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INDUCED REPRESENTATIONS OF GROUPS ON BANACH SPACES

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ABSTRACT. Let H be a closed subgroup of the locally compact group G and L a representation of H on the Banach space E. The notion of induced Banach representation is extended to the case where there exists an "inducing pair" (p, q) for L up to G. In particular, if L is bounded, then (p, p) is such a pair for any p in $[1, \infty]$. We construct an isometric representation U (depending on L, p, q) of G and resolve certain questions pertaining to this induction. We also consider some special cases of particular interest. Finally, we extend the Theorem on Induction-in-Stages and a version of the Frobenius Reciprocity Theorem to the context of inducing pairs.

Introduction. Let G be a locally compact group, H a closed subgroup of G and L a unitary representation of H on the Hilbert space E. In [13] G. W. Mackey constructed an induced unitary representation $U(L)$ of (second countable) G on a certain Hilbert space of functions. Later, R. J. Blattner [1] gave an equivalent construction for arbitrary G. It is natural to try to extend this notion of induced representation to the case where E is a Banach space or just a linear (topological) space and the operators $L(t)$, $t \in H$, are continuous and vary in a suitably smooth fashion. There are some good specific reasons for trying to do this. (1) It is well-known that the process of analytic continuation of Lie group representations forces one to consider Banach space representations and even linear system representations [8]. (2) It is also well-known that it is possible for an "induced" representation to be unitary while the original one L is not. Actually, what happens is that one constructs a bounded representation on a Hilbert space using an induction-like process and starting with a certain (generally unbounded) representation. The Hilbert space is then renormed to yield a unitary representation. This is how one obtains the so-called "complementary series" representations of semi-simple Lie groups (see [12]). Thus, the study of unitary representations itself forces one to consider induction for representations which need not be bounded. (3) In [15] C. C. Moore obtained a

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version of the Frobenius Reciprocity Theorem for isometric representations which involves a non-unitary Banach space induction. Hence, in view of these and other considerations, one is led to the question of what kind of representation should we consider L to be and how should we induce it up to G. In [7, 8] J. M. G. Fell made an extensive study of linear system representations, although he did not consider the inducing problem. This was considered by F. Bruhat [4], M. Rieffel [17], R. Rigelhoff [18] and briefly by Mackey [14]. Inducing in the Banach context was briefly considered by Bruhat in [4]. He assumed L to be dominated by a certain (generally non-constant) function and constructed a Banach representation of G (see § 3). This appears to be the most general construction of its type. In particular, all bounded and some unbounded representations are so dominated. However, this development was not pursued further in [4]. A less general but somewhat more comprehensive study of induced Banach representations was done by H. Kraljević in [11]. He assumed L to be isometric and constructed a family of isometric representations $U^p(L)$ of $G(1 \leqq p < \infty)$. In this setting, he established the Theorem on Inducing-in-Stages and described the "functions" comprising the induced Banach space. Despite these developments, we feel there is a need for considerable further progress in the theory of induced Banach representations, particularly in the case of unbounded L. This is our primary objective here. Inspired by Bruhat, we construct an isometric representation $U_q{}^p(L)$ of G whenever there exists an "inducing pair" (p, q) for L up to G (see 3.1). In particular, if L is bounded, then (p, p) is an inducing pair for each p in $[1, \infty]$. We then study its behavior in some important situations, amongst them Frobenius Reciprocity and Induction-in-Stages.

The remaining sections of this paper are as follows. §§ 1 and 2 are devoted to establishing the fundamental results and notions relating to Banach representations and measures on homogeneous spaces respectively which we shall require in what follows. The main section is § 3. Here we define what it means for the Banach representation L to be inducible up to G via an inducing pair (p, q) and we construct the corresponding induced (isometric) representation $U_q{}^p(L)$ of G on a certain Banach space of functions. The rest of § 3 is devoted to resolving (positively) certain technical questions concerning the behavior of $U_q{}^p(L)$. In particular, we show that our notion of induced Banach representation includes all others and preserves equivalence. In § 4 we consider some special inductions. Specifically, we (a) note a very useful consequence (4.1) of the trivial induction (i.e., $H = G$), (b) show how to construct induced unitary representations from non-unitary ones (4.3) and (c) extend a well-known result connecting inducing and regular representations to the "p-regular" case (4.5). § 5 is devoted to giving a concrete description of the induced Banach space. Any inducing process is expected to satisfy the well-known Theorem on Induction-in-Stages. In § 6 we show this is not quite the case in general; it is if L is bounded. For unbounded L, a certain modification is required. Finally, in § 7 we show that the above-mentioned version of the Frobenius Reciprocity Theorem is also not quite valid in general. If the representation being induced is bounded, then it is. If not, then the same modification is necessary.

Throughout this paper, G will be a locally compact group and H a closed subgroup of G with identity element denoted by e. All Haar measures and coset spaces will be of the right parity. For convenience we will let $Q = G/H$. The real numbers (resp. complex numbers) will be denoted by $\mathbf{R}$ (resp. $\mathbf{C}$). If f is a function defined on G, then fx will denote the right translate of f by x, i.e., $(fx)(y) = f(yx)$, $y \in G$. Similarly, xf will denote the left translate. If X is a locally compact Hausdorff space and E a Banach space, then $Cc(X, E)$ will denote the space of continuous functions from X into E having compact support. If $f: X \to E$ then $\|f\|_X$ will denote $\sup\{\|f(x)\| : x \in X\}$ and $\text{supp}(f)$ the usual support of f. Finally, it will be convenient to set Z denote a fixed compact symmetric neighborhood of e in G.

1. **Banach Space Representations.** In this section we discuss the preliminary ideas we require on the subject of Banach representations and their equivalence.

1.1 Definition. A (Banach) *representation* L of G on the Banach space E is a homomorphism of G into the bounded invertible operators on E which is strongly continuous. If there exists $b \geqq 1$ such that $\|L(x)\| \leqq b$, $x \in G$, then L is (uniformly) *bounded.* If each $L(x)$, $x \in G$, is an isometry, then L is *isometric*; in particular, if E is also a Hilbert space, then L is *unitary.*

Remark. It is worth mentioning that strong continuity of L is equivalent to weak continuity [**16**, p. 25], and if E is separable, these are in turn equivalent to weak measurability [**16**, p. 39].

1.2 Definition. Let L and M be representations of G on the spaces E and F respectively. Then $\text{Hom}_G(L, M)$ will denote the space of bounded operators $T: E \to F$ which intertwine L and M, i.e., T satisfies $M(x)T = TL(x)$, $x \in G$. If there exists an invertible element in $\text{Hom}_G(L, M)$, then L and M will be said to be *equivalent.* In particular, if there exists an invertible isometry in $\text{Hom}_G(L, M)$, then L and M will be *isometrically equivalent.*

1.3 REMARKS. (1) We shall see that isometric equivalence is too strong for our needs in general and that ordinary equivalence is the most suitable choice. In fact, S. Gaal has pointed out to the authors that in the category of Banach representations of G (with morphisms given by $\mathrm{Hom}_G(L, M)$), two representations are isomorphic precisely if they are equivalent.

(2) There exist weaker notions of relatedness for representations, namely Naimark-relatedness [8, § 6; 14, § 8] and "equal kernels", i.e., $\ker(L) = \ker(M)$. However, the former is known to be nontransitive in general, even for isometric representations [8, p. 307] and the latter is too weak to be generally useful. We will have more to say about equivalence in § 3.

The following are easy to verify:

1.4. LEMMA. *If L and M are representations of G, then* $\mathrm{Hom}_G(L, M)$ *is a Banach space relative to the operator norm.*

1.5. LEMMA. *If L_i, M_i are representations of G and L_i is equivalent to M_i, $i = 1, 2$, then* $\mathrm{Hom}_G(L_1, L_2)$ *is homeomorphically isomorphic to* $\mathrm{Hom}_G(M_1, M_2)$.

Before proceeding, we give a few examples of (non-unitary) representations.

1.6. EXAMPLE. Let E be an arbitrary Banach space and T a bounded operator on E. Let L be defined by $L(x) = e^{xT}$, $x \in \mathbf{R}$ [14, § 8]. Then L is a uniformly continuous representation of $\mathbf{R}$ on E which is in general not bounded.

1.7. EXAMPLE. Suppose γ is a non-trivial continuous homomorphism of G into the multiplicative group $\mathbf{R}^+$ of positive reals. If $q \in \mathbf{R}$ ($q \neq 0$), define $L(x) = \gamma(x)^q$, $x \in G$. Then L is also uniformly continuous, but not bounded.

1.8. EXAMPLE. For each $1 \leqq p \leqq \infty$, let E^p denote the usual completion of $\mathrm{Cc}(G)$ relative to $\| \ \|_p$, and define L^p by $L^p(x)f = fx$, $f \in E^p$, $x \in G$. Then L^p is an isometric representation of G on E^p. We call L^p the (right) p-regular representation of G and denote it by $R_G{}^p$ (see [14, § 8]).

2. **Measures on Homogeneous Spaces.** Throughout the rest of this paper we will make extensive use of the measure theory associated with the homogeneous space Q. This section is devoted to establishing the facts we will require. General references for this material are [3, 9].

Let dx (resp. dt) denote a (right) Haar measure for G (resp. H) and Δ_G (resp. Δ_H) the corresponding modular function. Since these quantities will in general be fixed, it will be convenient for us to denote the quotient map $\Delta_H/(\Delta_G \mid H)$ by δ. Note that δ is a continuous homomorphism of H into $\mathbf{R}^+$. It is well-known that there exists a continuous mapping $\rho : G \to (0, \infty)$ satisfying $\rho(tx) = \delta(t)\rho(x)$, $t \in H$, $x \in G$, which may be chosen such that $\rho(e) = 1$, i.e., $\rho \mid H = \delta$. Let $\pi : G \to Q$ be the canonical projection map. If C is a compact subset of Q, then there exists a compact subset C' of G for which $\pi(C') = C$.

