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MODULAR TRANSFORMATIONS AND GENERALIZATIONS OF SEVERAL FORMULAE OF RAMANUJAN

BRUCE C. BERNDT

1. In [7], the author presented a new method for deriving the transformation formulae of a large class of functions that includes the Dedekind eta-function $\eta(z)$. Here, and in the sequel, a transformation shall mean a modular transformation $Vz = V(z) = (az + b)/(cz + d)$, where a, b, c and d are rational integers with $c > 0$ and $ad - bc = 1$. The general theorem [7, Theorem 2] was shown to contain as special cases transformation formulae established by several other authors. In [9], we generalized the results of [7]; furthermore, one should consult [9] for a *complete* proof of the main theorems of [7] and [9]. Arising in the transformation formulae are various types of Dedekind sums, all of which may be shown to satisfy reciprocity theorems by the use of the transformation formulae.

In [8] and [11] we considered character analogues of the aforementioned classes of functions. Thus, transformation formulae were developed for a wide class of functions involving characters including the natural character generalizations of $\log \eta(z)$. The transformation formulae involve character generalizations of Dedekind sums which can be shown to obey reciprocity theorems by the employment of the transformation formulae. The results of these papers [8], [11] appear to be new. However, there is some overlap with papers of Katayama [31], [32].

We wish to show in this paper that the very general theorems of [7] and [11] contain many other interesting results as special cases. These results give the values of several interesting series and yield intriguing relations between various series. It is shown that a large mass of such results found in the literature can be deduced quite simply from a few general theorems given below.

One of the most interesting corollaries of Theorem 2 of [7] is Ramanujan's formula for $\zeta(2n + 1)$, $n \geqq 1$, a very interesting and striking formula found in his Notebooks [54] and proved by A. P. Guinand [22], E. Grosswald [20], [21], and others. In fact, Euler's formula for $\zeta(2n)$, $n \geqq 1$, and Ramanujan's formula for $\zeta(2n + 1)$ are both consequences of the same general theorem. Ramanujan's formula for $\zeta(2n + 1)$ contains a finite sum of products of 2 Bernoulli numbers. Ramanujan [54] also discovered a similar type of relation involving a

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finite sum of products of 2 Euler numbers.

For $\sigma = \text{Re}(s) > 0$, let

$$L(s) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-s}. \tag{1.1}$$

$L(s)$ is, in fact, an example of a Dirichlet L-function. Nothing arithmetically is known about $L(2n)$, $n \geqq 1$, but Ramanujan's Notebooks contain an interesting formula for $L(2n)$, similar to that for $\zeta(2n+1)$. We give a proof of this formula here, but, in fact, Katayama [30] and the author [11] have proven "Ramanujan formulas" for arbitrary Dirichlet L-functions.

From the principal theorems of [7] and [11], many other formulae of Ramanujan shall be deduced. Almost all of these can be found in his Notebooks, and some can be found in Ramanujan's letters to G. H. Hardy [53]. Many of these very peculiar formulae have been proved by S. L. Malurkar [44], Nanjundiah [47], M. B. Rao and M. V. Aiyar [55], S. Chowla [16], J. W. L. Glaisher [19], G. H. Hardy [24], G. N. Watson [67], H. F. Sandham [59], Grosswald [21] and others. However, a number of them have never been heretofore proved in print. Furthermore, many of Ramanujan's formulas are generalized here. Let us cite just 2 examples.

One of Ramanujan's most beautiful theorems, which, in fact, predates Ramanujan, is the representation (Proposition 2.7 below),

$$\sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{-2\pi k} - 1} = \frac{B_{2n}}{4n}, \tag{1.2}$$

where $n > 1$ is an odd, positive integer and B_j denotes the j-th Bernoulli number. We shall show that if V is an elliptic transformation with fixed point $\chi \in \mathcal{H} = \{z : \text{Im}(z) > 0\}$, then (Theorem 2.9 below)

$$\sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{-2\pi i \chi k} - 1} = \frac{B_{2n}}{4n}, \tag{1.3}$$

provided that $n > 1$ and $(c\chi + d)^{2n} \neq 1$. Since $Vz = -1/z$ has i as a fixed point, (1.2) is a special case of (1.3).

