1}^{\infty}$ of measurable events in Ω, we can identify the measure spaces $(\Omega, \mathcal{F})$ and $\{0, 1\}^{\infty}$ via $\omega \to \{I_{A_k}(\omega)\}_{k=1}^{\infty}$.

So from now on we take $\Omega = \{0, 1\}^{\infty}$ with product σ-algebra $\mathcal{F}$, so that μ is the probability law of the random sequence $\omega = (\omega_1, \omega_2, \cdots) \in \{0, 1\}^{\infty}$. We assume that $\{X_k(\omega)\}_{k=1}^{\infty} \equiv \{I_{A_k}(\omega)\}_{k=1}^{\infty} = \{\omega_k\}_{k=1}^{\infty}$ is exchangeable, and call the measure μ exchangeable as well.

A celebrated theorem of Bruno de Finetti [2] says that the most general exchangeable measure μ on $\{0, 1\}^{\infty}$ is a mixture of infinite-product measures. There are many ways to prove this, including a particularly elementary combinatorial one due to Feller [3, p. 228]. In this paper we give a proof intended to shed immediate light on a further analogy between exchangeable and independent sequences:

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the Strong Law of Large Numbers. To be sure, the Strong Law for exchangeable sequences is also known, but its natural and non-elementary setting seems to be as a corollary to the Ergodic Theorem of Birkhoff (or Doob's Martingale Convergence Theorem). We shall exhibit de Finetti's mixture of product measures as an invariant limit in the Ergodic Theorem on $\{0, 1\}^\infty$ for a particular transformation whose construction also yields the Strong Law. The transformation S we use was first defined by L. K. Arnold [1] and studied by Hajian, Ito, and Kakutani [4] as a measure-preserving transformation induced from a more complicated transformation, of interest in the context of σ-finite invariant measures. We define S directly, with a slight modification on the set of "terminating" sequences, borrowing the notation of the latter paper.

1. Any Borel measure μ on $\{0, 1\}^\infty$ is uniquely determined by its values on cylinder sets $I_a = \{\omega \in \{0, 1\}^\infty : \omega_i = a_i,\ i = 1, \cdots, n\}$, where $a \in \{0, 1\}^n$. We call a the *n-segment* of ω in I_a. For μ to be exchangeable, $\mu(I_a)$ must depend only on the length n of a and the number $s = \sum_{i=1}^n a_i$ of its 1's, hence $\mu(\{i\} \times I_a) = \mu(I_{a \times \{i\}})$ for $i = 0, 1$.

We define the set of "terminating" sequences $C = \{\omega \in \{0, 1\}^\infty :$ there exists $r \in \mathbf{Z}^+$ with $\omega_n = \omega_r$ for $n \geqq r\}$. By the above paragraph, $\mu(C) = \mu(\{(0,0,\cdots),(1,1,\cdots)\})$. On $\{0,1\}^\infty \backslash C = \bigcup_{p \geqq 0, q \geqq 1} B_{p,q}$, where $B_{p,q} \equiv \{\omega \in \{0, 1\}^\infty \backslash C : \omega_1 = \cdots = \omega_p = 0$ if $p \geqq 1,\ \omega_{p+1} = \cdots = \omega_{p+q} = 1,\ \omega_{p+q+1} = 0\}$, Arnold [1] defines the transformation S by $S\omega = \epsilon$, where $\omega \in B_{p,q}$ and $\epsilon_1 = \cdots = \epsilon_{q-1} = 1$ if $q \geqq 2$, $\epsilon_q = \cdots = \epsilon_{p+q} = 0$, $\epsilon_{p+q+1} = 1$, $\epsilon_r = \omega_r$ if $r \geqq p + q + 2$. S is the left shift on C, sending ω to $(\omega_2, \omega_3, \cdots)$. Since S on $\{0, 1\}^\infty \backslash C$ preserves numbers of 0's and 1's among coordinates and S fixes the sequences $(0, 0, \cdots)$ and $(1, 1, \cdots)$, it preserves any exchangeable measure μ, i.e., $\mu(S^{-1} I_a) = \mu(I_a)$ for all a. Clearly S is measurable, with measurable inverse on $\{0, 1\}^\infty \backslash C$.

As Hajian, Ito, and Kakutani [4] remark, for every $\omega \in \{0, 1\}^\infty \backslash C$ there exist infinitely many positive integers N_k with $\omega_{N_k+1} = 1$, $\omega_{N_k+2} = 0$, $s_k = \sum_{i=1}^{N_k} \omega_i \geqq s$, and $N_k - s_k \geqq n - s$. It is easy to check that there are $\binom{N_k}{s_k}$ distinct N_k-segments of $\{S^j \omega : j = 1, \cdots, \binom{N_k}{s_k}\}$. Therefore, writing χ_{I_a} for the indicator function of I_a,

$$\sum_{j=1}^{\binom{N_k}{s_k}} \chi_{I_a}(S^j \omega) = \binom{N_k - n}{s_k - s} \sim \binom{N_k}{s_k} \left(\frac{s_k}{N_k}\right)^s \left(\frac{N_k - s_k}{N_k}\right)^{n-s}.$$

Birkhoff's Ergodic Theorem states that as $N \to \infty$, for each a, $(1/N) \sum_{j=1}^{N} \chi_{I_a} (S^j\omega)$ converges *a.s.* (μ) for $\omega \in \{0, 1\}^\infty \backslash C$ to a number $\nu(I_a, \omega)$ in $[0, 1]$, depending measurably on a, ω, with $\int \nu(I_a, \omega)\, d\mu(\omega) = \mu(I_a)$, and $\nu(\cdot, \omega)$ can clearly be extended to a random Borel measure on $\{0, 1\}^\infty$.