Now let ϕ be an arbitrary element of $\mathrm{Cc}(G)$ and define $\phi^H(x) = \int_H \phi(tx)\, dt$, $x \in G$. Then ϕ^H is continuous on G, constant on right H-cosets and has compact support modulo H (i.e., $\pi(\mathrm{supp}(\phi^H))$ is compact). Hence, ϕ^H defines an element of $\mathrm{Cc}(Q)$ and we have a linear mapping $\phi \to \phi^H$ of $\mathrm{Cc}(G)$ *onto* $\mathrm{Cc}(Q)$. The function ρ enables us to define a positive Radon measure $d\bar{x}$ (where $\bar{x} = \pi(x)$) on Q as follows:

$$(*) \qquad \int_Q \int_H \phi(tx)\, dt d\bar{x} = \int_G \phi(x)\rho(x)\, dx, \ \phi \in \mathrm{Cc}(G).$$

This measure is unique up to null equivalence and quasi-invariant with respect to the right action of G on Q. Specifically, we have:

2.1. LEMMA. *If* $x \in G$, *then* $d(\bar{y}\bar{x}) = (\rho(yx)/\rho(y))\, d\bar{y}$, $y \in G$.

The previous double integral formula $(*)$ can be extended as follows:

2.2. LEMMA. *Suppose f is a Haar integrable function on G into the Banach space E. Then $(*)$ is valid for f.*

Now suppose ρ_1, ρ_2 are two ρ-functions on G relative to H with corresponding quasi-invariant measures $d_1\bar{x}, d_2\bar{x}$. Then these measures are null-equivalent; in fact, we have:

2.3. LEMMA. *The function ρ_1/ρ_2 is constant on cosets and* $d_1\bar{x} = (\rho_1(x)/\rho_2(x))\; d_2\bar{x}, x \in G$.

Finally, although the next result has nothing to do with homogeneous spaces, we include it here because it will prove to be extremely useful in the next section.

2.4. PROPOSITION. *Let C be a compact subset of G and suppose $f: G \to E$ is continuous. For $z \in G$ and $\epsilon > 0$, there exists a neighborhood V of e in G such that $V \subseteq Z$, and if $x \in zV$, then* $\|f(yx) - f(yz)\| < \epsilon, y \in C$.

PROOF. Let y be an element of C. Since f is continuous at yz, there exists a neighborhood W_y of yz in G such that $\|f(r) - f(yz)\|$

$< \epsilon/2$, $r \in W_y$. Because multiplication in G is continuous, there exists an open neighborhood U_y of yz (resp. V_y of e) such that $U_y \subseteq W$ (resp. $V_y \subseteq Z$), and $U_y \subseteq U_y V_y \subseteq W_y$. The collection $\{U_y : y \in C\}$ is an open cover of the compact space Cz. Thus, there exist $y_1, \cdots, y_n$ in C such that $Cz \subseteq \bigcup_{i=1}^n U_i$, where $U_i = U_{y_i}$, $1 \leqq i \leqq n$. Let $W_i = W_{y_i}$ and $V_i = V_{y_i}$, $1 \leqq i \leqq n$, and define $V = \bigcap_{i=1}^n V_i$. Then V is an open neighborhood of e in G which is contained in Z. Let $x \in zV$. Then $z^{-1}x \in V$ and $z^{-1}x \in V_i$, $1 \leqq i \leqq n$. If $y \in C$, then $yz \in U_j \subseteq W_j$, for some $1 \leqq j \leqq n$, so that $yx = (yz)\,(z^{-1}x) \in U_j V_j \subseteq W_j$. Hence, $\|f(yx) - f(yz)\| \leqq \|f(yx) - f(y_j z)\| + \|f(y_j z) - f(yz)\| < \epsilon$.

3. **Induced Banach Representations.** Let L be a representation of H on E. Observe that if r is any real number, then $\delta^r L$ is also a representation of H on E.

The most general setting for Banach induction seems to be that considered by Bruhat [4, pp. 132–133]. Suppose there exists k in $[-1/2, 1/2]$ and $b \geqq 1$ satisfying $\|L(t)\| \leqq b\delta(t)^k$, for all t in H, i.e., $\delta^{-k}L$ is bounded. If we let $p = 2/(2k + 1)$, then $1 \leqq p \leqq \infty$ and $k = 1/p - 1/2$. Thus, equivalently, we are supposing there exists p in $[1, \infty]$ for which $\delta^{1/2 - 1/p}L$ is bounded. Given such L and p, Bruhat constructed an isometric representation of G. However, in order to do this when $p = 2$, it is necessary that L be bounded. In view of some of our earlier comments (see Introduction) this is a shortcoming. Hence, we propose the following modification in Bruhat's approach. Suppose there exists q in $(0, \infty]$ and k in $[-1/q, 1 - 1/q]$ such that $\delta^{1/q - 1/p}L$ is bounded, where $p = q/(qk + 1)$. Given such L, p and q we will construct an isometric representation of G. Although this is a slight generalization of Bruhat's setting, it will have some interesting consequences (see 3.5, 3.19, 4.4).

3.1. Definition. An inducing pair for L up to G is a pair (p, q) in $[1, \infty] \times (0, \infty]$ having the property that $\delta^{1/q - 1/p}L$ is bounded, i.e., there exists $b \geqq 1$ such that

$$\delta(t)^{1/q}\|L(t)\| \leqq b\delta(t)^{1/p}, \quad t \in H.$$

If such a pair exists, we say that L is (Banach) *inducible* up to G.

3.2. Remark. We shall see later (4.2) that for most purposes we may assume $\delta(t)^{1/q}\|L(t)\| = \delta(t)^{1/p}$, $t \in H$.

The following are straightforward.

3.3. Properties. (a) *If L is inducible, then L is locally bounded, i.e., $\|L(\,\cdot\,)\|$ is bounded on compact sets.*

(b) *If δ is trivial, then L is inducible if and only if it is bounded.*

(c) *If (p, q) is an inducing pair for L, then*

$$1/b\delta(t)^{1/p} \leqq \delta(t)^{1/q}\|L(t)\| \leqq b\delta(t)^{1/p}, t \in H.$$

(d) *If L is bounded, then (p, p) is an inducing pair for each p in* $[1, \infty]$.

(e) *If (p, q) is an inducing pair for L, then $p = q$ if and only if L is bounded.*

(f) *The representation L is "Bruhat inducible" (as above) if and only if there exists p in* $[1, \infty]$ *such that $(p, 2)$ is an inducing pair for L in which case $k = (2 - p)/2p$.*

3.4. Example. If $G = \mathbf{R} \times \mathbf{R}$, $H = \{0\} \times \mathbf{R}$, $E = \mathbf{C}$ and L is defined by $L(0, b) = e^b$, $b \in \mathbf{R}$, then δ is trivial and L is not bounded. Hence, L is not inducible up to G.

3.5. Example. On the other hand, suppose H and L are as in 3.4, but G is the "$ax + b$" group, i.e., $G = \mathbf{R} \times \mathbf{R}$ with multiplication given by $(a, b)(c, d) = (a + e^b c, b + d)$, a, b, c, d in $\mathbf{R}$. Then $\Delta_G(a, b) = e^b$, $\delta(0, b) = e^{-b}$ and $(\infty, 1)$ is an inducing pair for L up to G. Thus, L is inducible in our sense but it is not Bruhat-inducible.

Remark. From these examples, it follows that:

(i) The overgroup G does affect the inducibility of L, particularly if L is unbounded.

(ii) In general, p and q need not be equal.

Fix an inducing pair (p, q) for L up to G. Let b be as in 3.1. We now construct an induced isometric representation of G corresponding to (p, q). For convenience, denote $\delta^{1/q}L$ by L_q.

3.6. Definition. A function $f : G \rightarrow E$ (L, q)-homogeneous (see [11]) if $f(tx) = L_q(t)f(x)$, $t \in H$, $x \in G$. Let $C_q(G, L)$ denote the linear space of all (L, q)-homogeneous functions on G which are continuous and have compact support modulo H. Now for each f in $\mathrm{Cc}(G, E)$ and x in G, note that the function $t \rightarrow L_q(t^{-1})f(tx)$ belongs to $\mathrm{Cc}(H, E)$. Hence, we may define a function $f_q{}^L : G \rightarrow E$ by

$$f_q{}^L(x) = \int_H L_q(t^{-1})f(tx)\, dt, x \in G.$$

3.7. Lemma. *The mapping $f \rightarrow f_q{}^L$* is a linear mapping of $\mathrm{Cc}(G, E)$ onto $C_q(G, L)$.

Proof. The function $f_q{}^L$ is easily seen to be (L, q)-homogeneous. Fix x in G and $\epsilon > 0$. If $y \in xZ$, then the support of $(fy) \mid H$ is contained in the compact subset $\mathrm{supp}(f)Zx^{-1} \cap H$ of H which we denote by C. Let $a = \max\{\delta(t^{-1})^{1/p} : t \in C\}$ and $c = \int_C dt$. Since f is uniformly continuous, there exists a neighborhood V of e in G such that $V \subseteq Z$ and $\|f(ty) - f(tx)\| < \epsilon/abc$, whenever $x^{-1}y \in V$. Therefore, if $y \in xV \subseteq xZ$, then

$$\begin{aligned}
\|f_q{}^L(y) - f_q{}^L(x)\| &= \left\| \int_H L_q(t^{-1})(f(ty) - f(tx))\, dt \right\| \\
&\leqq \int_C \|L_q(t^{-1})\| \, \|f(ty) - f(tx)\| \, dt \\
&\leqq \epsilon/ac \int_C \delta^{1/p}(t^{-1})\, dt \\
&< \epsilon,
\end{aligned}$$

so that $f_q{}^L$ is continuous. If $x \notin H\,\mathrm{supp}(f)$, then $f(tx) = 0$, $t \in H$. Thus, $\mathrm{supp}(f_q{}^L) \subseteq H\ \mathrm{supp}(f)$, so that $f_q{}^L \in C_q(G, L)$.

Since the mapping $f \to f_q{}^L$ is linear, it remains to show it is surjective. However, this is done as on pp. 184–185 of [11].

Now let $\phi \in Cc(G)$ and $v \in E$ and define $(\phi \otimes v)(x) = \phi(x)v$, $x \in G$. Then $(\phi \otimes v) \in Cc(G, E)$ and we may form $(\phi \otimes v)_q{}^L$. Let $T_q(G, L)$ denote the subset of $C_q(G, L)$ consisting of such functions.

