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ON THE UNIQUENESS AND GLOBAL ASYMPTOTIC STABILITY OF PERIODIC SOLUTIONS FOR A THIRD ORDER SYSTEM

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1. **Introduction.** Stability theory for periodic solutions of autonomous ordinary differential equations is, from some points of view, in a reasonably satisfactory state of development. All standard texts in the subject discuss the way in which local orbital stability is related to the Floquet multipliers or exponents, and there are important perturbation theorems which can be applied when $n - 1$ of these exponents have negative real parts. In practise, however, it is usually only in two dimensions, or near a "bifurcation point" that this theory can be applied. Even then, it may be difficult to determine the stability properties of the solution. In three or more dimensions the very existence, non-locally, of a periodic solution may be in question, and when this can be established, it is often by a fixed point theorem which leaves even the isolated nature of the solution in doubt.

Moreover, even if it were possible to determine the Floquet multipliers, this would at most establish the stability of the solution in a small region containing the trajectory. I know of very few examples where global asymptotic orbital stability of a periodic solution has been demonstrated in more than two dimensions. When this has been done, it has usually been for examples constructed particularly to have this stability property. Such constructions may indeed be difficult, because the researcher wishes to obtain the desired behavior within a certain restricted class of equations, e.g., [4], [5]. Sometimes the proof of stability depends on the existence of an attracting two dimensional submanifold on which limit cycle behavior is known to occur [3], [5].

In this paper I do not construct a system with stable periodic solutions. Instead, a particular set of equations, introduced over ten years ago in connection with certain biological phenomena, is considered. This system (given below) involves four real parameters, α, β, γ, and p, all positive. The main results is, roughly, that if p is sufficiently large and $|\alpha\beta\gamma - \frac{1}{2}|$ is sifficiently small, then there is a unique non-constant periodic solution such that all trajectories except those near the equilibrium point or starting on a particular one dimensional curve tend to the orbit of this solution. Moreover, the system is "structurally stable"

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away from the equilibrium point, which in this case is equivalent to saying that $n - 1$ Floquet exponents have negative real parts.

The restrictions on α, β, γ, and p make the result weaker than is desirable. On the other hand, this is not a bifurcation theorem, and the periodic solutions in question may have arbitrarily large amplitude. Nor is this a singular perturbation result, at least in the usual sense that there is a related "reduced" problem which is of lower order. I do treat a limiting case ($p = \infty$), but while the resulting problem is certainly more tractible than the case of finite p, it is still three dimensional.

The equations to be considered are

$$
\begin{aligned}
x' &= \frac{1}{1 + z^p} - \alpha x \\
y' &= x - \beta y \qquad\qquad (1) \\
z' &= y - \gamma z
\end{aligned}
$$

They were introduced by B. Goodwin as a qualitative model of a cellular control system with negative feedback [1], but the present analysis is mainly concerned with the mathematical features of the system. Let u denote the vector (x, y, z), and write (1) as

$$u' = f(u).$$

The positive octant $R_3{}^+ : x, y, z > 0$ is readily seen to be positively invariant and to contain a unique equilibrium point u_0. The Jacobian matrix $f_u(u_0)$ always has at least one real negative eigenvalue. However, the other two eigenvalues may have positive real parts. It can be verified that this occurs as described in the following result.

Proposition. *Suppose* $\alpha\beta\gamma < 1$. *Then there is a* p_0 *such that* $f_u(u_0)$ *has two eigenvalues with positive real parts whenever* $p > p_0$.

In [**6**] Tyson proves, as may be expected, that there is a periodic solution of (1) whenever $f_u(u_0)$ has some eigenvalues with positive real part. (In [**2**] the existence proof was extended to a class of n-dimensional systems, including (1) but possibly having nonlinear terms in every equation. Some of the present paper can probably be extended to n dimensions.) Numerical evidence suggests that the periodic solution is unique, and that every trajectory not on the one-dimensional stable manifold $\mathcal{S}$ at u_0 tends to the periodic orbit.

The system (1) appears quite simple, with only one non-linear term, and I hoped originally to give a direct proof of the stability and uniqueness of the periodic solution, perhaps by analyzing a particular "Poincaré map". Such a hope may have been naive; in any case, I have not achieved this goal. Instead I consider first the limiting value $p = \infty$.

If H denotes the Heaviside function:

$$H(w) = \begin{cases} 0, & w \leqq 0 \\ 1, & w > 0, \end{cases}$$

then the equations for $p = \infty$ take the form

$$\begin{aligned} x' &= H(1 - z) - \alpha x \\ y' &= x - \beta y \qquad (2) \\ z' &= y - \gamma z. \end{aligned}$$

To avoid unnecessary complications, the point $(\beta\gamma, \gamma, 1)$ will be excluded from consideration, since the nature of the flow defined by (2) near that point is a little unusual. If $\alpha\beta\gamma \geqq 1$, it can be shown that every trajectory tends to the point $(1/\alpha, 1/(\alpha\beta), 1/(\alpha\beta\gamma))$. However the main interest is in the case $\alpha\beta\gamma < 1$, when there are periodic solutions. (This will be apparent later.) Uniqueness and stability of these solutions are discussed separately.

THEOREM 1. *If $\alpha\beta\gamma < 1$, then every trajectory of* (2) *in the positive octant tends either* (I) *to some periodic orbit or* (II) *to* $(\beta\gamma, \gamma, 1)$.

REMARK. I think that it should be possible to eliminate the latter possibility, except for two distinguished orbits, because $(\beta\gamma, \gamma, 1)$ is close to the unstable equilibrium point of the continuous system, for $\alpha\beta\gamma < 1$ and p large. However I haven't found a quick way to do this, and since the main use of Theorem 1 is in conjunction with Theorem 2, where the possibility II can be eliminated, it doesn't seem worth a lot of effort.

Clearly this result is most useful when the uniqueness of the periodic solution can be established. I have been able to do this only when the point (α, β, γ) lies near a certain plane in the three-dimensional parameter space. On the other hand, near this plane the alternative II is eliminated, and uniqueness and stability are established for the continuous system (1), when p is large.

THEOREM 2. *Suppose that α^*, β^*, and γ^* are positive, with $\alpha^*\beta^*\gamma^* = 1/2$. Then* (2) *has a unique periodic solution in $R_+{}^3$. Also, there are two trajectories which tend to the point $(\beta\gamma, \gamma, 1)$, and in fact arrive at this point in finite time. Any solution in $R_+{}^3$ which does not lie on one of these trajectories tends to the periodic solution* (*orbitally*) *as t tends to infinity.*

Further, let $\hat{\mu}$ be a sufficiently small open neighborhood in $R_+{}^3$ of $(\beta\gamma, \gamma, 1)$. Then there are $p^* > 0$ and $\epsilon > 0$ such that if $p > p^*$ and $|\alpha - \alpha^*| + |\beta - \beta^*| + |\gamma - \gamma^*| < \epsilon$, then (1) *has a unique periodic solution in $R_+{}^3 - \hat{\mu}$, and every trajectory of* (1) *starting in $R_+{}^3 - \hat{\mu}$ and not on the stable manifold of* (1) *at u_0 tends to the periodic orbit as t approaches infinity.*

2. Stability.

PROOF OF THEOREM 1. First analyze the system (2) in the region $z > 1$, where

$$x' = -\alpha x$$
$$y' = x - \beta y \qquad (3)$$
$$z' = y - \gamma z.$$

The first two equations in (3) do not depend on z, and the x, y phase plane is easily sketched. Figure 1.a illustrates the case $\beta > \alpha$.

For any $\alpha, \beta > 0$, the origin is a stable node. Furthermore, if a solution of (3) begins in the region $z > 1$, $x(0) > 0$, $y(0) > 0$, it must eventually intersect the plane $z = 1$, with $0 < y \leqq \gamma$ at the point of intersection. It is, of course, easy to solve (3) explicitly, and by considering dy/dx and dz/dx, the "time" t can be eliminated. However, the resulting equations are not very illuminating, and it is better to use qualitative methods at this stage.

When $z < 1$, the system (2) becomes

$$x' = 1 - \alpha x$$
$$y' = x - \beta y \qquad (4)$$
$$z' = y - \gamma z$$

The x, y equations have a stable node at $(1/\alpha, 1/(\alpha\beta))$. If a solution of (2), or (4), starts in the region $z \leqq 1$, $0 < y < \gamma$, it must eventually enter the region $\beta\gamma < x < 1/\alpha$, $\gamma < y < 1/(\alpha\beta)$, and then intersect $z = 1$. Also, $y < x/\beta$ at the point of intersection. Figure 1.b is illustrative.