2. So for almost all ω in $\{0, 1\}^\infty \backslash C$, the subsequence

$$\binom{N_k}{s_k}^{-1} \sum_{j=1}^{\left[{N \atop s_k}\right]} \chi_{I_a} (S^j\omega)$$

converges, hence $(s_k/N_k)^s \cdot (1 - s_k/N_k)^{n-s}$ converges, and s_k/N_k converges to some number $p \in [0, 1]$. This is a restricted Strong Law. For a proof via martingales, see Loève [5, p. 400]; still another proof is contained in the Ergodic Theorem together with the Hewitt-Savage 0-1 Law (see Feller [3, p. 124]).

THEOREM. *If* $\{X_k\}_{k=1}^\infty$ *is an exchangeable sequence in* $\{0, 1\}^\infty$ *with law* μ, *then almost surely* (i.e., μ – *a.e.*) $m^{-1}\sum_{k=1}^{m} X_k$ *converges.*

PROOF. Without loss of generality we assume $\mu(C) > 0$. (Otherwise replace $\{X_k\}$ by an exchangeable sequence $\{\gamma_k\}$ with law τ a mixture of μ and the point mass at $(0, 0, \cdots)$, and $0 < \tau(\{(0, 0, \cdots, 0)\}) < 1$. Our theorem for $\{\gamma_k\}$ implies the same result for $\{X_k\}$.) Let $D = \{\omega : m^{-1} \sum_{k=1}^{m} X_k(\omega)$ does not converge as $m \to \infty\}$. Then D is invariant under finite permutations of indices, and by approximating it closely in measure by cylinder-sets, we have as in the Hewitt-Savage 0-1 Law that if $\mu(D) > 0$, then $\mu(D) = 1$. But $C \cap D = \emptyset$, therefore $\mu(C) > 0$ implies $\mu(D) = 0$. Since we previously showed s_k/N_k converges for μ – almost all ω, we have that $m^{-1}\sum_{k=1}^{m} X_k$ converges a.s. to the same limit.

For $\omega \in C$, we define $\nu(\cdot, \omega)$ to be the point mass at $(0, 0, \cdots)$ if ω terminates in 0's, at $(1, 1, \cdots)$ if in 1's; so that once again $\lim_{m\to\infty} m^{-1} \sum_{j=1}^{m} \chi_{I_a} (S^j\omega) = \nu(I_a, \omega)$.

So whatever $a \in \{0, 1\}^n$ and $n \in Z^+$ we choose, if $\omega \notin C$ and $s = a_1 + \cdots + a_n$, then $\lim_{m\to\infty} m^{-1} \sum_{j=1}^{m} \chi_{I_a}(S^j\omega) \equiv \nu(I_a, \omega) = (\lim_{k\to\infty} s_k/N_k)^s \cdot (1 - \lim_{k\to\infty} s_k/N_k)^{n-s} = \nu(I_1, \omega)^s \nu(I_0, \omega)^{n-s}$ a.s. (μ), where $I_1 = \{\omega : \omega_1 = 1\}$, $I_0 = \{\omega : \omega_1 = 0\}$. As is easily verified also for $\omega \notin C$, $\nu(\cdot, \omega)$ is a random *product* measure on $\{0, 1\}^\infty$, over the probability space $(\{0, 1\}^\infty, \mu)$, with $E_\mu(\nu(\cdot, \omega)) = \mu(\cdot)$. We denote the finite product-measure $\nu(\cdot, \omega)$ on $\{0, 1\}^\infty$ by $((1 - r)\delta_0 + r\delta_1)^\infty$, where $r = \nu(I_1, \omega)$ and δ_0 and δ_1 are respectively the point masses at $\{0\}$ and $\{1\}$ on $\{0, 1\}$.

We define for $0 \leqq t \leqq 1$, $F(t) \equiv \mu(\{\omega : s_k/N_k \to \text{some } r \leqq t\}) = \mu(\{\omega : \nu(I_1, \omega) \leqq t\})$. Then $\mu(I_a) = \int_0^1 \nu(I_a, \omega)\, d\mu(\{\omega : \nu(I_1, \omega) \leqq t\}) = \int_0^1 t^s(1-t)^{n-s}\, dF(t)$, which is precisely de Finetti's (1931) representation.

REMARK. De Finetti's theorem gives a one-to-one correspondence $\mu \leftrightarrow F$ between exchangeable Borel probability measures on $\{0, 1\}^\infty$ and probability distribution functions on $[0, 1]$. Since $\mu(I_a)$ is expressed in terms of the moments of F, this gives an amusing no-calculation proof of the

PROPOSITION. *A probability distribution function on a compact real interval is uniquely determined by its moments.*

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