3.8. Lemma. *For each x in G, the subset $\{f(x) : f \in T_q(G, L)\}$ is dense in E.*

Proof. This is proved as in the case of Theorem 1(c) of [11].

Next we introduce a p-norm on $C_q(G, L)$. Let f be an element of $C_q(G, L)$. For each x in G, the function

$$t \to \rho(x)^{-1/p}\delta(t)^{-1/p}\|f(tx)\|$$

is bounded on H. Thus, as in [4] we define:

$$N_p(f, x) = \rho(x)^{-1/p}\sup\{\delta(t)^{-1/p}\|f(tx)\| : t \in H\}, x \in G.$$

3.9. Lemma. *The function $N_p(f, \cdot)$ is bounded, constant on cosets, vanishes outside $H\,\mathrm{supp}(f)$ (i.e., has compact support modulo H) and is lower semi-continuous.*

Proof. Straightforward.

Consequently, for each f in $C_q(G, L)$ we may define:

$$N_p(f) = \left(\int_Q N_p(f, x)^p \, d\bar{x} \right)^{1/p}, \quad 1 \leqq p < \infty,$$

and

$$N_\infty(f) = \sup\{N_\infty(f, x) : x \in G\}.$$

3.10. Lemma. *N_p is a norm on $C_q(G, L)$.*

Proof. Straightforward.

3.11. Remark. Note that for $p = \infty$, N_p does not depend on the choice of ρ. Also in this case, the representation L_q is bounded. Thus, for each f in $C_q(G, L)$, the function $\|f(\,\cdot\,)\|$ is bounded; in fact, $\|f\|_G = N_\infty(f)$.

3.12. Lemma. *Suppose L is isometric, so that for each $1 \leqq p < \infty$, (p, p) is an inducing pair for L up to G. Then for each f in $C_p(G, L)$, we have:*

$$N_p(f) = \left(\int_G \|f(x)\|^p \psi(x)\, dx \right)^{1/p},$$

where ψ is any non-negative element of Cc(G) *satisfying $\psi^H(x) = 1$, $x \in \operatorname{supp}(f)$* (see [6]).

Proof. In this case we have $N_p(f, x) = \rho(x)^{-1/p}\|f(x)\|$, so that

$$\begin{aligned} N_p(f)^p &= \int_Q \rho(x)^{-1}\|f(x)\|^p\, d\bar{x} \\ &= \int_Q \psi^H(x)\|f(x)\|^p \rho(x)^{-1}\, d\bar{x} \\ &= \int_Q \int_H \psi(tx)\|f(tx)\|^p \rho(tx)^{-1}\, dt d\bar{x} \\ &= \int_G \psi(x)\|f(x)\|^p\, dx, \end{aligned}$$

by 2.2.

Let $F_q{}^p(G, L)$ denote the completion of $C_q(G, L)$ with respect to the norm N_p. We will describe $F_q{}^p(G, L)$ specifically in § 5. Observe that $C_q(G, L)$ is closed under right translation.

3.13. Lemma. *For each x in G, the mapping $f \to fx$ is an onto isometry of $C_q(G, L)$ relative to the norm N_p and thus passes to an onto isometry of $F_q{}^p(G, L)$ which we denote by $U_q{}^p(L)(x)$.*

Proof. If $p = \infty$, then the proof is clear. Suppose $p < \infty$. Then

$$N_p(fx, y) = (\rho(yx)/\rho(y))^{1/p} N_p(f, yx)$$

so that

$$
\begin{aligned}
N_p(fx)^p &= \int_Q N_p(f, yx)^p(\rho(yx)/\rho(y))\, d\bar{y} \\
&= \int_Q N_p(f, yx)^p\, d\overline{yx} \qquad\qquad \text{(see 2.1)} \\
&= N_p(f)^p, \qquad f \in C_q(G, L).
\end{aligned}
$$

3.14. THEOREM. *If (p, q) is an inducing pair for L up to G, then $U_q{}^p(L)$ is an isometric representation of G on the Banach space $F_q{}^p(G, L)$.*

PROOF. It is clear that $U_q{}^p(L)$ is a group homomorphism. Thus, we need only show it is strongly continuous. To do this, it suffices to show that if $f \in C_q(G, L)$, then $N_p(fx - f) \to 0$ as $x \to e$. Fix f in $C_q(G, L)$ and note that $H\,\mathrm{supp}(f)Z$ is a compact subset of G modulo H. Let C be a compact subset of G such that $H\,\mathrm{supp}(f)Z = HC$. Since we may assume $x \in Z$, we see that $\mathrm{supp}(fx - f) \subseteq HC$. Now let $\epsilon > 0$ and first assume $p = \infty$, so that $\|L_q(t)\| \leqq b$, $t \in H$, for some $b \geqq 1$. Then by 3.11

$$
\begin{aligned}
N_\infty(fx - f) &= \sup\{\|f(tyx) - f(ty)\| : t \in H, y \in C\} \\
&\leqq \sup\{\|L_q(t)\|\ \|f(yx) - f(y)\| : t \in H, y \in C\} \\
&\leqq b \sup\{\|f(yx) - f(y)\| : y \in C\}.
\end{aligned}
$$

By 2.4, there exists a neighborhood V of e in G such that $V \subseteq Z$ and $\|f(yx) - f(y)\| < \epsilon/b$, for all y in C, provided $x \in V$. This completes the proof for the case $p = \infty$.

Now assume $p < \infty$. Then

$$
\begin{aligned}
N_p(fx - f)^p &= \int_Q N_p(fx - f, y)^p\, d\bar{y} \\
&\leqq \int_Q \rho(y)^{-1} \sup\{\delta(t^{-1})\|L_q(t)\|^p : t \in H\} \\
&\qquad \cdot \|f(yx) - f(y)\|^p\, d\bar{y} \\
&\leqq ab^p \int_{\pi(C)} \|f(yx) - f(y)\|^p\, d\bar{y},
\end{aligned}
$$

where $a = \min\{\rho(y) : y \in C\}$. Once again by 2.4, there exists a neighborhood V of e in G such that $V \subseteq Z$ and $\|f(yx) - f(y)\| < \epsilon/b(ac)^{1/p}$, where $c = \int_{\pi(C)} d\bar{x}$, as long as $x \in V$. In this event, $N_p(fx - f) < \epsilon$, and the proof is complete.

3.15. Notation. In § 6 we will have occasion to use the more informative notation $U_q{}^p(L : H, G)$ in place of $U_q{}^p(L)$. However, in the opposite direction, if $p = q$, then we will write $U^p(L)$ and $F^p(G, L)$ for $U_p{}^p(L)$ and $F_p{}^p(G, L)$ respectively.

We saw earlier (3.11) that the definition of $U_q{}^p(L)$ depends on ρ for $p < \infty$ only. However, we shall now show that (up to isometric equivalence), $U_q{}^p(L)$ is independent of the choice of ρ. Let ρ_1, ρ_2 be two ρ-functions on G relative to H and (p, q) an inducing pair for L up to G. We will use the subscript $i = 1, 2$ to designate the ρ-function being used in what follows.

3.16. Lemma. *The representations* $U_q{}^p(L)_i$, $i = 1, 2$, *are isometrically equivalent.*

Proof. The space $C_q(G, L)$ is dense in $F_q{}^p(G, L)_i$, $i = 1, 2$. If $f \in C_q(G, L)$, then

$$N_p(f, x)_i = \rho_i(x)^{-1} \sup\{\delta(t^{-1})\|f(tx)\|^p : t \in H\}, x \in G,$$

and by 2.3 we have:

$$\begin{aligned} N_p(f)_1{}^p &= \int_Q \sup\{\delta(t^{-1})\|f(tx)\|^p : t \in H\}\rho_1(x)^{-1}\, d_1\bar{x} \\ &= \int_Q \sup\{\delta(t^{-1})\|f(t)\|^p : t \in H\}\rho_2(x)^{-1}\, d_2\bar{x} \\ &= N_p(f)_2{}^p. \end{aligned}$$

Therefore, the identity mapping on $C_q(G, L)$ is an isometry which intertwines the two representations.

Suppose now that L is an isometric representation of H on E. For each $1 \leqq p < \infty$, Kraljević [11] constructed an induced (isometric) representation $V^p(L)$ of G. We show that $U^p(L)$ and $V^p(L)$ are isometrically equivalent. Kraljević considered the space $C_\infty(G, L)$ with norm given by

$$\|f\|_p = \left(\int_Q \|f(x)\|^p\, d\bar{x}\right)^{1/p}, f \in C_\infty(G, L).$$

(Note that the function $x \rightarrow \|f(x)\|$ is constant on cosets.) Let $E^p(G, L)$ denote the Banach space completion of $C_\infty(G, L)$ relative to this norm. Then $V^p(L)$ is given by $V^p(L)(x)f = (\rho x/\rho)^{1/p} fx$, $x \in G$, $f \in C_\infty(G, L)$. (Note also that Kraljević's Haar measures are left invariant.)

3.17. Lemma. *For each* $1 \leqq p < \infty$, $V^p(L)$ *is isometrically equivalent to* $U^p(L)$.

PROOF. For each f in $C_\infty(G, L)$, define $Tf = \rho^{1/p}f$. Then T is a linear mapping of $C_\infty(G, L)$ onto $C_p(G, L)$. Furthermore, $\|f(tx)\| = \|f(x)\|$, so that

$$
\begin{aligned}
N_p(Tf)^p &= \int_Q N_p(Tf, x)^p \, d\bar{x} \\
&= \int_Q \sup\{\|f(tx)\|^p : t \in H\} \, d\bar{x} \\
&= \int_Q \|f(x)\|^p \, d\bar{x} \\
&= \|f\|_p{}^p.
\end{aligned}
$$

Thus, T is an isometry. Finally,

$$
\begin{aligned}
U^p(L)(x)Tf &= (\rho^{1/p}f)x \\
&= \rho^{1/p}(\rho x/\rho)^{1/p}f \\
&= TV^p(L)\,(x)f, f \in C_\infty(G, L),
\end{aligned}
$$

which completes the proof.