Another curious result found in Ramanujan's Notebooks is (Proposition 4.8 below):

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{4n-1}}{\cosh\{(2k+1)\pi/2\}} = 0, \tag{1.4}$$

where n is a positive integer. We shall show that if $0 \leqq r < 1$ (Proposition 4.20 below),

$$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^{2M-1}\cos\{(2k+1)\pi r/2\}}{\cosh\{(2k+1)\pi/2\}}$$
$$+ (-1)^M \sum_{k=0}^{\infty} \frac{(+1)^k(2k+1)^{2M-1}\cosh\{(2k+1)\pi r/2\}}{\cosh\{(2k+1)\pi/2\}} = 0,$$

where M is a positive integer. Thus, if $r = 0$ and $M = 2n$, the above reduces to (1.4).

By no means, is the work which follows exhaustive. We have chosen those examples which we think are the most interesting, the most striking, and perhaps the most surprising. General results have been stated in such a way that the interested reader may simply specify certain transformations and/or parameters to produce further identities with almost no further calculation or manipulation. For the most part, our examples arise from the modular transformation $Vz = -1/z$. We have also derived specific examples for the transformation $Vz = -(z+1)/z$. Additional formulae may be obtained by differentiation. Except for a few instances, we have not emphasized this. Furthermore, we have not given any series relations arising from the more general theorem in [9], other than those which can be derived from [7]. Those examples arising from [9] appear to be not quite as interesting or elegant as most of the others given here.

In comparing some of our results with the formulations given in Ramanujan's Notebooks, one should keep in mind that the conventions used by Ramanujan for the Bernoulli and Euler numbers are not those customarily used today. We employ the even suffix notations, used in [1], for example. Furthermore, some of our formulations, like that of Ramanujan's formula for $\zeta(2n+1)$, for example, are equivalent to those found in the Notebooks after very elementary manipulation. Lastly, as was customary in his time, Ramanujan used the notation S_n for $\zeta(n)$, $n \geqq 2$.

In the following, we choose that branch of $\log z$ with $-\pi \leqq \arg z < \pi$.

2. We shall first formulate Theorem 2 of [7] when $s = -m$, where m is an integer. To do this, we must introduce some notation and make a few definitions. If $z \in \mathcal{H}$ and r_1 and r_2 are real, let [7, p. 496].

$$H(z, -m, r_1, r_2) = A(z, -m, r_1, r_2) + (-1)^m A(z, -m, -r_2, -r_2), \tag{2.1}$$

where

$$A(z, -m, r_1, r_2) = \sum_{n>-r_1} \sum_{k=1}^{\infty} k^{-m-1} e^{2\pi i k(nz+r_1z+r_2)} \quad . \tag{2.2}$$

Let α be real and let $\lambda(\alpha)$ denote the characteristic function of the integers. Define for α real and $\sigma > 1$,

$$\zeta(s, \alpha) = \sum_{n>-\alpha} (n+\alpha)^{-s} = \zeta(s, \{\alpha\} + \lambda(\alpha)), \tag{2.3}$$

where here, and in the sequel, we denote the fractional part of α by $\{\alpha\}$. Of course, $\zeta(s, \{\alpha\} + \lambda(\alpha))$ denotes the Hurwitz zeta-function which has an analytic continuation to the full complex s-plane. Define $R_1 = ar_1 + cr_2$ and $R_2 = br_1 + dr_2$. Let

$$\begin{aligned} g(z, -m, r_1, r_2) = \lim_{s\to -m} \{&-\lambda(r_1)e^{\pi i s}(2\pi i)^{-s}(cz+d)^{-s}\Gamma(s)(\zeta(s, r_2) \\ &+ e^{\pi i s}\zeta(s, -r_2)) + \lambda(R_1)(2\pi i)^{-s}\Gamma(s)(\zeta(s, -R_2) + e^{\pi i s}\zeta(s, R_2))\}. \end{aligned} \tag{2.4}$$

Furthermore, $B_n(x)$ denotes the n-th Bernoulli polynomial, $\bar{B}_n(x) = B_n(\{x\})$, and $B_n = B_n(0)$ denotes the n-th Bernoulli number, where $0 \leqq n < \infty$. If $\rho = \{R_2\}c - \{R_1\}d$, let

$$\begin{aligned} &h(z, -m, r_1, r_2) \\ &= \sum_{j=1}^{c} \sum_{k=0}^{m+2} (-1)^{k-1} B_k \left(\frac{j - \{R_1\}}{c}\right) \bar{B}_{m+2-k}\left(\frac{jd+\rho}{c}\right) \frac{(cz+d)^{k-1}}{k!(m+2-k)!}, \end{aligned} \tag{2.5}$$

where if $m + 2 < 0$, we shall understand that $h(z, -m, r_1, r_2) = 0$.