The general picture of the flow determined by (2) is now clear. Every trajectory in the positive octant eventually enters, and remains in, the region $0 < x < 1/\alpha$, $0 < y < 1/(\alpha\beta)$, $0 < z < 1/(\alpha\beta\gamma)$, if $\alpha\beta\gamma < 1$. If $z(0) > 1$ then the trajectory follows an orbit of (3), and eventually intersects $z = 1$, in the region $0 < y \leqq \gamma$. Unless $x = \beta\gamma$ and $y = \gamma$ at this point, in which case the solution can be continued no further, it then switches to an orbit of (4) and again intersects $z = 1$, this time in the region $\beta\gamma < x < 1/\alpha$, $\gamma < y < x/\alpha\beta$). (A sequence of successive

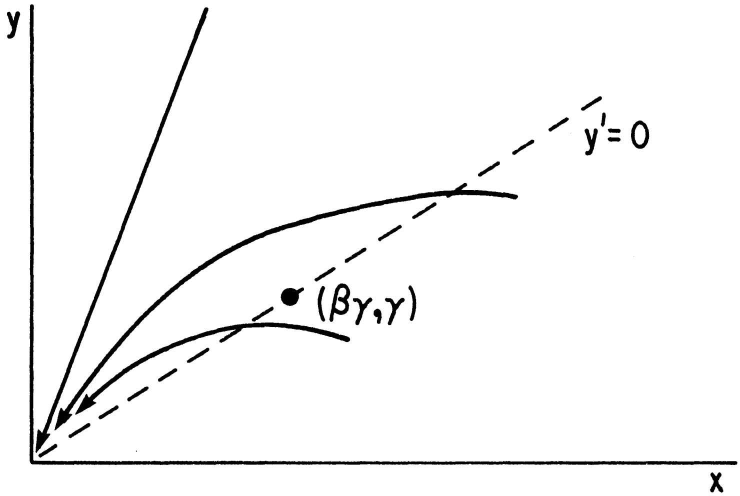

Figure 1a. Trajectories of (3).

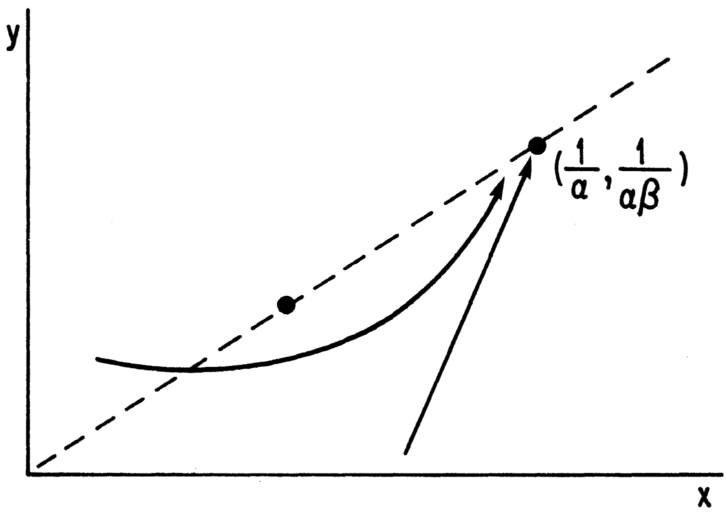

Figure 1b. Trajectories of (4).

switch points is shown in Figure 2.) Except possibly for the first one, the switch points alternate between the regions $\{\beta\gamma < x < 1/\alpha, \gamma < y < x/\beta\}$, and $\{0 < x < \beta\gamma, x/\beta < y < \gamma\}$. Only two trajectories, one starting in $z > 1$ and the other starting in $z < 1$, fail to alternate eventually in this fashion.

It will be shown that any sequence $\{P_{2j}\}$ of "even" switch points must tend to a limit P^* in the region $0 < x < 1/\alpha$, $0 < y < 1/(\alpha\beta)$. If $P^* \neq (\beta\gamma, \gamma, 1)$, then P^* clearly is a switch point of a periodic solution.

2.1. **Basic Comparison Lemma.** The principle tool in proving Theorem 1 is a comparison Lemma relating switch points for certain pairs of solutions starting on $z = 1$ in the set $\beta\gamma < x < 1/\alpha$, $\gamma < y < x/\beta$.

NOTATION: Suppose that $u(t) = (x(t), y(t), z(t))$ is a solution of (3) in the positive octant, with $u(0) = u_0$. Then $x'(t) < 0$ for all t, so $y(t)$ and $z(t)$ can be expressed as functions of $x(t)$. Let these functions be $y(x, u_0)$ and $z(x, u_0)$.

LEMMA 1. *Let* $u^i(t) = (x^i(t), y^i(t), z^i(t))$, $i = 1, 2$, *be two solutions of* (3), *with distinct initial conditions* $u_0^i = (x_0^i, y_0^i, 1)$ *which satisfy*

$$\beta\gamma < x_0^i, \quad \gamma < y_0^i < x_0^i/\beta.$$

Suppose further that $x_0^1 \leqq x_0^2$, *and that*

$$y(x_0^2, u_0^1) \leqq y_0^2.$$

Define x_1^i *by the conditions* $z^i(x_1^i, u_0^i) = 1, i = 1, 2$, *and let* $y_1^i = y(x_1^i, u_0^i)$. *Then*

$$x_1^2 < x_1^1, y_1^2 < y_1^1 \tag{5}$$

PROOF. Change notation slightly by letting $y^i(x) = y(x, u_0^i)$, $z^i(x) = z(x, u_0^i)$. That is, in the proof of Lemma 1, $y^i(\cdot)$ and $z^i(\cdot)$ will denote "functions of x", rather than t. Both $y^1(x)$ and $y^2(x)$ satisfy the equation

$$\frac{dy}{dx} = \frac{x - \beta y}{-\alpha x}, \tag{6}$$

so their graphs in the x, y plane do not intersect. Hence $y^2(x) > y^1(x)$ for $0 < x < \infty$. Also

$$\frac{dz^i}{dx} = \frac{y^i - \gamma z^i}{-\alpha x}, \tag{7}$$

which is negative if $z = 1, y > \gamma$. Hence

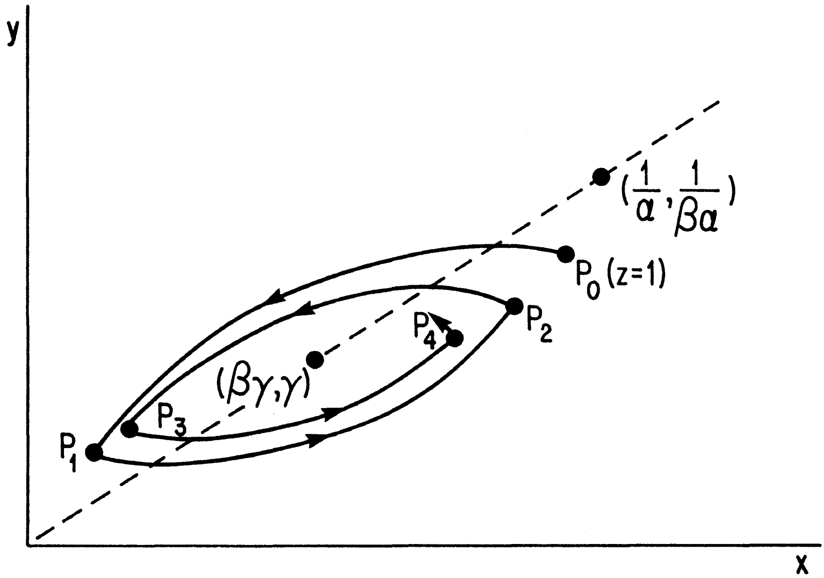

Figure 2. (Trajectories of (2)).

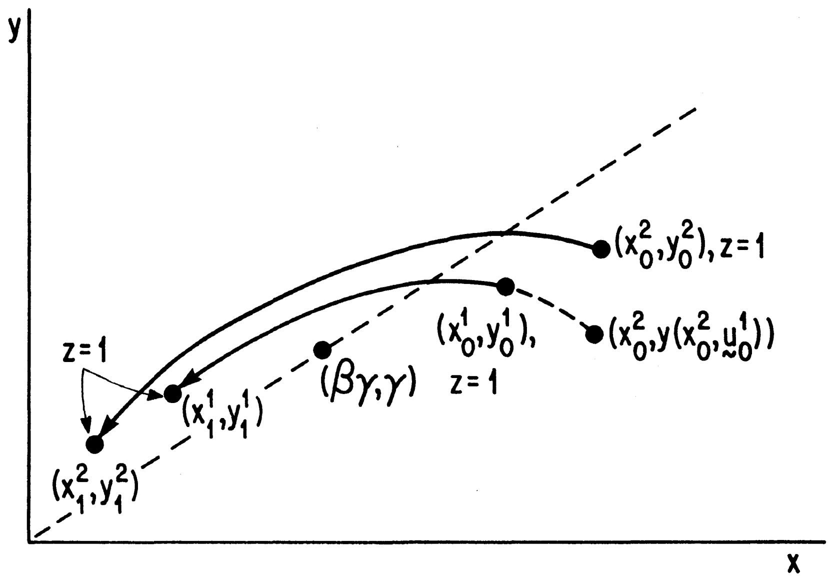

Figure 3. (Two trajectories of (3)).