Suppose L is a unitary representation of H on the Hilbert space E. Then $(2, 2)$ is an inducing pair for L up to G and $U^2(L)$ is a unitary representation of G on the Hilbert space $F^2(G, L)$ which is precisely the same as that constructed by Blattner [1].

We saw in 3.3(f) that a representation L of H is Bruhat-inducible if and only if there exists an inducing pair for L of the form (p, 2). We have the following "converse".

3.18. PROPOSITION. *Suppose (p, q) is an inducing pair for L. Then $(p, 2)$ is an inducing pair for the representation $\delta^{1/q - 1/2}L$ of H and $U_2{}^p(\delta^{1/q-1/2}L) = U_q{}^p(L)$.*

PROOF. The first part is clear. The second part is also clear once we observe that $C_2(G, \delta^{1/q-1/2}L) = C_q(G, L)$.

3.19. REMARK. It is important to note here that the representations $\delta^{1/q-1/2}L$ and L are quite different in general. In particular, they need not even have equal kernels (e.g., let $L = \delta^{1/2-1/q}$).

Now let us turn to the question of how equivalent representations behave under our inducing process.

3.20. PROPOSITION. *Let L and M be representations of H on the Banach spaces E and F respectively. Suppose (p, q) is an inducing pair for both L and M. up to G. If L is equivalent (resp. isometrically equivalent) to M, then $U_q{}^p(L)$ is equivalent (resp. isometrically equivalent) to $U_q{}^p(M)$.*

PROOF. Let $T: E \to F$ be a bounded invertible operator which intertwines L and M. For each f in $C_q(G, L)$, define $T_0(f) = T \circ f$. Then T_0 is a linear isomorphism of $C_q(G, L)$ onto $C_q(G, M)$ which commutes with right translation on these spaces. Hence, we need only see that T_0 is bounded. For each $1 \leqq p \leqq \infty$, we have

$$N_p(T_0(f)) \leqq \|T\| N_p(f)$$

and

$$N_p(f) \leqq \|T^{-1}\| N_p(T_0(f)), f \in C_q(G, L).$$

Therefore, $\|T_0\| \leqq \|T\|$, and T_0 is an isometry if T is.

4. **Some Special Inductions.** In this section we discuss three very important kinds of induced Banach representation.

Suppose $H = G$. In order for a representation L of H to be inducible "up to G," it is necessary that L be uniformly bounded. Assume this is the case, so that each (p, p) $(1 \leqq p \leqq \infty)$ is an inducing pair for this trivial induction of L. Then $C_p(G, L)$ consists of functions $f_v: G \to E$ of the form $f_v(x) = L(x)v$, $x \in G$, $v \in E$. Thus, $C_p(G, L)$ is linearly isomorphic to E under the correspondence $f_v \leftrightarrow v$ and is independent of p. Furthermore, $N_p(f_v) = \sup\{\|L(x)v\| : x \in G\} \geqq \|v\|$, so that $C_p(G, L)$ is complete relative to the norm N_p (which is also independent of p). Let E_L denote the Banach space obtained by giving E the norm $\| \ \|_L$ defined by $\|v\|_L = \sup\{\|L(x)v\| : x \in G\}$, so that $\|v\| \leqq \|v\|_L \leqq \|L\| \, \|v\|$, $v \in E$. Denote the induced representation $U^p(L)$ by L', and observe that $L'(x)f_v$ corresponds to $L(x)v$. We have thus obtained the following well-known result (see [17, 3.1], for example):

4.1. PROPOSITION. *If L is a uniformly bounded representation of G on E, then:*

(i) *There exists a larger norm $\| \ \|_L$ on E which makes it a Banach space E_L.*

(ii) *L becomes an isometric representation L' of G on E_L which is equivalent to L.*

This result allows us to make a useful simplification in the definition of inducing pair.

4.2. PROPOSITION. *If (p, q) is an inducing pair for L up to G, then there exists a representation L' of H such that:*

(i) *L' is euqivalent to L.*

(ii) $\delta(t)^{1/q}\|L'(t)\| = \delta(t)^{1/p}, t \in H$.

(iii) *$U_q{}^p(L')$ is equivalent (perhaps not isometrically) to $U_q{}^p(L)$.*

Proof. By definition of inducing pair, the representation $M = \delta^{1/q - 1/p} L$ is bounded. Thus, by 4.1, there exists an isometric representation M' of H which is equivalent to M. Consequently, $L' = \delta^{1/p - 1/q} M'$ is equivalent to L, where $\|L'(t)\| = \delta(t)^{1/p - 1/q}$, $t \in H$. Now apply 3.20.

Next we consider the question of when $U_p{}^q(L)$ is essentially unitary.

4.3. Proposition. *If L is a representation of H and q in* $(0, \infty]$ *is such that* $\delta^{1/q - 1/2}L$ *is equivalent to a unitary representation of H, then* $(2, q)$ *is an inducing pair for L and* $U_q{}^2(L)$ *is equivalent to a unitary representation of G.*

Proof. Let M be a unitary representation of H which is equivalent to $N = \delta^{1/q - 1/2}L$. Then N is bounded, so that $(2, q)$ is an inducing pair for L. But (2.2) is an inducing pair for N and $U^2(N) = U_q{}^2(L)$ (3.18). Furthermore, $U^2(M)$ is equivalent to $U^2(N)$ (3.20), which completes the proof.

4.4. Remarks. (a) We shall see at another time that the situation in 4.3 is essentially what it takes for $U_q{}^p(L)$ to "be" unitary in general. Thus, roughly speaking, whenever $U_q{}^p(L)$ is unitary and L isn't, there exists a unitary representation M of H for which $U^2(M)$ is equivalent to $U_q{}^p(L)$. However, M can be very different from L. (b) Although $U_q{}^2(L)$ is essentially unitary, L need not even be bounded (e.g., $L = \delta^{1/2 - 1/q}$). This can't happen in Bruhat's setting. (c) The connection between the above and the existence of "complementary series" representations (see [12] for example) is unclear at this time.

Finally, suppose H is normal in G, so that δ is trivial. For each p in $[1, \infty]$, $C_p(G, 1_H)$ is precisely equal to $\{\phi \circ \pi : \phi \in \mathrm{Cc}(Q)\}$. Also, for each $1 \leqq p \leqq \infty$, $N_p(\phi \circ \pi) = \|\phi\|_p$, $\phi \in \mathrm{Cc}(Q)$. Therefore, the mapping $\phi \to \phi \circ \pi$ passes to an isometry of $L^p(Q)$ onto $F^p(G, 1_H)$ which intertwines $R_Q{}^p \circ \pi$ and $U^p(1_H)$, $1 \leqq p \leqq \infty$. Consequently:

4.5. Proposition. *If H is normal in G, then for each* $1 \leqq p \leqq \infty$, $U^p(1_H)$ *is isometrically equivalent to* $R_Q{}^p \circ \pi$. *In particular, if* $H = \{e\}$, *then* $U^p(1_H) = R_G{}^p$.

5. **The Induced Banach Space.** Let (p, q) be an inducing pair for L up to G. In § 3, we defined $F_q{}^p(G, L)$ to be the abstract completion of $C_q(G, L)$ relative to the norm N_p. Our objective here is to give a concrete realization of $F_q{}^p(G, L)$ along traditional lines (as in [2]). This was partially done in [11] for the isometric case. Consequently, we will leave most of the details for the interested reader to verify.

Let $\mathcal{E}_q(G, L)$ denote the linear space of all (L, q)-homogeneous function $f: G \to E$. For each such f, define $N_p(f, \cdot\,)$ as in § 3.

5.1. LEMMA. (i) *The function* $N_p(f, \cdot\,)$ *is constant on cosets and non-negative (possibly infinity-valued).*

(ii) *For each* x *in* G, $N_p(f, x) = 0$ *if and only if* $f(tx) = 0$, *all* t *in* H.

First consider the case where $p = v$. Let $\mathcal{V}_q{}^\infty(G, L)$ denote the subset of $\mathcal{E}_q(G, L)$ consisting of continuous functions f which vanish at ∞ modulo H, i.e., given $\epsilon > 0$, there exists a compact subset C of G such that $\|f(x)\| < \epsilon$, for x not in HC.

5.2. LEMMA. (i) *If* $f \in \mathcal{V}_q{}^\infty(G, L)$, *then* f *is bounded.*

(ii) *If* $f \in \mathcal{E}_q(G, L)$, *then* $f \in \mathcal{V}_q{}^\infty(G, L)$ *if and only if* $N_\infty(f, \cdot\,)$ *vanishes at* ∞ *as a function of* Q.

PROOF. (i) follows from the fact that $\delta^{1/q}L$ is bounded and (ii) from 3.11.

The space $\mathcal{V}_q{}^\infty(G, L)$ is linear and can be normed equally by $\| \;\|_G$ (same as N_∞; see 3.11). Relative to $\| \;\|_G$, the space $\mathcal{V}_q{}^\infty(G, L)$ can be shown to be closed in the Banach space of continuous bounded functions $f: G \to E$. Consequently, $(\mathcal{V}_q{}^\infty(G, L), N_\infty)$ is a Banach space in which $C_q(G, L)$ is dense. Therefore:

5.3. PROPOSITION. *The space* $F_q{}^\infty(G, L)$ *is isometrically isomorphic to* $(\mathcal{V}_q{}^\infty(F, L), N_\infty)$.

Now suppose $1 \leqq p < \infty$ and for f in $\mathcal{E}_q(G, L)$, define

$$N_p(f) = \left(\int_Q^* N_p(f, x)^p \, d\bar{x} \right)^{1/p}.$$

Also, let

$$\mathcal{E}_q{}^p(G, L) = \{f \in \mathcal{E}_q(G, L)\,; N_p(f) < \infty\},$$

and

$$\mathcal{N}_q{}^p(G, L) = \{f \in \mathcal{E}_q(G, L) : N_p(f) = 0\}.$$

5.4. LEMMA. (i) *The function* N_p *is a semi-norm on the linear space* $\mathcal{E}_q{}^p(G, L)$.

(ii) *The space* $\mathcal{E}_q{}^p(G, L)$ *is complete relative to* N_p.