THEOREM 2.1 [7, p. 498]. *For $z \in \mathcal{H}$ and an arbitrary integer m, we have*

$$\begin{aligned} (cz+d)^m H(Vz, -m, r_1, r_2) &= H(z, -m, R_1, R_2) \\ &+ g(z, -m, r_1, r_2) + (2\pi i)^{m+1} h(z, -m, r_1, r_2). \end{aligned} \tag{2.6}$$

We next specialize Theorem 2.1 by setting $r_1 = r_2 = 0$. To calculate $g(z, -m) \equiv g(z, -m, 0, 0)$, we must separate the case $m = 0$. From (2.3), $\zeta(s, 0) = \zeta(s)$, where $\zeta(s)$ designates the Riemann zeta-function. From the functional equation of $\zeta(s)$ [68, p. 269],

$$\lim_{s\to -m} (1 + e^{\pi i s})\Gamma(s)\zeta(s) = \lim_{s\to -m} (2\pi)^s e^{\pi i s/2}\zeta(1-s)$$
$$= (2\pi)^{-m} e^{-\pi i m/2}\zeta(1+m),$$

provided that $m \neq 0$. Thus, from (2.4) for $m \neq 0$,

$$g(z, -m) = \{1 - (-cz - d)^m\}\zeta(m+1). \tag{2.7}$$

Since $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1, we find that

$$\lim_{s\to 0} \Gamma(s)(1 - e^{\pi i s}(cz + d)^{-s}) = \log(cz + d) - \pi i.$$

Since $\zeta(0) = -1/2$ [68, p. 268], we deduce from the above and (2.4) that

$$g(z, 0) = \pi i - \log(cz + d). \tag{2.8}$$

Put $H(z, -m) = H(z, -m, 0, 0)$. From (2.1) and (2.2), we see that

$$H(z, -m) = (1 + (-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{-2\pi i k z} - 1}. \tag{2.9}$$

Furthermore, suppose that $Vz = -1/z$ or $Vz = -(z+1)/z$, so that in either instance $c = 1$ and $d = 0$. We then deduce from Theorem 2.1 the following result.

THEOREM 2.2. *For $z \in \mathcal{H}$, m integral, and $Vz = -1/z$ or $Vz = -(z+1)/z$, we have*

$$z^m(1 + (-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{-2\pi i k Vz} - 1}$$
$$= (1 + (-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{-2\pi i k z} - 1} + g(z, -m) \tag{2.10}$$
$$+ (2\pi i)^{m+1} \sum_{k=0}^{m+2} \frac{B_k(1)}{k!} \frac{B_{m+2-k}}{(m+2-k)!} (-z)^{k-1},$$

where $g(z, -m)$ is given by (2.7) *and* (2.8).

Many consequences of Theorem 2.2 will now be deduced.

THEOREM 2.3 (EULER'S FORMULA FOR $\zeta(2N)$). *Let N be a positive integer. Then*

$$\zeta(2N) = \frac{(-1)^{N+1}(2\pi)^{2N}}{2(2N)!} B_{2N}.$$

PROOF. Let $m = 2N - 1$ in (2.10). Trivially, from (2.9), $H(z, -2N + 1) \equiv 0$. Using (2.7), we see that (2.10) reduces to

$$(1 + z^{2N-1})\zeta(2N) = \frac{(2\pi)^{2N}(-1)^{N+1}}{(2N)!} \{B_1(1)B_{2N} - B_{2N}B_1 z^{2N-1}\},$$

where we have used the fact that $B_{2k+1} = 0, k \geqq 1$. Since $B_1(1) = 1/2$ and $B_1 = -1/2$, Euler's formula now follows.

Note that if we set $r_1 = r_2 = 0$ and $m = 2N - 1$, $N > 0$, in (2.6), we obtain an infinitude of formulas for $\zeta(2N)$.

THEOREM 2.4 (RAMANUJAN'S FORMULA FOR $\zeta(2N + 1)$). *Let α and β be positive numbers such that $\alpha\beta = \pi^2$. Let N be a positive integer. Then,*

$$\alpha^{-N}\left\{(1/2)\zeta(2N+1) + \sum_{k=1}^{\infty} \frac{k^{-2N-1}}{e^{2\alpha k} - 1}\right\}$$

$$= (-\beta)^{-N}\left\{(1/2)\zeta(2N+1) + \sum_{k=1}^{\infty} \frac{k^{-2N-1}}{e^{2\beta k} - 1}\right\} \tag{2.11}$$

$$- 2^{2N} \sum_{k=0}^{N+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!} \alpha^{N+1-k}\beta^k.$$