(8) $$z^2(x_0{}^1) \geqq z^1(x_0{}^1) = 1.$$

LEMMA 2. *$z^2(x) > z^1(x)$ for $0 < x < x_0{}^1$.*

PROOF. If not, then from (8) there is a largest x^* in $(0, x_0{}^1)$ with $z^2(x^*) = z^1(x^*)$. Then, however,

$$\begin{aligned}\frac{dz^2}{dx}(x^*) &= \frac{y^2(x^*) - \gamma z^2(x^*)}{-\alpha x^*}\\ &< \frac{y^1(x^*) - \gamma z^1(x^*)}{-\alpha x^*}\\ &= \frac{dz^1}{dx}(x^*),\end{aligned}$$

which contradicts the definition of x^*.

COROLLARY. *$z^1 = 1$ "before" (i.e., for a larger value of $x < x_0{}^1$) $z^2 = 1$.*

This proves the first half of (5). For the second half, define $\hat{x} < x_0{}^1$ by the equation

$$\frac{dz^1}{dx}(\hat{x}) = 0.$$

Then

(9) $$z^2(\hat{x}) = y^1(\hat{x})/\gamma = \max_{x>0} z^1(x) > 1.$$

By Lemma 2,

(10) $$z^2(\hat{x}) > y^1(\hat{x})/\gamma, \text{ and } y^2(\hat{x}) > y^1(\hat{x}).$$

Also, suppose that $\hat{\hat{x}} < \hat{x}$ is defined by

$$y^2(\hat{\hat{x}}) = y^1(\hat{x}).$$

If $dz^2/dx(\hat{\hat{x}}) < 0$, let $\bar{\bar{x}}$ be the unique zero of dz^2/dx, while if $dz^2/dx)\hat{\hat{x}}) \geqq 0$, let $\bar{\bar{x}} = \hat{\hat{x}}$. Then $0 \leqq \bar{\bar{x}} \leqq \hat{\hat{x}}$. Also, define $\bar{x}$ by

$$y^1(\bar{x}) = y^2(\bar{\bar{x}}),$$

and let $\bar{y} = y^1(\bar{x})$.

LEMMA 3. *$z^2(\bar{\bar{x}}) \geqq z^1(\bar{x})$.*

PROOF. First suppose that $dz^2/dx(\hat{x}) > 0$, so that $\bar{x} = \hat{x}, \bar{\bar{x}} = \hat{\hat{x}}$. Since $y^2(x) \geqq y^1(x)$ for $\hat{\hat{x}} \leqq x \leqq \hat{x}$, it follows from (10) that $z^2(x) \geqq y^1(x)/\gamma$ on this interval, and the conclusion of Lemma 3 follows from (9). If, on the contrary, $dz^2/dx(\hat{\hat{x}}) < 0$, then $z^2(x)$ increases as x decreases from $\bar{x}$ to $\bar{\bar{x}}$, and the result follows from Lemma 2.

Continuing with the proof of Lemma 1, observe that in the interval $0 < x < \bar{x}$, both $y^1(x)$ and $y^2(x)$ increase with x. Hence we can express x and z^i as functions of y^i:

$$z^i(x) = Z^i(y^i(x)),$$
$$(y^i)^{-1} = X^i.$$

Then

$$\frac{dZ^i}{dy} = \frac{y - \gamma Z^i}{X^i(y) - \beta y} > 0, \tag{11}$$

with $y - \gamma Z^i < 0$, $X^i(y) - \beta y < 0$ because $dx/dt < 0$.

Also, $X^2(y) < X^1(y)$ in $0 < y < \bar{y}$, and $Z^2(\bar{y}) > Z^1(\bar{y})$. Suppose that $Z^2(\tilde{y}) = Z^1(\tilde{y})$ for some largest $\tilde{y}$ in $(0, \bar{y})$. Since $y - \gamma Z^i(y) < 0$ and $X^2(y) - \beta y < X^1(y) - \beta y < 0$, (11) implies that

$$\frac{dZ^2}{dy} < \frac{dZ^1}{dy} \text{ at } \tilde{y},$$

a contradiction which proves Lemma 1.

2.2. **Proof of Theorem 1.** Now consider $Z < 1$. Then $x(t)$, $y(t)$, and $z(t)$ solve (4). It is easily verified, however, that $(1/\alpha - x(t))$, $(1/(\alpha\beta) - y(t))$, and $(1/(\alpha\beta\gamma) - z(t))$ solve (3). Hence Lemma 1 has the following implication.

LEMMA 4. *Let $u^i(t) = (x^i(t), y^i(t), z^i(t))$, $i = 1, 2$, be two solutions of (2) such that $z^i(0) = 1$, and $x^i(0) = x_1^i$, $y^i(0) = y_1^i$, with*

$$x_1^2 < x_1^1, \quad y_1^2 < y\left(\frac{1}{\alpha} - x_1^2, U_0^1\right),$$

where $y(\,\cdot\,,\,\cdot\,)$ is the same function as in Lemma 1, and $U_0^i = (1/\alpha - x_1^i,\ 1/(\alpha\beta) - y_1^i, 1/(\alpha\beta\gamma) - 1)$. Define x_2^i by $z(1/\alpha - x_2^i, U_0^i) = 1$, $i = 1, 2$, and let

$$y_2^i = y(1/\alpha - x_2^i, U_0^i).$$

Then

$$x_2^2 > x_2^1, y_2^2 > y_2^1.$$

Now let $u(t) = (x(t), y(t), z(t))$ be a solution of (1) with $u(0) = (x_0, y_0, 1)$, and

$$0 < x_0 < \frac{1}{\alpha}, \quad 0 < y_0 < \frac{1}{\alpha\beta}, \quad u(0) \neq (\beta\gamma, \gamma, 1).$$

Define $A(x_0, y_0) = (x_1, y_1)$ to be the next switch point of the solution u. (Thus $(x_1, y_1) \neq (x_0, y_0)$). If $\beta\gamma < x_0 < 1/\alpha, \gamma < y_0 < x_0/\beta$, then $0 < x_1 < \beta\gamma, x_1/\beta < y_1 < \gamma$. Also, define a "Poincare map" B by $B = A^2$. It will be shown that if $B(x_i, y_i) = (x_{i+2}, y_{i+2})$, then the sequences $\{x_{2j}\}$, $\{y_{2j}\}$ are eventually monotone, as well as bounded, and that $p^* = \lim_{j\to\infty}(x_{2j}, y_{2j})$ is either $(\beta\gamma, \gamma, 1)$, or a switch point of a periodic solution. This will prove Theorem 1. Let

$$M = \left\{(x, y) \mid \beta\gamma < x < \frac{1}{\alpha}, \gamma < y < \frac{x}{\beta}\right\},$$

and for each $Q = (x_0, y_0)$ in M, define four subsets of M as follows, where $Q^* = (x_0, y_0, 1) \in R^3$.

$$S^1(Q) = \left\{(x, y) \mid x_0 < x < \frac{1}{\alpha}, y(x, Q^*) < y < x/\beta\right\},$$

$$S^2(Q) = \left\{(x, y) \mid \beta\gamma < x \leqq x_0, y(x, Q^*) < y < x/\beta\right\},$$

$$S^3(Q) = \left\{(x, y) \mid \beta\gamma < x \leqq x_0, \gamma < y \leqq y(x, Q^*)\right\},$$

$$S^4(Q) = \left\{(x, y) \mid x_0 < x < \frac{1}{\alpha}, \gamma < y \leqq y(x, Q^*)\right\}.$$

Similarly, for a point $R = (x_1, y_1)$ in the region $0 < x < \beta\gamma$, $x/\beta < y < \gamma$, let

$$T^i(R) = S^i\left(\left(\frac{1}{\alpha} - x_1, \frac{1}{\alpha\beta} - y_1\right)\right).$$

Let "$\circ$" and "$-$" denote the interior and closure, respectively, of a set in R^n. Then the following result follows from Lemma 1 and 4.

LEMMA 5. *If $B(Q) \in \overline{S^1(Q)}$, but $B(Q) \neq Q$, then $AB(Q) \in T^1(\mathring{A}(Q))$, and in fact,*

$$B^{j+1}(Q) \in S^1(B^j(Q)), j = 1, 2, 3, \cdots.$$

This implies that if $(x_0, y_0) = Q \in M$ and $B^{k+1}(Q) \in S^1(B^k(Q))$ for some $k \geqq 0$, then $B^{j+1}(Q) \in S^1(B^j(Q))$ for all $k \geqq j$, and hence $p^* =$

$\lim_{k\to\infty} B^k(Q)$ exists. The trajectory starting at Q^* "spirals out" to the trajectory starting at p^*.