(iii) *The space* $\mathcal{N}_q{}^p(G, L)$ *is the closure of* $\{0\}$ *in* $\mathcal{E}_q{}^p(G, L)$.

5.5. LEMMA. $\mathcal{N}_q{}^p(G, L)$ *is the linear subspace of* $\mathcal{E}_q{}^p(G, L)$ *consisting of functions* f *having the property that* $N_p(f, \cdot\,) = 0$, *almost everywhere on* Q.

5.6. LEMMA. *The space* $C_q(G, L)$ *is contained in* $\mathcal{E}_q{}^p(G, L)$, *and* $C_q(G, L) \cap \mathcal{N}_a{}^p(G, L) = \{0\}$.

PROOF. If $f \in C_q(G, L)$ and $N_p(f) = 0$, then by 3.9 and 5.5 it follows that $N_p(f, \cdot)$ must be identically 0. Now apply 5.1.

Next let $\mathcal{F}_q{}^p(G, L)$ denote the closure of $C_q(G, L)$ in $\mathcal{E}_q{}^p(G, L)$ so that $\mathcal{N}_q{}^p(G, L) \subseteq \mathcal{F}_q{}^p(G, L)$. As usual, define $E_q{}^p(G, L) = \mathcal{E}_q{}^p(G, L)/\mathcal{N}_q{}^p(G, L)$, which is then a Banach space with norm given by $N_p(f + \mathcal{N}_q{}^p(G, L)) = N_p(f)$. By 5.6, we see that $F_q{}^p(G, L)$ is (isometrically isomorphic to) $\mathcal{F}_q{}^p(G, L)/\mathcal{N}_q{}^p(G, L)$. Therefore, we will be finished once we (i) obtain a more satisfactory description of the elements of $\mathcal{N}_q{}^p(G, L)$ and (ii) determine the elements of $\mathcal{E}_q{}^p(G, L)$ which belong to $\mathcal{F}_q{}^p(G, L)$.

As a first response, we have the following:

5.7 LEMMA. *If* $f \in \mathcal{F}_q{}^p(G, L)$, *then* f *is Haar-measurable (in the sense of Bourbaki).*

PROOF. This is proved as in [**11**].

In order to settle (i) and (ii), we need the following. For convenience, let ν denote the right Haar measure dx on G. Let β be a Bruhat function for G and H [3, pp. 57–58]. Define a positive Radon measure ν_H on G as follows:

$$\int_G f(x)\, d\nu_H(x) = \int_G f(x)\rho(x)\beta(x)\, d\nu(x), f \in \mathrm{Cc}(G).$$

Of course, ν_H depends on the particular choice of ρ and β.

5.8 DEFINITION. A function $f: G \to E$ is *Haar-measurable modulo* H if given $\epsilon > 0$ and compact $K \subseteq Q$, there exists a compact subset K_ϵ of K such that $\nu_H(\pi^{-1}(K - K_\epsilon)) < \epsilon$ and $f \mid \pi^{-1}(K_\epsilon)$ is continuous (compare with [**2**, p. 169]. The function f is Haar-null modulo H if it is ν_H-null. These definitions are justified by the fact that although the conditions depend on H, they are independent of the particular choice of ρ and β.

Since we have not had the opportunity here to develop the machinery necessary to prove the next theorem, we state it here without proof and will return to it elsewhere.

5.9 THEOREM. *The space* $\mathcal{F}_q{}^p(G, L)$ *consists precisely of those Haar-measurable functions* $f: G \to E$ *satisfying*:

(i) f *is Haar-measurable modulo* H.

(ii) f *is* $(L. q)$*-homogeneous.*

(iii) $N_p(f) < \infty$.

Moreover, $\mathcal{N}_q{}^p(G, L)$ is the subspace of $\mathcal{F}_q{}^p(G, L)$ consisting of those f which are Haar-null modulo H.

There is a particular case which is worthy of note.

5.10 THEOREM. *Suppose G is a second-countable. If L is an isometric representation of H on the separable Banach space E, then for each p in $[1, \infty)$, the space $F_q{}^p(G, L)$ is isometrically isomorphic to $L^p(Q, E)$.*

PROOF. This is proved in the more-or-less usual way, making use of the existence of a Borel cross-section from Q into G.

Since we have not had the opportunity here to develop the machinery necessary to prove the next theorem, we state it here without proof and will return to it elsewhere.

5.9. THEOREM. *The space $\mathcal{F}_q{}^p(G, L)$ consists precisely of those Haar-measurable functions $f : G \rightarrow E$ satisfying:*

(i) *f is Harr-measurable modulo H.*

(ii) *f is (L, q)-homogeneous.*

(iii) $N_p(f) < \infty$.

There is a particular case which is worthy of note.

5.10. THEOREM. *Suppose G is second-countable. If L is an isometric representation of H on the separable Banach space E., then for each p in $[1, \infty)$, the space $F_q{}^p(G, L)$ is isometrically isomorphic to $L^p(Q, E)$.*

PROOF. This is proved in the more-or-less usual way, making use of the existence of a Borel cross-section from Q into G.

6. **Induction in Stages.** Suppose K is a closed subgroup of G containing H. If L is a representation of H on E which is inducible up to K, then the resulting (isometric) representation of K is certainly inducible up to G. If, in addition, L is inducible directly up to G, then it is natural to expect that the two induced representations of G are equivalent. This is known to be the case for isometric representations [11]; hence, it's true for bounded representations as a consequence of 3.20 and 4.1. However, in view of some of our comments and results, it is desirable to have a result of this type for unbounded representations as well, i.e., for the case $p \neq q$. This is our purpose here. We shall see that "induction-in-stages" holds in the setting of inducing pairs if appropriately modified (6.3). First however, we require some preliminary facts.

Suppose that (p, q) is an inducing pair for L up to G with b as in 3.1. Let f be an element of $Cc(G, E)$ and C a compact subset of G containing $\mathrm{supp}(f)$. Let μ (resp. ν) denote the measure of $CC^{-1} \cap H$ (resp. $\pi(C)$) in H (resp. Q). Define

$$\alpha = \mu b \sup\{\delta(t^{-1})^{1/p} : t \in CC^{-1} \cap H\}$$

and

$$\beta = \nu^{1/p} b \sup\{\rho(x)^{-1} : x \in C\}^{1/p}.$$

6.1. Lemma. *If $p < \infty$, then $N_p(f_q^L) \leqq \alpha\beta\|f\|_G$. If $p = \infty$, then $N_\infty(f_q^L) \leqq \alpha b\|f\|_G$.*

Proof. We first show that $\|f_q^L\|_C \leqq \alpha\|f\|_G$. If $x \in C$, then

$$\begin{aligned} \|f_q^L(x)\| &= \left\| \int_H L_q(t^{-1})f(tx)\, dt \right\| \\ &\leqq \int_{Cx^{-1}\cap H} \|L_q(t^{-1})\|\, \|f(tx)\|\, dt \\ &\leqq b\int_{CC^{-1}\cap H} \delta^{1/p}(t^{-1})\|f(tx)\|\, dt \\ &\leqq \alpha\|f\|_G. \end{aligned}$$

Suppose that $p < \infty$. Noting that $\mathrm{supp}(f_q^L) \subseteq HC$ we have:

$$\begin{aligned} N_p(f_q^L)^p &= \int_Q \rho(x)^{-1}\sup\{\delta(t)^{-1}\|f_q^L(tx)\|^p : t \in H\}\, d\bar{x} \\ &\leqq \int_{\pi(C)} \rho(x)^{-1}\sup\{\delta(t^{-1})\|L_q(t)\|^p : t \in H\}\|f_q^L(x)\|^p\, d\bar{x} \\ &\leqq b^p \int_{\pi(C)} \rho(x^{-1})\|f_q^L(x)\|^p\, d\bar{x} \\ &\leqq (\alpha\beta\|f\|_G)^p, \end{aligned}$$

which completes the case $p < \infty$. If $p = \infty$, then

$$\begin{aligned} N_\infty(f_q^L) &= \sup\{\|f_q^L(tx)\| : t \in H, x \in C\} \\ &\leqq \sup\{\|L_q(t)\|\, \| f_q^L(x) : t \in H, x \in C\} \\ &\leqq \mathrm{b}\|f_q^L\|_C \\ &\leqq \alpha b\|f\|_G. \end{aligned}$$

6.2. Proposition. *If D is total in E, then $\{(\phi \oplus v_q^L.\ \phi \in Cc(G), v \in D\}$ is total in $F_q^p(G, L)$.*

PROOF. Since $C_q(G, L)$ is dense in $F_q{}^p(G, L)$, it suffices to show the given subset is total in $C_q(G, L)$. Let $f \in Cc(G, E)$ so that $f_q{}^L$ is a typical element of $C_q(G, L)$ (see 3.7). Suppose $\epsilon > 0$ and let C be a compact subset of G whose interior contains the support of f. It is well-known that the span of $Cc(G) \otimes E$ is sup-norm dense in $Cc(G, E)$. Consequently, there exists $\phi_1, \cdots, \phi_n$ in $Cc(G)$ and $v_1, \cdots, v_n$ in E such that $\|f - \sum_{i=1}^n \phi_i \otimes v_i\|_G < \epsilon/2\alpha\beta$, with α, β as in 6.1. Let $w_1, \cdots, w_n$ in D be such that $\|v_i - w_i\| < \epsilon/2nc\alpha\beta$, $1 \leqq i \leqq n$, where $c = \max\{\|\phi_i\|_G : 1 \leqq i \leqq n\}$. Then

$$\begin{aligned}\left\| f(x) - \sum_{i=1}^n (\phi_i \otimes w_i)(x) \right\| &\leqq \left\| f(x) - \sum_{i=1}^n \phi_i(x) v_i \right\| \\ &\quad + \left\| \sum_{i=1}^n \phi_i(x)(v_i - w_i) \right\| \\ &< \epsilon/2\alpha\beta + \sum_{i=1}^n |\phi_i(x)|\, \|v_i - w_i\| \\ &< \epsilon/2\alpha\beta + c \sum_{i=1}^n \|v_i - w_i\| \\ &< \epsilon/\alpha\beta,\end{aligned}$$

so that $\|f - \sum_{i=1}^n \phi_i \otimes w_i\|_G < \epsilon/\alpha\beta$. Now let $\psi : G \to [0, 1]$ be continuous such that the support of ψ is contained in the interior of C and $\psi = 1$ on $\mathrm{supp}(f)$. Define $\psi_i = \psi\phi_i$, so that $\psi_i \in Cc(G)$, $\mathrm{supp}(\psi_i) \subseteq C$ (i.e., $\mathrm{supp}(\psi_i \otimes w_i) \subseteq C$) and $\psi_i = \phi_i$ on $\mathrm{supp}(f)$, $1 \leqq i \leqq n$. Also,

$$\left\| \psi(x) f(x) - \sum_{i=1}^n \psi(x)\phi_i(x) w_i \right\| < \epsilon \psi(x)/\alpha\beta$$

i.e.,

$$\left\| \psi f - \sum_{i=1}^n \psi_i \otimes w_i \right\|_G < \epsilon/\alpha\beta.$$