PROOF. In (2.10), let $m = 2N$, where N is *any* integer except 0. Also, let $Vz = -1/z$ and $z = i\pi/\alpha$. Since $\pi^2/\alpha = \beta$, we obtain upon the use of (2.7),

$$2(-1)^N(\pi/\alpha)^{2N} \sum_{k=1}^{\infty} \frac{k^{-2N-1}}{e^{2\alpha k} - 1}$$

$$= 2\sum_{k=1}^{\infty} \frac{k^{-2N-1}}{e^{2\beta k} - 1} + \{1 - (-1)^N(\pi/\alpha)^{2N}\}\zeta(2N+1) \tag{2.12}$$

$$+ (2\pi)^{2N+1}(-1)^{N+1} \sum_{k=0}^{N+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!} (\pi/\alpha)^{2k-1}.$$

Multiplying both sides of (2.12) by $(-\beta)^{-N}/2$, we arrive at (2.11).

Theorem 2.4 is quite remarkable. First, we see that this formula is the natural complement of Euler's formula for $\zeta(2N)$ in that both formulae are instances of the application of the transformation $Vz = -1/z$ to the function $H(z, -m)$. Secondly, appearing in the summands of the infinite series of (2.11) are generating functions for the Bernoulli numbers:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| < 2\pi).$$

Thus, Ramanujan's formula gives a formula for $\zeta(2N + 1)$ involving a finite sum of Bernoulli numbers and two doubly infinite series involving Bernoulli numbers. If α is a rational multiple of π, we see, especially from (2.12), that $\zeta(2N + 1)$ is a rational multiple of π^{2N+1} plus two very rapidly convergent series.

If N is even and $\alpha = \beta = \pi$, equation (2.11) yields no information on $\zeta(2N + 1)$. If N is odd and $\alpha = \beta = \pi$, (2.11) or (2.12) reduces to

$$\zeta(2N+1) = (2\pi)^{2N}\pi \sum_{k=0}^{N+1} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!} - 2 \sum_{k=1}^{\infty} \frac{k^{-2N-1}}{e^{2\pi k} - 1}. \tag{2.13}$$

Formula (2.13) has an analogue for even N which can be gotten by differentiating (2.10), as we shall see later.

Theorem 2.4 has an interesting history. Formula (2.11) is stated twice by Ramanujan in his Notebooks [54, vol. I, p. 259, no. 15; vol. II, p. 177, no. 21]. However, the special case (2.13) appears to have been first proven by Lerch [38]. The first published proof of Ramanujan's formula (2.11) appears to have been given by S. L. Malurkar [44]. Grosswald has given a proof of (2.13) [20] and the more general formula (2.11) [21]. These two papers [20], [21] of Grosswald brought Ramanujan's formula to light long after it had apparently been forgotten. Smart [65] has recently proved (2.13), and Katayama [29], [32] and Riesel [57] have recently established (2.11).

Theorem 2.2 in the case $Vz = -1/z$ and $m = 2N$ with $N > 0$ was first proven by Guinand [22] and the resulting formula for $\zeta(2N + 1)$ is explicitly stated by him. In another paper [23], Guinand discusses his results again and gives an equivalent formulation of (2.13). Chandrasekharan and Narasimhan [15, pp. 15–17] have also given a proof of Guinand's result. Theorem 2.1 in the case $r_1 = r_2 = 0$ and $m = 2N > 0$ was first proven by Apostol [3], but due to a miscalculation the term involving $\zeta(2N + 1)$ was omitted, and hence a formula for $\zeta(2N + 1)$ was not ascertained. Many years later, Apostol [5] and the present author independently realized that Ramanujan's formula could be deduced from a corrected version of Apostol's theorem [3] or from the author's result [7, equation (30)]. A corrected version of Apostol's result was first given by Mikolás [45] and shortly thereafter by Iseki [28]. Bodendiek [12] and Bodendiek and Halbritter [13] have also given proofs. Guinand's result is a special case of more general results of Mikolás [45] and Glaeske [17], [18]. Essentially then, several proofs of Ramanujan's formula have been given in the literature, although at the time most of the authors did not realize that they were giving proofs of Ramanjan's formula.

In fact, Ramanujan's formula for $\zeta(2N+1)$ is but one of an infinite class of such formulas, as we can see by setting $r_1 = r_2 = 0$ and $m = 2N$ in Theorem 2.1. The most elegant formulae arise by letting V be an elliptic transformation and then letting z be a fixed point of V. We give one additional example, which is indicated by Smart [65], to illustrate this. Let $\rho = (-1 + i\sqrt{3})/2$. We also use ρ in another sense in this paper (See the sentence prior to Theorem 2.1.), but there should not be any cause for confusion in the sequel.