It is clear that p^* lies in the closed triangle $\overline{M}$ and that $p^* \neq (\beta\gamma, \gamma)$. As long as

$$p^* \neq \left(\frac{1}{\alpha}, \frac{1}{\alpha\beta}\right), \tag{12}$$

it must be a switch point for a periodic solution. To check (12), consider the solution u^* of (2) with initial conditions $x(0) = 1/\alpha$, $y(0) = 1/(\alpha\beta)$, $z(0) = 1$. This solution first follows a trajectory of (3). The second switch point $B(1/\alpha, 1/(\alpha\beta))$ lies in M, so $(1/\alpha, 1/(\alpha\beta)) \in S^1(B(1/\alpha, 1/(\alpha\beta)))$. (Hence by repeated application of Lemma 1, u^* "spirals in".) By using Lemma 1 to compare u^* and the trajectory starting at $\bar{Q}$, it is easily seen that P^* lies below and to the left of $\lim_{n\to\infty} B^n(1/\alpha, 1/(\alpha\beta))$, which implies (12).

Next consider a $Q \in M$ such that for some j,

$$B^{j+1}(Q) \in \overline{S^3(B^j(Q))}.$$

This is equivalent to

$$B^j(Q) \in \overline{S^1(B^{j+1}(Q))},$$

and unless the trajectory is periodic, it must eventually spiral in. Again

$$\lim_{n\to\infty} \beta^n(Q) = p^*$$

exists. Either $p^* = (\beta\gamma, \gamma)$, or it is the switch point of a periodic solution.

Next solutions which neither spiral in nor spiral out must be considered. This would occur if, for some $Q \in M$,

$$B^{j+1}(Q) \notin \overline{S^1(B^j(Q))} \cup \overline{S^3(B^j(Q))}$$

for $j = 1, 2, 3, \cdots$. If $B^{j+1}(Q) \in S^2(B^j(Q))$ for all large j, then again $p^* = \lim_{n\to\infty} B^n(Q)$ is well defined. This is also the case if $B^{j+1}(Q) \in S^4(B^j(Q))$ for all large j. The only problem arises if $B^{j+1}(Q)$ sometimes lies in $S^4(B^j(Q))$ and sometimes in $S^2(B^j(Q))$, for arbitrarily large j. This possibility is eliminated by

LEMMA 6. *If* $B(Q) \in S^2(Q)$, *then* $B^2(Q) \notin S^4(B(Q))$.

PROOF. Since two solutions of (3) cannot have trajectories which cross when projected onto the x, y plane, it is clear that if $B(Q) \in S^2(Q)$, then

$$AB(Q) \in T^1(A(a)) \cup T^4(A(Q)),$$

If $AB(Q) \in \overline{T^1(A(Q))}$, then $A(AB(Q)) = B^2(Q) \in S^1(B(Q))$, and the solution spirals out. If $AB(Q) \in T^4(A(Q))$, then $B^2(Q) \in S^1(B(Q)) \cup S^2(B(Q))$, because solutions of (4) cannot cross in the x, y plane. Hence, in any case, $B^2(Q) \notin S^4(B(Q))$, which proves Lemma 6 and completes the proof of Theorem 1.

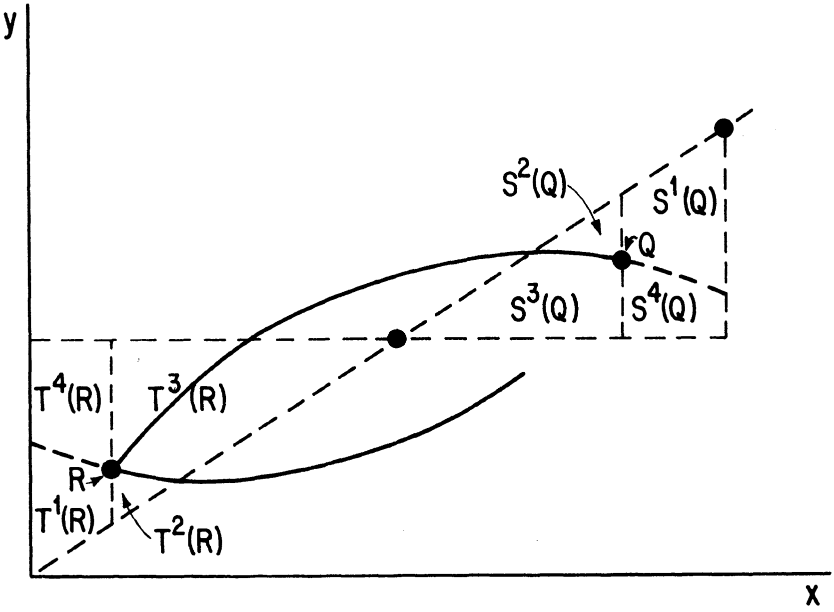

Figure 4.

3. **Uniqueness** (α, β, γ **distinct**). To study uniqueness it appears necessary to solve equations (3) and (4) explicitly. As before, (3) can be solved for y and z as functions of x, because x is monotone. I proceed for now under the assumption that α, β, and γ are distinct. This gives the following equations, for some constant c.

$$y(x) = \frac{x}{\beta - \alpha} + cx^{\beta/\alpha}$$

$$\frac{Z(x)}{x^{\gamma/\alpha}} - \frac{z(x_0)}{x_0^{\gamma/\alpha}} = \frac{x^{-\gamma/\alpha+1} - x_0^{-\gamma/\alpha+1}}{(\alpha - \beta)(\alpha - \gamma)} - \frac{c}{\beta - \gamma}\left(x^{(\beta-\gamma)/\alpha} - x_0^{(\beta-\gamma)/\alpha}\right).$$

Suppose that $z = 1$ at $x = x_0, y = y_0$ and again at $x = x_1, y = y_1$, with $z > 1$ in between. Then the following three equations must be satisfied.

$$y_0 = \frac{x_0}{\beta - \alpha} + c x_0^{\beta/\alpha}, \tag{13}$$

$$y_1 = \frac{x_1}{\beta - \alpha} + c x_1^{\beta/\alpha} \tag{14}$$

$$\frac{1}{x_1^{\gamma/\alpha}} - \frac{1}{x_0^{\gamma/\alpha}} = \frac{x_1^{-\gamma/\alpha+1} - x_0^{-\gamma/\alpha+1}}{(\alpha - \beta)(\alpha - \gamma)} - \frac{c}{\beta - \gamma}(x_1^{(\beta-\gamma)/\alpha} - x_0^{(\beta-\gamma)/\alpha}). \tag{15}$$

This is without any requirement of periodicity. Continuing from $(x_1, y_1, 1)$ with a solution of (4), recall that $1/\alpha - x$, $1/(\alpha\beta) - y$, $1/(\alpha\beta\gamma) - z$ solves (3). The solution will be periodic if $x = x_0$ and $y = y_0$ at the next point where $z = 1$. This results in three further equations:

$$\frac{1}{\alpha\beta} - y_1 = \frac{1/\alpha - x_1}{\beta - \alpha} + d(1/\alpha - x_1)^{\beta/\alpha} \tag{16}$$

$$\frac{1}{\alpha\beta} - y_0 = \frac{1/\alpha - x_0}{\beta - \alpha} + d(1/\alpha - x_0)^{\beta/\alpha}, \tag{17}$$

and

$$\begin{aligned} &\left(\frac{1}{\alpha\delta\gamma} - 1\right)\left(\frac{1}{(1/\alpha - x_0)^{\gamma/\alpha}} - \frac{1}{(1/\alpha - x_1)^{\gamma/\alpha}}\right) \\ &= \frac{(1/\alpha - x_0)^{-\gamma/\alpha+1} - (1/\alpha - x_1)^{-\gamma/\alpha+1}}{(\alpha - \delta)(\alpha - \gamma)} \\ &\quad - \frac{d((1/\alpha - x_0)^{(\beta-\gamma)/\alpha} - (1/\alpha - x_1)^{(\beta-\gamma)/\alpha})}{\beta - \gamma}, \end{aligned} \tag{18}$$

where c and d are constants.

By adding equation (13) and (17) and equations (14) and (16), y_0 and y_1 can be eliminated. Then c and d can be eliminated to give two equations which must be satisfied by the pair (x_0, x_1):

$$\begin{aligned} x_1^{-n} - x_0^{-n} &= \frac{1}{\alpha^2(m-1)(n-1)}(x_1^{1-n} - x_0^{1-n}) \\ &+ \frac{1}{m\alpha^3(m-1)(m-n)} \frac{((1/\alpha - x_0)^m - (1/\alpha - x_1)^m)(x_1^{m-n} - x_0^{m-n})}{(x_1^m(1/\alpha - x_0)^m - x_0^m(1/\alpha - x_1)^m)}, \end{aligned} \tag{19}$$

where $m = \beta/\alpha$, $n = \gamma/\alpha$, and

$$\left(\frac{1}{\alpha^3 mn}-1\right)((1/\alpha - x_0)^{-n} - (1/\alpha - x_1)^{-n})$$

$$(20) \qquad = \frac{(1/\alpha - x_0)^{1-n} - (1/\alpha - x_1)^{1-n}}{\alpha^2(m-1)(n-1)}$$

$$+\frac{(x_1{}^m - x_0{}^m)}{\alpha^3(m-1)(m-n)}\ \frac{((1/\alpha - x_0)^{m-n} - (1/\alpha - x_1)^{m-n})}{(x_1{}^m(1/\alpha - x_0)^m - x_0{}^m(1/\alpha - x_1)^m)}.$$

To prove that (2) has a unique periodic solution it must be shown that (19)-(20) has a unique solution (x_0, x_1) with

$$(21) \qquad 0 < x_1 < \beta\gamma < x_0 < 1/\alpha.$$

I have been able to do this only under the assumption that $\alpha\delta\gamma = 1/2$, or $|\alpha\beta\gamma - 1/2|$ small. However some preliminary results do not have this restriction, so I continue with the case of general (distinct) $\alpha\beta\gamma$ satisfying $\alpha\beta\gamma < 1$.