But clearly, $\psi f = f$, so that $\|f - \sum_{i=1}^n \psi_i \otimes w_i\|_G < \epsilon/\alpha\beta$. Moreover, letting $g = f - \sum_{i=1}^n \psi_i \otimes w_i$, we see that g is an element of $Cc(G, E)$ with support contained in C. By 3.7 and 6.1 we see that for $p < \infty$,

$$N_p\left(f_q^L - \sum_{i=1}^{n} (\psi_i \otimes w_i)_q^L\right) < \alpha\beta \left\| f - \sum_{i=1}^{n} \psi_i \otimes w_i \right\|_G$$
$$< \epsilon,$$

and for $p = \infty$,

$$N_\infty\left(f_q^L - \sum_{i=1}^{n} (\psi_i \otimes w_i)_q^L\right) < \alpha b \left\| f - \sum_{i=1}^{n} \psi_i \otimes w_i \right\|_G$$
$$< \epsilon b/\beta.$$

This completes the proof.

The following is our Theorem on Induction-in-Stages. For this purpose, we will find it convenient to use the more informative notation of 3.15.

6.3. THEOREM. *Let K be a closed subgroup of G containing H. Suppose L is a representation of H on E and (p, q) is an inducing pair for L up to K. Then (p, q) is also an inducing pair for the representation $M = (\Delta_H/(\Delta_K \mid H))^{1/p-1/q_L}$ of H up to G and $U_q{}^p(M : H, G)$ is isometrically isomorphic to $U^p(U_q{}^p(L : H, K) : K, G)$.*

PROOF. The first part is straightforward. For the second part, we require some additional notation. Let ρ_K(resp. ρ_H) denote a ρ-function for G relative to K (resp. H) and let σ be a ρ-function for K relative to H. Furthermore, let $\delta_K = \rho_K \mid K$, $\delta_H = \rho_H \mid H$ and $\gamma = \sigma \mid H$ (recall § 2). For convenience, we also denote $U_q{}^p(L : H, K)$ by V.

Now for each f in $C_q(G, M)$ and x in G, define $Tf(x) : K \to E$ by $Tf(x)\ (s) = \delta_K(s)^{-1/p} f(sx)$, $s \in K$. Using the properties of δ_K and f, we may verify that $Tf(x)$ is in $C_q(K, L)$. Hence, we have a mapping $Tf : G \to C_q(K, L)$ which we show belongs to $C_p(G, V)$.

For k in K, we have

$$\begin{aligned} Tf(kx)(s) &= \delta_K(s)^{-1/p} f(skx) \\ &= \delta_K(k)^{1/p} \delta_K(sk)^{-1/p} f(skx) \\ &= \delta_K(s)^{1/p} Tf(x)\ (sk) \\ &= \delta_K(k)^{1/p} (V(k) Tf(x))(s), \quad s \in K. \end{aligned}$$

Thus, Tf is (V, p)-homogeneous. Next let x and y be in G with x fixed. Since we will let y approach x, we may assume y is in xZ. Thus, if $\mathrm{supp}(f) \subseteq HC$, for C compact, then (for such y) we have

$\mathrm{supp}(fy \mid K) \subseteq HB$, where $B = CZx^{-1} \cap K$ is also compact. Suppose $p = \infty$. Then

$$\begin{aligned}
N_\infty(Tf(x) - Tf(y)) &= b \sup\{\|Tf(x)(s) - Tf(y)(s)\| : s \in K\} \\
&= b \sup\{\|f(sx) - f(sy)\| : s \in K\} \\
&= b \sup\{\|f(tsx) - f(tsy)\| : t \in H, s \in B\} \\
&= b \sup\{\delta_H(t)^{1/q}\delta_K(t)^{-1/q} \\
&\quad \cdot \|L(t)(f(sx) - f(sy))\| : t \in H, s \in B\} \\
&\leqq b \sup\{\gamma(t)^{1/q}\|L(t)\| : t \in H\} \\
&\quad \cdot \sup\{\|f(sx) - f(sy)\| : s \in B\} \\
&\leqq b^2 \sup\{\|f(sx) - f(sy)\| : s \in B\}.
\end{aligned}$$

The continuity of Tf then follows from 2.4. Now let $p < \infty$. Then

$$\begin{aligned}
N_p(Tf(x) - Tf(y))^p &= \int_{K/H} N_p(Tf(x)(s) - Tf(y)(s))^p \, d\bar{s} \\
&= \int_{K/H} \sigma(s)^{-1} \sup\{\gamma(t)^{-1/p}\|Tf(x)(ts) \\
&\quad - Tf(y)(ts)\| : t \in H\}^p \, d\bar{s} \\
&= \int_{K/H} \sigma(s)^{-1} \sup\{\gamma(t)^{-1}\delta_K(ts)^{-1}\|f(tsx) \\
&\quad - f(tsy)\|^p : t \in H\} \, d\bar{s} \\
&\leqq \int_{K/H} \sigma(s)^{-1}\delta_K(s)^{-1}\sup\{\gamma(t)^{-1}\delta_K(t)^{-1}\delta_H(t)^{p/q} \\
&\quad \cdot \delta_K(t)^{1-p/q}\|L(t)\|^p : t \in H\}\|f(sx) - f(sy)\|^p \, d\bar{s} \\
&= \int_{K/H} \sigma(s)^{-1}\delta_K(s)^{-1}\sup\{\gamma(t)^{1/q-1/p}\|L(t)\|\}^p \\
&\quad \cdot \|f(sx) - f(sy)\|^p \, d\bar{s} \\
&\leqq b^p \int_{K/H} \sigma(s)^{-1}\delta_K(s)^{-1}\|f(sx) - f(sy)\|^p \, d\bar{s} \\
&= b^p \int_K \beta(s)\delta_K(s)^{-1}\|f(sx) - f(sy)\|^p \, ds,
\end{aligned}$$

where β is a Bruhat-function for K and H [3, pp. 57–58]. Recall that $\mathrm{supp}(fy \mid K) \subseteq HB$, so that $A = \mathrm{supp}(\beta) \cap HB$ is a compact subset of K. Let $a = \max\{\beta(s) : s \in A\}$, and $c = \max\{\delta_K(s^{-1}) : s \in A\}$.

Then

$$N_p(Tf(x) - Tf(y))^p \leqq ab^pc \int_A \|f(sx) - f(sy)\|^p \, ds,$$

and once again the continuity of Tf follows from 2.4. Finally, since $\operatorname{supp}(Tf) \subseteq K \operatorname{supp}(f) \subseteq KHC = KC$, it follows that $Tf \in C_p(G, V)$.

Therefore, we have a linear mapping $T : C_q(G, M) \to C_p(G, V)$ which we now show is isometric. For $p = \infty$, we have

$$\begin{aligned} N_\infty(Tf) &= \sup\{N_\infty(Tf, x) : x \in G\} \\ &= \sup\{\sup\{N_\infty(Tf(sx)) : s \in K\} : x \in G\} \\ &= \sup\{N_\infty(Tf(x)) : x \in G\} \\ &= \sup\{\sup\{N_\infty(Tf(x), s) : s \in K\} : x \in G\}, \end{aligned}$$

where

$$\begin{aligned} N_\infty(Tf(x), s) &= \sup\{\|Tf(x)\,(ts)\| : t \in H\} \\ &= \sup\{\|f(tsx)\| : t \in H\} \\ &= N_\infty(f, sx). \end{aligned}$$

Hence,

$$\begin{aligned} N_\infty(Tf) &= \sup\{\sup\{N_\infty(f, sx) : s \in K\} : x \in G\} \\ &= \sup\{N_\infty(f, x) : x \in G\} \\ &= N_\infty(f), \end{aligned}$$

so that T is isometric for $p = \infty$. Now suppose $p < \infty$. Then for f in $C_q(G, M)$, we have

$$\begin{aligned} N_p(Tf)^p &= \int_{G/K} N_p(Tf, x)^p \, d\bar{x} \\ &= \int_{G/K} \rho_K(x)^{-1} \sup\{\delta_K(s)^{-1/p} N_p(Tf(sx)) : s \in K\}^p \, d\bar{x} \\ &= \int_{G/K} \rho_K(x)^{-1} \sup\{\delta_K(s)^{-1} N_p(\delta_K(s)^{1/p} V(s) \\ &\qquad \cdot Tf(x))^p : s \in K\} \, d\bar{x} \\ &= \int_{G/K} \rho_K(x)^{-1} \sup\{N_p(V(s) Tf(x))^p : s \in K\} \, d\bar{x} \\ &= \int_{G/K} \rho_K(x)^{-1} N_p(Tf(x)) \, d\bar{x} \qquad (V \text{ is isometric}) \end{aligned}$$

$$= \int_{G/K} \int_{K/H} \rho_K(x)^{-1}\sigma(s)^{-1}\sup\{\gamma(t)^{-1/p}$$
$$\cdot \|Tf(x)\,(ts)\| : t \in H\}^p d\bar{s}d\bar{x}$$
$$= \int_{G/K} \int_{K/H} \rho_K(x)^{-1}\sigma(s)^{-1}\sup\{\gamma(t)^{-1}$$
$$\cdot \delta_K(ts)^{-1}\|f(tsx)\|^p : t \in H\}\, d\bar{s}d\bar{x}$$
$$= \int_{G/K} \int_{K/H} \rho_K(x)^{-1}\sigma(s)^{-1}\delta_K(s^{-1})$$
$$\cdot \sup\{\gamma(t)^{1/q - 1/p}\|L(t)f(sx)\| : t \in H\}^p\, d\bar{s}d\bar{x}.$$