Lemma 7. *For each fixed x_0 in $(\beta\gamma, 1/\alpha)$, equation* (19) *has at most one solution x_1 in $(0, \beta\gamma)$.*

Proof. Suppose that (x_0, x_1) is a solution pair for (19), alone, which also satisfies (21). Then there are corresponding values of y_0, y_1, c, and d such that (13), (14), (15), (16), and (17) are all satisfied. This means that there is a solution $u(t)$ of (2) such that $u(0) = (x_0, y_0, 1)$, and $u(t_1) = (x_1, y_1, 1)$ for some first $t_1 > 0$. Furthermore, the "return trajectory, along an orbit of (4), must intersect some point (x_0, y_0, z^*), but not necessarily for $z^* = 1$.

Nevertheless, Lemma 1 implies that for any given x_0, this is possible for at most one y_0, and hence at most one x_1.

Lemma 8. *For any x_0 in $(\beta\gamma, 1/\alpha)$, there is exactly one x_1, such that* (19) *is satisfied.*

Proof. Again this makes use of Lemma 1. Let $v(t)$ be the solution of (4) such that $v(0) = (0, 0, 1)$, and let Γ denote the projection of the trajectory of v onto the x, y plane. For a fixed pair (x_0, y_0) with $\beta\gamma < x_0 < 1/\alpha, 0 < y_0 < x_0/\beta$, there are unique "downward" and "upward" solutions, solving the first two equations of (3) and (4), respectively, which begin at (x_0, y_0). Let γ_1 and γ_2 denote the trajectories of these solutions. If (x_0, y_0) lies between Γ and the line $y = x/\beta$, then γ_1 and γ_2 will intersect at precisely one other point, (x_1, y_1), in the x, y plane. Consider a fixed x_0 in $(\beta\gamma, 1/\alpha)$, and let $\zeta(y_0)$ denote the value of z when the solution of (3) starting at $(x_0, y_0, 1)$ reaches the plane $x = x_1$. It is easily checked that $\zeta(y_0) \to 0$ as $(x_0, y_0) \to \Gamma$ from above, while

$\zeta(y_0) > 1$ for y_0 sufficiently close to x_0/β. The result therefore follows by continuity.

Observe also that if (x_0, x_1) solves (19), then so does (x_1, x_0).

LEMMA 9. *The set* Λ *of solutions* (x_0, x_1) *of* (19) *is the graph of a continuous function* $x_1 = f(x_0), 0 \leqq x_0 \leqq 1/\alpha$. (Clearly, $f(\beta\gamma) = \gamma$, $f(1/\alpha) = 1/(\alpha\beta)$. It is only necessary to consider $\Lambda_1 = \Lambda \cap [\beta\gamma, 1/\alpha]$.)

PROOF. It has been shown that Λ_1 is the graph of a function F_1 defined on $[\beta\gamma, 1/\alpha]$. The continuity of F_1 therefore follows because it is bounded and is the inverse image of 0 under a continuous function, and hence closed.

To obtain further results, it will now be assumed that $\alpha\beta\gamma = 1/2$. I continue also to consider the case α, β, γ distinct. This restriction will eventually be removed.

If $\alpha\beta\gamma = 1/2$, then $1/\alpha\beta\gamma - 1 = 1$, so (19) and (20) become symmetric in the following sense: If (x_0, x_1) solves (19) and (20), then so do (x_1, x_0), $(1/\alpha - x_0, 1/\alpha - x_1)$, and $(1/\alpha - x_1, 1/\alpha - x_0)$.

COROLLARY. *If* $\alpha\beta\gamma = 1/2$, *then all solutions of* (19)–(20) *lie on the line* $x_0 + x_1 = 1/\alpha$.

PROOF. The solution set of (19) alone is the graph of $x_1 = f(x_0)$. Suppose (x_0^*, x_1^*) solves (19) and (20), and $x_0^* + x_1^* \neq 1/\alpha$. Then the graph of f passes through four distinct points which are symmetric with respect to the line $x_1 + x_0 = 1/\alpha$. The entire graph is symmetric with respect to the line $x_1 = x_0$. Assume for definiteness that $x_0 < 1/\alpha - x_1 < 1/(2\alpha) < x_1 < 1/\alpha - x_0$. Then the arc of the graph of f connecting (x_0, x_1) with (x_1, x_0) cannot contain $(1/\alpha - x_0, 1/\alpha - x_1)$ and therefore cannot contain $(1/\alpha - x_1, 1/\alpha - x_0)$, which gives a contradiction.

If follows that when $\alpha\beta\gamma = 1/2$, I can set $x_1 = 1/\alpha - x_0$ and consider a single equation $\lambda(x_0) = \lambda(1/\alpha - x_0)$, where

$$\begin{aligned}\lambda(x) = x^{-n} - \frac{1}{\alpha^2(m-1)(n-1)} x^{1-n} \\ - \frac{x^{m-n}}{\alpha^3(m-1)(m-n)} \cdot \frac{1}{(x^m + (1/\alpha - x)^m)}.\end{aligned} \tag{22}$$

I did not find the analysis of this equation easy. However, it turns out that a simple way to proceed is to set $x_0 = 1/\alpha \cdot r$. Then (22) becomes

$$\frac{(1-r)^n}{r^n} = \frac{1 - K(1-r) - L\dfrac{(1-r)^m}{(r^m + (1-r)^m)}}{1 - Kr - L\dfrac{r^m}{(r^m + (1-r)^m)}}$$

where

$$K = \frac{2mn}{(m-1)(n-1)}, \; L = \frac{2n}{(m-1)(m-n)}.$$

(Recall, $m = \beta/\alpha$, $n = \gamma/\alpha$.)

Letting $s = (1-r)/r$, this reduces to

$$\frac{k}{1+s^n} = 1 - \frac{K}{1+s} - \frac{L}{1+s^m},$$

with

$$k = \frac{-2m}{(n-1)(m-n)}.$$

Write this as

$$\varphi(s) = 0, \tag{23}$$

where

$$\varphi(s) = a_1 s^{m+n+1} + a_2 s^{m+n} + a_3 s^{m+1} + a_4 s^m$$
$$+ a_5 s^{n+1} + a_6 s^n + a_7 s + a_8$$

for certain constants $a_1, \cdots, a_8$. ($a_1 = 1$.) A computation, using the specific values of $a_1, \cdots, a_8$ in terms of α, m, and n, shows that

$$\varphi(1) = \varphi'(1) = \varphi''(1) = 0, \text{ and}$$
$$\varphi'''(1) < 0.$$

Since $\varphi \to +\infty$ as $s \to +\infty$, φ must have an odd number, say $2k+1$, of zeroes on $1 < s < \infty$, counting multiplicity.

Also, the symmetry of (22) implies that if $s^* > 0$ is a root of $\varphi(s) = 0$, then so is $1/s^*$. Hence (23) has $2k+1$ roots on $0 < x < 1$.

It is necessary to show that (23) has exactly one root on $1 < s < \infty$; i.e., that $k = 0$. If $k \geqq 1$, then $\varphi(s)$ must have at least 9 zeroes on $0 < s < \infty$, counting the triple root at $s = 1$.

However an easy inductive proof shows that a function of the form

$$\sum_{j=1}^{N} a_i s^{k_i} = 0,$$

k_i's any real numbers, has at most $N - 1$ real zeroes on $0 < s < \infty$, and in this case $N = 8$.

4. **Completion of proof of Theorem 2.** The proof when α, β, and γ are not distinct is similar, except that the function $\varphi(s)$ involves logarithmic terms. For example, if $\alpha = \beta \neq \gamma$, then $m = 1, n \neq 1$, and $\varphi(s)$ becomes of the form

$$\varphi(s) = s^{n+2} + a_2 s^{n+1} + a_3 s^n + a_4 s^2 + a_5 s + a_6$$
$$+ a_7 s \ln s + a_8 s^{n+1} \ln s.$$

This may be seen, for instance, by taking $\lim_{m \to 1} \varphi(s)$ with $m \neq 1, n \neq m$ and $n \neq 1$. Once again, φ, φ', and φ'' all vanish at $s = 1$, and $\varphi'''(1)$ is negative. A similar induction to that used for $m \neq 1$ shows that φ cannot have more than seven zeros in $(0, \infty)$.

This completes the uniqueness proof if $\alpha\beta\gamma = 1/2, p = \infty$. There remains to finish proving stability by showing that $B^n(x_0, y_0) \nrightarrow (\beta\gamma, \gamma)$, for any $(x_0, y_0) \in M$.

I point out the following consequence of Lemma 1: If $u_i(t)$ is a solution of (2) starting at $(x_i, y_i, 1), i = 1, 2$, and $(x_2, y_2) \in S^1(x_1, y_1)$, then $B^{n+1}(x_2, y_2) \in S^1(B^n(x_n, y_n)), n = 1, 2, 3, \cdots$. Hence if $(x_0{}^*, y_0{}^*) \in M$ is a switch point of the periodic solution u_p of (2), and if $x(0) > x_0{}^*, y(0) > y_0{}^*$, then $B^n(x_0, y_0) \nrightarrow (\beta\gamma, \gamma)$, and so $B^n(x_0, y_0) \to (x_0{}^*, y_0{}^*)$. It must be shown that solutions beginning near enough to $(\beta\gamma, \gamma)$, in M, spiral out to u_p.

To prove this, observe that when $\alpha\beta\gamma = 1/2$, (19) and (20) take the forms

$$G(x_0, x_1) = 0 \tag{24}$$

and

$$G(1/\alpha - x_0, 1/\alpha - x_1) = 0. \tag{25}$$

Also, $G(a, b) = -G(b, a)$.

The partial derivatives $G_1(a, b) = \partial G_1(a, b)/\partial a$ and $G_2(a, b) = \partial G(a, b)/\partial b$ certainly are continuous on $0 < a, b < 1/\alpha$. As previously described, (24) defines x_1 as a continuous function of x_0 — say $x_1 = r(x_0), \beta\gamma < x_0 < 1/\alpha$. Also the equation

$$G(1/\alpha - x_2, 1/\alpha - x_1) = 0 \tag{26}$$

defines x_2 as a continuous function $s(\,\cdot\,)$ of $x_{1,\,0} < x_1 < \beta\gamma$.

The switch point $x_0{}^*$ of u_p in $(\beta\gamma, 1/\alpha)$ is the unique point in this interval such that $r(x_0{}^*) = 1/\alpha - x_0{}^*$. Similarly, $x_1{}^*$ in $(0, \beta\gamma)$ is characterized by

$$s(x_1^*) = 1/\alpha - x_1^*, \text{ or}$$
$$s(x_1^*) = x_0^*.$$

Since $\varphi'(s^*) \neq 0$ at the unique root s^* of (23) in $1 < s < \infty$, it follows that

$$\frac{d}{dx_0} G(x_0, 1/\alpha - x_0) \, |x_0 = x_0^* \neq 0, \text{ or}$$

$$(27) \qquad G_1(x_0^*, 1/\alpha - x_0^*) - G_2(x_0^*, 1/\alpha - x_0^*) \neq 0.$$

I shall assume that $G_2(x_0^*, 1/\alpha - x_0^*) \neq 0$. If this is not the case, the roles of x_0 and x_1 should be interchanged, expressing x_0 as a function of x_1 near (x_0^*, x_1^*).

Now let $p(x) = s \circ r(x)$, $\beta\gamma < x < 1/\alpha$. Because solutions starting in $S^1(x_0^*, y_0^*)$ spiral in, it is seen that

$$p(x_0) < x_0 \text{ if } x_0^* < x_0 < 1/\alpha.$$

Also, $p'(x_0^*) - 1 \leqq 0$, $p(x_0^*) = x_0^*$.

LEMMA 10. $p'(x_0^*) < 1$.

PROOF. Clearly,

$$p'(x_0^*) = s'(r(x_0^*))r'(x_0^*).$$

Using (24) and (25), it is easily shown that

$$r'(x_0^*) = \frac{-G_1(x_0^*, 1/\alpha - x_0^*)}{G_2(x_0^*, 1/\alpha - x_0^*)} = s'(r(x_0^*)),$$

and hence $p'(x_0^*) = r'(x_0^*)^2$. But Lemma 1 shows that $r'(x_0^*) \leqq 0$, and (27) implies that $r'(x_0^*) \neq -1$. Hence $p'(x_0^*) \neq 1$, as desired.

Lemma 10 clearly shows that $p(x_0) > x_0$ if $x_0^* - x_0$ is positive but small. Because x_0^* is the unique solution of $p(x_0) = x_0$ in $(\beta\gamma, 1/\alpha)$, it can only be that $p(x_0) > x_0$ for $\beta\gamma < x_0 < x_0^*$, so solutions starting on $(x_0, f(x_0), 1)$, near $x_0 = \beta\gamma$, must spiral out. From Lemma 1 it is seen that if $(x_0, y_0) \in M$, then $B^n(x_0, y_0) \to (x_0^*, y_0^*)$ as $n \to \infty$. This proves the global asymptotic orbital stability of u_p.

This completes the discussion of uniqueness and stability for (2), and I now indicate how the assertions in Theorem 2 about the continuous system (1) are obtained. Some of this is a technical exercise, and will not be given in complete detail.

The treatment proceeds by considering a certain Poincaré map defined by the flow of (1). Let $q = 1/p$, and for each $q > 0$ let

$$u_c(g) = (x_c(q), y_c(q), z_c(q))$$

be the unique critical point of (1) in the positive octant. Then $z_c(q)$ satisfies the equation

$$1 = \alpha\beta\gamma(z + z^p), (p = q^{-1}),$$

from which it follows that if $\alpha\beta\gamma < 1$, then $z_c(q) \to 1$ as $q \to 0$. Hence, $x_c(q) \to \beta\gamma$, $y_c(q) \to \gamma$.

For $q \ll 1$, the trajectories of (1) have high curvature near the plane $z = z_c(q)$, as in a "corner layer". For this reason it seems more convenient to study a different Poincaré map from that discussed earlier, in connection with (2). Thus solutions are thought of as starting on, and returning to, the plane $x = x_c(q)$. To be precise, let $\hat{R}^q$ denote the rectangle

$$x = x_c(q), z_c(q) \leqq z \leqq \frac{1}{\alpha\beta\gamma}, y_c(q) \leqq y \leqq \frac{1}{\alpha\beta},$$
$$(y, z) \neq (y_c(q), z_c(q)).$$

In [6] it is shown that for small q, every trajectory of (1) in the positive octant except those on the (one dimensional) stable manifold at $u_c(q)$ eventually intersects $\hat{R}^q$ non-tangentially. These trajectories define a diffeomorphism $\hat{\Pi}^q$ of $\hat{R}^q$ into itself, as follows:

If $u(t) = u(t, y_0, z_0)$ is the solution of (1) with $u(0) = (x_c(q), y_0, z_0)$, and $u(0) \in \hat{R}^q$, then there is a first $T_q = T_q(y_0, z_0) > 0$ such that $u(T_q) \in \hat{R}^q$. Let

$$\hat{\Pi}^q(x_c(q), y_0, z_0) = u(T_q(y_0, z_0), y_0, z_0).$$

If R^q is the usual projection of $\hat{R}^q$ onto the (y, z) plane, let Π^q be the mapping of R^q into itself induced by $\hat{\Pi}^q$. It will be useful to write Π^q as the composition of four mappings, defined in an obvious way by following trajectories of (1) from the rectangle $\hat{R}^q$ first to the plane $z = z_c(q)$, then to the rectangle

$$\hat{S}^q : x = x_c(q), 0 \leqq y \leqq y_c(q), 0 \leqq z \leqq z_c(q),$$
$$(y, z) \neq (y_c(q), z_c(q)),$$

then again to the plane $z = z_c(q)$, and finally back to the rectangle $\hat{R}^q$. Thus

$$\Pi^q = \Pi_4{}^q \circ \Pi_3{}^q \circ \Pi_2{}^q \circ \Pi_1{}^q,$$

where $\Pi_1{}^q$ maps R^q into the set $0 \leqq x \leqq x_c(q), 0 \leqq y \leqq y_c(q)$, etc.

All of these definitions of sets and mappings have obvious extensions to the case $q = 0$, where they are done with reference to the system (2). Observe that a map considered previously, starting in the region

$$z = 1, (x, y) \in M$$

and returning to this set, is the restriction of

$$P = \Pi_3{}^0 \circ \Pi_2{}^0 \circ \Pi_1{}^0 \circ \Pi_4{}^0$$

to M. It has been shown that if $\alpha = \alpha^*, \beta = \beta^*, \gamma = \gamma^*$, with $\alpha^*\beta^*\gamma^* = 1/2$, then

(a) P has a unique fixed point, $(x_0{}^*, y_0{}^*)$, and

(b) For any $(x_0, y_0) \in \{\beta\gamma < x_0 < 1/\alpha, \gamma < y_0 < 1/(\alpha\beta)\}$, $\lim_{n\to\infty} P^n(x_0, y_0) = (x_0{}^*, y_0{}^*)$.