On the other hand,

$$N_p(f)^p = \int_{G/H} N_p(f, x)^p\, d\bar{x}$$
$$= \int_{G/H} \rho_H(x)^{-1}\sup\{\delta_H(t)^{-1/p}\|f(tx)\| : t \in H\}^p\, d\bar{x}$$
$$= \int_{G/H} \rho_H(x)^{-1}\sup\{[\delta_H(t)/\delta_K(t)]^{1/q - 1/p}$$
$$\cdot \|L(t)f(x)\| : t \in H\}^p\, d\bar{x}$$
$$= \int_G \beta(x)\sup\{\gamma(t)^{1/q - 1/p}\|L(t)f(x)\| : t \in H\}^p\, dx$$

(where this time β is a Bruhat function for G and H [3, pp. 57–58])

$$= \int_{G/K} \int_K \beta(sx)\rho_K(sx)^{-1}\sup\{\gamma(t)^{1/q - 1/p}$$
$$\cdot \|L(t)f(sx)\| : t \in H\}^p\, dsd\bar{x}$$
$$= \int_{G/K} \int_{K/H} \int_H \sup\{\gamma(t)^{1/q - 1/p}\|L(t)f(rsx)\| : t \in H\}$$
$$\cdot \beta(rsx)\rho_K(x)^{-1}\,\sigma(rs)^{-1}\, drd\bar{s}d\bar{x}$$
$$= \int_{G/K} \int_{K/H} \int_H \sup\{\gamma(tr)^{1/q - 1/p}\|L(tr)f(sx)\| : t \in H\}^p$$
$$\cdot \beta(rsx)\rho_K(x)^{-1}\sigma(s)^{-1}\delta_K(s)^{-1}\, drd\bar{s}d\bar{x}$$
$$= \int_{G/K} \int_{K/H} \sup\{\gamma(t)^{1/q - 1/p}\|L(t)f(sx)\| : t \in H\}^p$$
$$\cdot \rho_K(x)^{-1}\sigma(s)^{-1}\delta_K(s)^{-1}\int_H \beta(rsx)\, drd\bar{s}d\bar{x}$$

$$= \int_{G/K} \int_{K/H} \sup\{\gamma(t)^{1/q-1/p}\|L(t)f(sx)\| : t \in H\}^p$$
$$\cdot \rho_K(x)^{-1}\sigma(s)^{-1}\delta_K(s^{-1})\, d\bar{s}d\bar{x}$$
$$= N_p(Tf)^p.$$

Therefore, T is an isometry for $p < \infty$ as well and thus passes to an isometry of $F_q{}^p(G, M)$ into $F^p(G, V)$. Next we show that the range of T is dense in $F^p(G, V)$, i.e., T is onto.

Let $\psi \in Cc(K)$, $\phi \in Cc(G)$ and $v \in E$. Then $w = (\psi \otimes v)_q{}^L$ is an element of $C_q(K, L) \subseteq F_q{}^p(K, L)$, and consequently $(\phi \otimes w)_p{}^V$ is an element of $C_p(G, V)$. Define $\theta : G \to \mathbf{C}$ by

$$\theta(x) = \int_K \delta_K(s)^{-1/p}\psi(s^{-1})\phi(sx)\, ds$$
$$= \int_C \delta_K(s)^{-1/p}\psi(s^{-1})\phi(sx)\, ds, \quad x \in G,$$

where $C = \operatorname{supp}(\psi)^{-1}$ is a compact subset of K. It follows from the uniform continuity of ϕ that θ is in $Cc(G)$. Thus, $(\theta \otimes v)_q{}^M$ is in $C_q(G, M)$. In view of 6.2, in order to complete the proof that T is onto, it suffices to show that $T(\theta \otimes v)_q{}^M = (\phi \otimes w)_p{}^V$. We have:

$$(T(\theta \otimes v)_q{}^M)(x)(s) = \delta_K(s)^{-1/p}(\theta \otimes v)_q{}^M(sx)$$
$$= \delta_K(s)^{-1/p} \int_H \delta_H(t^{-1})^{1/q}\delta_K(t^{-1})^{1/p-1/q}$$
$$\cdot\, \theta(tsx)L(t^{-1})v\, dt$$
$$= \delta_K(s)^{-1/p} \int_H \int_K \delta_H(t^{-1})^{1/q}\delta_K(t^{-1})^{1/p-1/q}$$
$$\cdot\, \delta_K(k)^{-1/p}\, \psi(k^{-1})\phi(ktsx)L(t^{-1})v\, dkdt,$$

for $x \in G$, $s \in K$. On the other hand,

$$(\phi \otimes w)_p{}^V(x)(s) = \left(\int_K \delta_K(k^{-1})^{1/p}V(k^{-1})(\phi \otimes w)(kx)\, dk\right)(s)$$
$$= \int_K \delta_K(k^{-1})^{1/p}(V(k^{-1})((\phi \otimes w)(kx)))(s)\, dk$$
$$= \int_K \delta_K(k^{-1})^{1/p}((\phi \otimes w)(kx))(sk^{-1})\, dk$$

$$= \int_K \delta_K(k^{-1})^{1/p}\phi(kx) \int_H \gamma(t^{-1})^{1/q}\psi(tsk^{-1}) \cdot L(t^{-1})v\, dtdk$$

$$= \int_H \int_K \delta_K(kts)^{-1/p}\phi(ktsx)\gamma(t^{-1})^{1/q}\psi(k^{-1}) \cdot L(t^{-1})v\, dkdt$$

$$= \delta_K(s)^{-1/p} \int_H \int_K \delta_K(t^{-1})^{1/p-1/q}\delta_K(k)^{-1/p} \cdot \delta_H(t^{-1})^{1/q}\psi(k^{-1})\phi(ktsx)\, L(t^{-1})v\, dkdt$$

$$= T(\theta \otimes w)_q{}^M(x)(s),\ x \in G, s \in K.$$

Hence, $F_q{}^p(G, M)$ is isometrically isomorphic to $F^p(G, V)$, and it remains only to show that T intertwines $U_q{}^p(M : H, G)$ and $U^p(V : K, G)$. Since $\{Tf : f \in C_q(G, M)\}$ is dense in $F^p(G, V)$, it suffices to check this on the elements of $C_q(G, M)$. We have:

$$(TU_q{}^p(M : H, G)(x))(f) = T(fx),$$

where

$$(T(fx))(y)(s) = \delta_K(s)^{-1/p}f(syx), \quad y \in G, s \in K.$$

On the other hand,

$$(U^p(V : K, G)(x)T)(f) = (Tf)x,$$

where

$$((Tf)x)(y)(s) = (Tf)(yx)(s) = \delta_K(s)^{-1/p}f(syx), \quad y \in G, s \in K,$$

which completes the proof.

As in the case of unitary inductions, we have the following well-known consequence of induction-in-stages.

6.4. Proposition. *If K is a closed subgroup of G and p is an arbitrary element of $[1, \infty]$, then $R_G{}^p$ is isometrically equivalent to $U^p(R_K{}^p)$.*

Proof. Let $H = \{e\}$ and $L = 1_H$. Then apply 4.4 and the previous theorem.

7. **Frobenius Reciprocity.** In [15] Moore obtained a version of the Frobenius Reciprocity Theorem for isometric and hence bounded

representations (see 3.20 and 4.1). This result has since been generalized in different directions by different people [**10**, **17**, **18**]. In this section we will give another generalization which is consistent with our study of inducing pairs. The following is our Frobenius Reciprocity Theorem. Once again, let the notation be as in § 2.

7.1. THEOREM. *Let L be a representation of H on E and V a representation of G on F. Suppose:*

(i) *G is σ-compact.*
(ii) *F is the dual of a Banach space.*
(iii) *E and F are separable.*

If $(1, q)$ is an inducing pair for L up to G and V is bounded, then $\mathrm{Hom}_H(L, \delta^{1-1/q}V \mid H)$ *is topologically isomorphic to* $\mathrm{Hom}_G(U_q{}^1(L), V)$.

PROOF. Let b be as in 3.1 for L and let a be such that $\|V(x)\| \leqq a$, $x \in G$. For convenience, write $M = \delta^{1-1/q}V| \, H$ and $U = U_q{}^1(L)$. Let B be an element of $\mathrm{Hom}_H(L, M)$, and for each f in $C_q(G, L)$, define $f_B : E \rightarrow F$ by

$$f_B(x) = \rho(x)^{-1}V(x^{-1})Bf(x), \quad x \in G.$$

Then f_B is constant on cosets and has compact support modulo H since $\mathrm{supp}(f_B) \subseteq \mathrm{supp}(f)$. The function f_B is also continuous as we now verify.

Let x and y be in G with x fixed and y in xZ. Then

$$\begin{aligned} \|f_B(x) - f_B(y)\| &\leqq \|\rho(x)^{-1}V(x^{-1})Bf(x) - \rho(x)^{-1}V(y^{-1})Bf(x)\| \\ &\quad + \|\rho(x)^{-1}V(y^{-1})Bf(x) - \rho(y^{-1})V(y^{-1})Bf(y)\| \\ &\leqq \|(V(x^{-1}) - V(y^{-1}))B(\rho(x)^{-1}f(x))\| \\ &\quad + \|B\| \, \|V(y^{-1})\| \, \|\rho(x)^{-1}f(x) - \rho(y)^{-1}f(y)\|. \end{aligned}$$

which implies the desired continuity since V is bounded and strongly continuous. Hence, we may form $\Phi(B)(f) = \int_Q f_B(x)\, d\bar{x}$, which is an element of F. Therefore, we have a linear mapping $\Phi(B)$: $C_q(G, L) \rightarrow F$. Since $\|f_B(x)\| \leqq a\|B\|N_1(f, x)$, it follows that

$$\begin{aligned} \|\Phi(B)(f)\| &\leqq \int_Q \|f_B(x)\| \, d\bar{x} \\ &\leqq a\|B\|N_1(f), f \in C_q(G, L), \end{aligned}$$

i.e., $\|\Phi(B)\| \leqq a\|B\|$. Thus, $\Phi(B)$ extends to a bounded linear map of $F_q{}^1(G, L)$ into F with the same norm. We may verify that $\Phi(B)$ intertwines U and V. Consequently, Φ is a linear map of $\mathrm{Hom}_H(L, M)$ into $\mathrm{Hom}_G(U, V)$ satisfying $\|\Phi(B)\| \leqq a\|B\|$, $B \in \mathrm{Hom}_H(L, M)$. Next we show that Φ is surjective.