This implies that Π^0 also has a unique fixed point, $(\tilde{y}_0, \tilde{z}_0)$, and $\lim_{n\to\infty}(\Pi^0)^n(y, z) = (\tilde{y}_0, \tilde{z}_0)$, for any $(y, z) \in R^0$. It must be shown that if U is a given open neighborhood of $(\gamma, 1)$, then Π^q has essentially these properties also, if (y, z) and $(\tilde{y}, \tilde{z})$ are restricted to $R^q - U$ and if $q > 0$ and $|\alpha\beta\gamma - 1/2|$ are sufficiently small.

In the following I assume that $\alpha\beta\gamma = 1/2$, and verify the desired conclusion for small $q > 0$. The extension to small $|\alpha\beta\gamma - 1/2|$ will be obvious, because slight changes in α, β, and γ result in a C^1-small perturbation of Π^q.

There are two main steps in the following argument. First it is shown that in $R^0 - U, \Pi^q$ is a C^1-small perturbation of Π^0. Then it is verified that $D\Pi^0(\tilde{y}_0, \tilde{z}_0)$ has eigenvalues lying strictly inside the unit circle. These two steps complete the proof of Theorem 2.

It is not hard to show that the mappings Π^q, for $q > 0$ and small, can be extended to the set $R^0 - U$, which may be larger than $R^q - U$, and that $\Pi^q(y, z) \to \Pi^0(y, z)$ as $q \to 0$, uniformly in $R^0 - U$. It is more difficult to show that

$$D\Pi^q(y, z) \to D\Pi^0(y, z) \tag{28}$$

uniformly in $R^0 - U$, so I shall outline how this is done. Because $R^0 - U$ is compact, it is only necessary to prove (28) in a neighborhood of any point (y, z) in $R^0 - U$.

Write Π^q as the composition of two maps, $A^q = \Pi_2{}^q \circ \Pi_1{}^q$, and $B^q = \Pi_4{}^q \circ \Pi_3{}^q$. These are essentially symmetric with each other, and I shall show that $DA^q(y_0, z_0) \to DA^0(y_0, z_0)$, if $(y_0, z_0) \in R^0$. (The extension to uniform convergence in a neighborhood of (y_0, z_0) will be clear.)

Let $u^q(t, y_0, z_0) = (x^q(t, y_0, z_0), y^q(t, y_0, z_0), z^q(t, y_0, z_0))$ solve (1), or (2) if $q = 0$, with initial condition $(x_c(q), y_0, z_0)$,(y_0, z_0) in R^0. Let $\tau_q = \tau_q(y_0, z_0)$ denote the first $t > 0$ where $z^q(t, y_0, z_0) = z_c(q)$, and let $t^q(y_0, z_0)$ be the first $t > 0$ where $x^q(t, y_0, z_0) = x_c(q)$. Thus $0 < \tau_q < t_q$. It must be shown that

$$\frac{\partial}{\partial y_0}(u^q(t_q, y_0, z_0)) \to \frac{\partial}{\partial y_0}u^0(t_0, y_0, z_0) \text{ and} \tag{29}$$

$$\frac{\partial}{\partial z_0}(u^q(t_q, y_0, z_0)) \to \frac{\partial}{\partial z_0}u^0(t_0, y_0, z_0) \tag{30}$$

as $q \to 0$.

Concentrating on (29), it is seen that

$$\frac{\partial}{\partial y_0} \; u^q(t_q, y_0, z_0) = (u^q)'(t_q, y_0, z_0)\,\frac{\partial t_q}{\partial y_0} + \frac{\partial u^q}{\partial y_0}(t_q, y_0, z_0). \tag{31}$$

Also,

$$(u^q)'(t_q, y_0, z_0) = \left(\frac{1}{1 + z^q(t_q, y_0, z_0)^p} - \alpha x^q, x^q - \beta y^q, y^q - \gamma z^q\right).$$

Since $z^0(t_0, y_0, z_0) < 1$, it follows that

$$(u^q)'(t_q, y_0, z_0) \to (u^0)'(t_0, y_0, z_0).$$

Observe that $t_q(y_0, z_0)$ is defined, locally, by the equation $x^q(t_q, y_0, z_0) = x_c(q)$. Therefore

$$\frac{\partial t_q}{\partial y_0} = -\,\frac{\partial x^q}{\partial y_0}\bigg/\!\left(\frac{1}{1 + z^q(t_q)^p} - \alpha x_c(q)\right). \tag{32}$$

The denominator tends to $1 - \alpha x_c(0) = 1 - \alpha\beta\gamma$. Referring to (31) and (32) it is seen that (29) will be proved if it is shown that

$$W^q(t_q) = \frac{\partial u^0}{\partial y_0}(t_0, y_0, z_0)$$

tends to $W^0(t_0) = \partial y^0/\partial y_0(t_0, y_0, z_0)$ as $q \to 0$.

For any q, $W^q(0) = (0, 1, 0)$. If $q > 0$, then W^q satisfies

$$W' = \Omega W + \psi_q(t)\begin{pmatrix} W_3 \\ 0 \\ 0 \end{pmatrix}, \tag{33}$$

where W_i, $i = 1, 2, 3$ are the components of W, $\psi_q(t) = pz^q(t, y_0, z_{0)}{}^{p-1}/(1 + z^q(t, y_0, z_0)^{p)2}$, and

$$\Omega = \begin{pmatrix} -\alpha & 0 & 0 \\ 1 & -\beta & 0 \\ 0 & 1 & -\alpha \end{pmatrix}.$$

Also, W^0 satisfies $W' = \Omega W$, except at $t = \tau_0$, where $z^0 = 1$.

The function ψ_q acts like a "delta function" near τ_0, and it must be shown that the effect of this is, in the limit, the same as the effect of the discontinuity in $(x^0)'$ at τ_0. Away from τ_0 the analysis is straight forward, because $\psi_q(t) \to 0$ as $q \to 0$, uniformly on any interval $[0, \tau_0 - \delta]$ or $[\tau_0 + \delta, t_0]$.

First, consider the function $W^0 = \partial u^0/\partial y_0(t, y_0, z_0)$. It is necessary to determine

$$\lim_{\epsilon_1, \epsilon_2 \to 0^+} (W^0(\tau_0 + \epsilon_1) - W^0(\tau_0 - \epsilon_2)).$$

This is done by computing with u^0 directly. On $[0, \tau_0)$, $u^0(t, y_0, z_0) = e^{\Omega t}(\beta\gamma, y_0, z_0)$, so

$$\lim_{\epsilon \to 0^+} \frac{\partial u^0}{\partial y_0}(\tau_0 - \epsilon, y_0, z_0) = e^{\Omega\tau_0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{34}$$

On the other hand, for $t > t_0$,

$$(u^0)'(t, y_0, z_0) = \Omega u^0 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this equation, one easily computes that

$$\begin{aligned} \frac{\partial}{\partial y_0} u^0(t, y_0, z_0) = & -\Omega e^{\Omega(t-\tau_0)} u^0(\tau_0 - 0, y_0, z_0) \frac{\partial \tau_0}{\partial y_0} \\ & + e^{\Omega(t-\tau_0)} (u^0)'(\tau_0 - 0, y_0, z_0) \frac{\partial \tau_0}{\partial y_0} \\ & + e^{\Omega(t-\tau_0)} \frac{\partial u_0}{\partial y_0} \tau_0 - 0, y_0, z_0) - e^{\Omega(t-\tau_0)} \frac{\partial \tau_0}{\partial y_0} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \tag{35}$$

But $(u^0)'(\tau_0 - 0, y_0, z_0) = \Omega u_0(\tau_0 - 0, y_0, z_0)$, so the first two terms on the right hand side of (35) cancel.

From the definition of $\tau_0 = \tau_0(y_0, z_0)$ by the equation $z^0(\tau_0, y_0, z_0) = 1$, and (34), one finds in addition that

$$\frac{\partial \tau_0}{\partial y_0}(y_0, z_0) = - \left[e^{\Omega \tau_0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]_3 /(y_1 - \gamma)$$

$$= - W_3{}^0(\tau_0)/(y_1 - \gamma),$$

where $y_1 = y_0(\tau_0, y_0, z_0)$.

In dealing with W^q near τ_q it is necessary to have a uniform a priori bound on $\|W^q\|$, (any norm), over $[0, t_q]$. This can easily be obtained from (33), if an estimate can be found for $\int_0^t \psi_q(s)\, ds$ over this interval. To derive such an estimate, observe that because $\|u^q(t) - u^0(t)\| \to 0$ as $q \to 0$, uniformly in $[0, t_0]$, there are $Q > 0$ and $\delta > 0$ such that

$$|(z^q)'(t, y_0, z_0)| = |y^q(t, y_0, z_0) - \gamma z^q(t, y_0, z_0)|$$
$$> K = \frac{|y_1 - \gamma|}{2} > 0$$

for $0 < q < Q$ and $\tau_0 - \delta \leqq t \leqq \tau_0 + \delta$. On $[0, \tau_0 - \delta]$ and $[\tau_0 + \delta, t_0]$, $\psi_q(t) \to 0$ uniformly as $q \to 0$. Also

$$\int_{-\delta}^{\delta} \psi_q(\tau_0 + s)\, ds \leqq \int_{-\delta}^{\delta} \frac{p(1 + Ks)^{p-1}}{(1 + (1 + KS)^p)^2} ds \tag{36}$$
$$= \frac{1}{1 + (1 - K\delta)^p} - \frac{1}{1 + (1 + K\delta)^p}$$
$$\leqq N$$

for some constant N, as $q \to 0$.

Continuing to choose N as some sufficiently large number independent of q, a bound on $\|W^q\|$ implies, in turn, a bound $\|(W_3{}^q)'\| \leqq N$ on the derivative of the third component of W^q.

Now let $\epsilon > 0$ be given. Integrating (33) gives a relation of the form

$$W_1{}^q(\tau_0 + \delta) - W_1{}^q(\tau_0 - \delta) = h^q(t)$$
$$+ \int_{\tau_0 - \delta}^{\tau_0 + \delta} \psi_q(t) W_3{}^q(t)\, dt.$$

where $|h^q(t)| \leqq N\epsilon$ for some N independent of ϵ, if $0 < q \leqq Q(\epsilon)$, $\delta = \delta(\epsilon) > 0$. Also, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Write the integral as

$$\int_{\tau_0 - \delta}^{\tau_0 + \delta} \psi_q(t)(z^q)'(t) \frac{W_3{}^q(t)}{(z^q)'(t)}\, dt$$

and use the mean value theorem for integrals to express this as

$$\frac{W_3{}^q(t^*)}{(z^q)'(t^*)}\left[\frac{1}{1+z^q(\tau_0-\delta)^p}-\frac{1}{1+z^q(\tau_0+\delta)^p}\right]$$

for some t^* in $[\tau_0-\delta, \tau_0+\delta]$. This enables one to prove the following result.

LEMMA 11. *For each pair* $\delta > 0, \epsilon > 0$, *there is a* $Q > 0$ *such that if* $0 < q < Q$, *then*

$$\left\| W^q(\tau_0+\delta) - W^q(\tau_0-\delta) - \frac{W_3{}^0(\tau_0)}{y_1-\gamma}\begin{pmatrix}1\\0\\0\end{pmatrix}\right\| < \epsilon.$$

Also, if $\tau_0 + \delta \leqq t \leqq t_0$, then

$$W^q(t) \to e^{\Omega(t-\tau_0-\delta)}W^0(\tau_0+\delta)$$

as $q \to \infty$.

To show, as desired, that

$$\lim_{q\to 0}\|W^q(t_q) - W^0(t_0)\| = 0,$$

first choose $\delta > 0$ so that

$$\left\| W^0(\tau_0+\delta) - W^0(\tau^0-\delta) - \frac{W_3{}^0(\tau_0)}{y_1-\gamma}\begin{pmatrix}1\\0\\0\end{pmatrix}\right\|$$

is small. Then apply Lemma 11 and the remark following. I omit further details.

Finally, it is shown that $D\Pi^0(\tilde{y}_0, \tilde{z}_0)$ has eigenvalues λ_1, λ_2 of absolute value less than one. Because solutions of (2) tend to the periodic solution, it follows that $|\lambda_i| \leqq 1$. (It is necessary to use Lemma 1 once again, to show that a solution starting near u_p cannot first tend away from u_p, and then, eventually, approach the closed orbit from a different direction.) Hence the desired "hyperbolicity" of $D\Pi^0$ is equivalent to the local structural stability of the flow near u_p. It must be demonstrated that for q sufficiently small, Π^q has a unique fixed point in a neighborhood of $(\tilde{y}_0, \tilde{z}_0)$ which is independent of q.

The techniques for doing this are largely adaptations of those used earlier. Perhaps the major change is to "perturb" the previous analysis of (2) at the switch points $z = 1$ to nearby planar regions $z = 1 + \mu(x, y) \in M$, and $z = 1 - \mu$, $(1/\alpha - x, 1/(\alpha\beta) - y) \in M$.

To be more precise, let $v_\mu = (x_\mu, y_\mu, z_\mu)$ be the solution of (2) such that

$$x_\mu(0) = \chi_0, y(0) = \eta_0, z(0) = 1 + \mu, \mu > 0. \tag{37}$$

Let $(\chi_1, \eta_1, 1 - \mu)$ be the first subsequent point of intersection of this solution with the plane $z = 1 - \mu$, say at $t = \tau_\mu$, and let the next intersection with $z = 1 + \mu$ be when $t = t_\mu$. By choosing (χ_0, η_0) in a sufficiently small neighborhood $\mathcal{O}$ of (x_0^*, y_0^*), the point (χ_1, η_1) will be close to (x_1^*, y_1^*), for small μ, and the trajectory will be transverse to $z = 1 + \mu$ at (χ_0, η_0) and to $z = 1 - \mu$ at (χ_1, η_1).

For small $\mu > 0$, the conditions (19) and (20), written as in (24) when $\mu = 0$, can be expressed in the form

$$\begin{aligned} G_\mu(\chi_0, \chi_1) &= 0 \\ H_\mu(\chi_0, \chi_1) &= 0 \end{aligned} \tag{38}$$

for certain functions G_μ and H_μ such that

$$\begin{aligned} G_\mu(\chi_0, \chi_1) &\to G(\chi_0, \chi_1) \\ H_\mu(\chi_0, \chi_1) &\to G(1/\alpha - \chi_0, 1/\alpha - \chi_1) \end{aligned}$$

as $\mu \to 0$, uniformly in $\mathcal{O}$. Also, the first partial derivatives of G_μ and H_μ approach the appropriate derivatives of G uniformly in $\mathcal{O}$, as $\mu \to 0$.

Assume again that $G_2(x_0^*, y_0^*) \neq 0$. (Otherwise, interchange x_0 and x_1.) Because $|r'(x_0)| < 1$, and $d/dx_0\, G(x_0, 1/\alpha - x_0)|_{x_0 = x_0^*} \neq 0$, it can be shown that when $\mu > 0$ is sufficiently small, (38) has a unique solution, which is non-degenerate, in a neighborhood of (x_0^*, y_0^*). Furthermore, if $\mu > 0$ is small, then comparison results like Lemma 1 hold for two solutions starting on $z = 1 + \mu$ near (x_0^*, y_0^*) and intersecting $z = 1 - \mu$ at (χ_i, η_1). In fact, one can prove that $\partial\chi_1/\partial\eta_0 < 0$. From this it follows that the uniqueness of the solution (χ_0, χ_1) of (38) near (x_0^*, y_0^*) implies the uniqueness of the periodic solution of (2) near u_p.

Fix $\mu > 0$ small enough for these remarks to apply. Let $v^q(t, \chi_0, \eta_0)$ denote the solution of (1) satisfying the initial conditions (37), and suppose that this solution first intersects $z = 1 - \mu$ at $t = \tau_q$, and $z = 1 + \mu$ at $t = t_q$. Just as was done previously, in analyzing the map Π^q, one can verify that $v^q(t, \chi_0, \eta_0) \to v_\mu(t, \chi_0, \eta_0)$ uniformly on $0 \leqq t \leqq t_0$ and in the neighborhood $\mathcal{O}$ of (x_0^*, y_0^*). Further, the partial derivatives $\partial v^q/\partial\chi_0(\tau_q)$ and $\partial v^q/\partial\eta_0(\tau_q)$ tend to $\partial v_\mu/\partial\chi_0(\tau_0)$ and $\partial v_\mu/\partial\eta_0(\tau_0)$ and similarly, the partial derivatives at $t = t_q$ approach the desired limits. Hence for small $q > 0$ one can conclude that the periodicity conditions (38) undergo a further C^1 perturbation, and the resulting equations still have a locally unique solution for small $q > 0$.

Some final remarks are perhaps in order. The heart of this paper is in the analysis of the discontinuous system (2), more particularly, in analyzing the equations (19) and (20). The symmetry when $\alpha\beta\gamma = 1/2$ results in further simplification, so that it is only necessary to prove one strict derivative inequality ($|r'(x_0^*)| < 1$). This implies that the two curves $G = 0$ and $H = 0$ in the (x_0, x_1) plane defined by (19) and (20) intersect each other non-tangentially at the point (x_0^*, x_1^*). The argument sketched in the last few pages shows that these curves undergo a 0^1-small perturbation for small $\mu > 0$ and $q > 0$, enabling one to avoid the necessity of verifying a second derivative inequality, as long as a neighborhood of the point $(\beta\gamma, \gamma, 1)$ is omitted. This point corresponds to a tangential intersection of the curves $G = 0$ and $H = 0$.

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