Let C be an element of $\mathrm{Hom}_G(U, V)$ and define $D : \mathrm{Cc}(G, E) \to F$ by $D(h) = C(h_q{}^L)$, $h \in \mathrm{Cc}(G, E)$. Since we may verify that $N_1(h_q{}^L) \leqq b^2\|h\|_1$, we have that $\|D(h)\| \leqq b^2\|C\| \; \|h\|_1$. Thus, D extends to a bounded linear mapping of $L^1(G, E)$ into F with $\|D\| \leqq b^2\|C\|$. It follows from a result of N. Dinculeanu [5] that there eixsts a mapping A of G into the bounded linear operators from E into F satisfying:

(i) A is strongly measurable.
(ii) A is essentially bounded with $\|A\|_\infty = \|D\|$.
(iii) $D(h) = \int_G A(x)h(x)\, dx$, $h \in L^1(G, E)$.
(iv) A is essentially unique.

Now let t be in H and h in $\mathrm{Cc}(G, E)$. Then we can verify that

$$(th)_q{}^L = \delta(t)^{1/q}\, \Delta_H(t^{-1})(L(t)h)_q{}^L.$$

Hence,

$$D(th) = \delta(t)^{1/q}\, \Delta_H(t^{-1})D(L(t)h).$$

But

$$D(th) = \Delta_G(t^{-1}) \int_G A(t^{-1}x)h(x)\, dx$$

and

$$D(L(t)h) = \int_G A(x)L(t)h(x)\, dx,$$

so that

$$\delta(t) \int_G A(t^{-1}x)h(x)\, dx = \delta(t)^{1/q} \int_G A(x)L(t)h(x)\, dx,$$

for $t \in H$, $h \in \mathrm{Cc}(G, E)$. The mapping $h \to L(t)h$ is a bijection of $\mathrm{Cc}(G, E)$ with itself. Thus, for each t in H, we have:

$$D(h) = \delta(t)^{1-1/q} \int_G A(t^{-1}x)L(t^{-1})f(x)\, dx, \quad h \in \mathrm{Cc}(G, E),$$

i.e., the mapping $x \to \delta(t)^{1-1/q}A(t^{-1}x)L(t^{-1})$ has the same properties as A. Consequently, for each t in H, $\delta(t)^{1-1/q}A(t^{-1}x) = A(x)L(t)$, for almost all x in G. An application of Fubini's theorem yields: for almost all x in G, $\delta(t)^{1-1/q}A(t^{-1}x) = A(x)L(t)$, for almost all t in H. Next we show that the mapping $x \to V(x)A(x)$ is constant almost everywhere on G.

For each y in G and h in $\mathrm{Cc}(G, E)$, we have $(V(y)D)(h) = V(y)(C(h_q{}^L)) = (CU(y))(h_q{}^L) = C((h_q{}^L)y) = C(hy)_q{}^L = D(hy)$. But

$$(V(y)D)(h) = V(y) \int_G A(x)h(x)\, dx,$$

while

$$D(hy) = \int_G A(xy^{-1})h(x)\,dx,$$

i.e., for each y in G, we see that

$$D(h) = V(y^{-1}) \int_G A(xy^{-1})h(x)\,dx, \quad h \in \mathrm{Cc}(G, E).$$

As before, it must be that for each y in G, $A(x) = V(y)A(xy)$, for almost all x in G. By Fubini's theorem, for almost all x in G, $A(x) = V(y)A(xy)$, for almost all y in G. Let x_0 be such an x, and define $B = V(x_0)A(x_0)$. Then $B = V(x_0)V(y)A(x_0y) = V(x_0y)A(x_0y)$, for almost all y in G. Since null sets are translation invariant, we may redefine A on a null set (if necessary) so that $B = V(y)A(y)$, for all y in G. Thus, $\|B\| \leqq a\|A\|_\infty$. In order to show that B is in $\mathrm{Hom}_H(L, M)$, fix t in H and choose x in G such that:

(i) $V(x)A(x) = B$,

(ii) $V(t^{-1}x)A(t^{-1}x) = B$,

(iii) $\delta(t)^{1-1/q}A(t^{-1}x) = A(x)L(t)$.

This is possible since (for fixed t) the set of x for which (i), (ii) and (iii) fail is a null set. For such x, we have: $BL(t) = V(x)A(x)L(t) = \delta(t)^{1-1/q}V(x)A(t^{-1}x) = \delta(t)^{1-1/q}V(t)V(t^{-1}x)A(t^{-1}x) = \delta(t)^{1-1/q}V(t)B = M(t)B$, $t \in H$. Thus, $B \in \mathrm{Hom}_H(L, M)$. Next we verify that $\Phi(B) = C$.

Let h be in $\mathrm{Cc}(G, E)$. Then

$$\begin{aligned}
\Phi(B)(h_q{}^L) &= \int_Q (h_q{}^L)_B(x)\,d\bar{x} \\
&= \int_Q \rho(x)^{-1}V(x^{-1})B\left(\int_H \delta(t)^{-1/q}\right. \\
&\qquad \left.\cdot L(t^{-1})h(tx)\,dt\right)d\bar{x} \\
&= \int_Q \int_H \rho(x)^{-1}\delta(t)^{-1/q}A(x)L(t^{-1}) \\
&\qquad \cdot h(tx)\,dt d\bar{x} \\
&= \int_Q \int_H \rho(tx)^{-1}(\delta(t)^{1-1/q}A(x) \\
&\qquad \cdot L(t^{-1}))h(tx)\,dt d\bar{x},
\end{aligned}$$

where the inner integral is constant on cosets. Let X denote the set of ξ in Q such that for all x in $\pi^{-1}(\xi)$, the equation $\delta(t)^{1-1/q}A(x)L(t^{-1}) =$

$A(tx)$ fails to hold for almost all t in H. Since $\pi^{-1}(X)$ is contained in a null set and is a union of cosets, it follows from [3, p. 55] that X is a null subset of Q. Hence, for almost all ξ in Q, there exists x in $\pi^{-1}(\xi)$ such that $\delta(t)^{1-1/q}A(x)L(t^{-1}) = A(tx)$, for almost all t in H. Therefore,

$$\begin{aligned}\Phi(B)(h_q{}^L) &= \int_Q \int_H \rho(tx)^{-1}A(tx)h(tx)\,dtd\bar{x} \\ &= \int_G A(x)h(h)\,dx \\ &= D(h) \\ &= C(h_q{}^L), \quad h \in \mathrm{Cc}(G, E).\end{aligned}$$

In view of 6.2, it follows that $\Phi(B) = C$, i.e., Φ is onto.

Finally, $\|\Phi(B)\| = \|C\| \geqq b^{-2}\|D\| = b^{-2}\|A\|_\infty \geqq (ab^2)^{-1}\|B\|$, so that

$$\begin{aligned}(ab^2)^{-1}\|B\| &\leqq \|\Phi(B)\| \\ &\leqq a^{-1}\|B\|, \quad B \in \mathrm{Hom}_H(L, M).\end{aligned}$$

Hence, Φ is injective also, and the proof is complete.

7.2. Corollary. *If L is bounded, then* $\mathrm{Hom}_H(L, V \mid H)$ *is topologically isomorphic to* $\mathrm{Hom}_G(U_q{}^1(L), V)$. *In particular, if L and V are isometric, then the isomorphism is isometric.*

References

1. R. J. Blattner, *On induced representations,* Amer. J. Math. 83 (1961), 79–98.

2. N. Bourbaki, *Éléments de Mathématique, Intégration,* Chap. 1–4, Hermann, Paris, 1965.

3. ——, *Éléments de Mathématique, Intégration,* Chap. 7–8, Hermann, Paris, 1963.

4. F. Bruhat, *Sur les représentations induites des groupes de Lie,* Bull. Soc. Math. France 84 (1956), 97–205.

5. N. Dinculeanu, *Sur la représentation intégrale des certaines opérations linéaires* II, Comp. Math. 14 (1959), 1–22.

6. R. A. Fakler, *On Mackey's tensor product theorem,* Duke Math. J. 40 (1973), 689–694.

7. J. M. G. Fell, *The dual spaces of Banach algebras,* Trans. A.M.S. 114 (1965), 227–250.

8. ——, *Non-unitary dual spaces,* Acta Math. 114 (1965), 267–310.

9. S. A. Gaal, *Linear analysis and representation theory,* Springer-Verlag, New York, 1973.

10. A. Kleppner, *Intertwining forms for summable induced representations,* Trans. A.M.S. 112 (1964), 164–183.

11. H. Kraljević, *Induced representations of locally compact groups on Banach spaces,* Glasnik Mat. 4 (1969), 183–196.

12. R. Lipsman, *Group representations,* Lec. Notes Math. No. 338, Springer-Verlag, New York, 1974.

13. G. W. Mackey, *Induced representations of locally compact groups* I, Ann. Math. **55** (1952), 101-139.

14. ——, *Infinite dimensional group representations,* Bull A.M.S. **69** (1963), 628-686.

15. C. C. Moore, *On the Frobenius reciprocity theorem for locally compact groups,* Pac. J. Math. **12** (1962), 359-365.

16. R. T. Moore, *Measurable, continuous and smooth vectors for semigroups and group representations,* Mem. A.M.S. No. 78 (1968).

17. M. A. Rieffel, *Induced Banach representations of Banach algebras and locally compact groups,* J. Functional Anal. **1** (1967), 443-491.

18. R. Rigelhoff, *Induced representations of locally compact groups,* Acta Math. **125** (1970), 155-187.

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