PROPOSITION 2.5. *If N is a positive integer, then*

$$(1-\rho^{2N})\zeta(2N+1) = 2(\rho^{2N}-1)\sum_{k=1}^{\infty}\frac{k^{-2N-1}}{(-1)^k e^{\pi k\sqrt{3}}-1}$$

$$+ (-1)^N(2\pi)^{2N+1}i\sum_{k=0}^{N+1}\frac{B_{2k}}{(2k)!}\frac{B_{2N+2-2k}}{(2N+2-2k)!}\rho^{2k-1}.$$

PROOF. In Theorem 2.2, let $Vz = -(z+1)/z$ and $z = \rho$ which is a fixed point of V. If $m = 2N$, where N is *any* integer except 0, we get upon the use of (2.7),

$$2(\rho^{2N}-1)\sum_{k=1}^{\infty}\frac{k^{-2N-1}}{(-1)^k e^{\pi k\sqrt{3}}-1} = (1-\rho^{2N})\zeta(2N+1)$$

(2.14)

$$+ (2\pi)^{2N+1}(-1)^{N+1}i\sum_{k=0}^{N+1}\frac{B_{2k}}{(2k)!}\frac{B_{2N+2-2k}}{(2N+2-2k)!}\rho^{2k-1}.$$

The proposition now follows.

If we use the relation

$$\frac{1}{e^x - 1} = (1/2)\coth(x/2) - 1/2, \tag{2.15}$$

Ramanujan's formula (2.11) may be transformed into

$$\alpha^{-N} \sum_{k=1}^{\infty} \frac{\coth(\alpha k)}{k^{2N+1}} = (-\beta)^{-N} \sum_{k=1}^{\infty} \frac{\coth(\beta k)}{k^{2N+1}}$$

$$- 2^{2N+1} \sum_{K=0}^{N+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!} \alpha^{N+1-k}\beta^k,$$

where $\alpha\beta = \pi^2$ and $N \geqq 1$. Proofs of the above have been given by Nanjundiah [47] and Riesel [57]. In particular, if $\alpha = \beta = \pi$ and N is odd, we get from the above, or alternatively, from (2.13),

$$\sum_{k=1}^{\infty} \frac{\coth(\pi k)}{k^{2N+1}} = 2^{2N}\pi^{2N+1} \sum_{k=0}^{N+1} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!}.$$

The last result in the case $n = 3$ was communicated by Ramanujan in the first of his now famous letters to Hardy [53, p. xxvi]. The cases $n = 1$ and $n = 3$ can also be found in Ramanujan's Notebooks [54, vol. II, p. 180, ex. i, ii]. However, in fact, the general result was initially established by Lerch [38] in 1901. Sandham [58] proved the case given in Ramanujan's letter. Lerch's result has also been proven by Watson [67], Sandham [59], Smart [65], and Sayer [60].

If we use (2.15) in Proposition 2.5, we find that for $N \geqq 1$ and $N \not\equiv 0 \pmod 3$

$$\sum_{k=1}^{\infty} \frac{\coth(\pi i k \rho)}{k^{2N+1}} = \frac{(2\pi)^{2N+1}(-1)^N i}{\rho^{2N} - 1} \sum_{k=0}^{N+1} \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!} \rho^{2k-1}.$$

In fact, let V be any elliptic transformation with fixed point $\gamma \in \mathcal{H}$.

Then, unless $(c\gamma + d)^{2N} = 1$, Theorem 2.1, in the case $r_1 = r_2 = 0$, enables us to find in closed form the value of

$$\sum_{k=1}^{\infty} \frac{\coth(\pi i k \gamma)}{k^{2N+1}},$$

where N is a positive integer. In this connection, see also another paper of Lerch [39].

By the use of (2.15), other results in the sequel may be transformed, but we shall not specifically mention this.

As the proofs show, Theorem 2.4 and Proposition 2.5 are valid for any non-zero integer N. We would like now, however, to examine the results for $N < 0$ in more detail.

PROPOSITION 2.6. *Let α and β be positive numbers with $\alpha\beta = \pi^2$. If $n > 1$ is an integer, then*

$$\alpha^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\alpha k} - 1} - (-\beta)^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\beta k} - 1} = \{\alpha^n - (-\beta)^n\} B_{2n}/4n. \tag{2.16}$$

PROOF. In (2.12) let $N = -n$, where $n > 1$. The finite sum on the right side of (2.12) is then empty. Multiply both sides of (2.12) by $(-\beta)^n/2$ and use the fact that [68, p. 268]

$$\zeta(1 - 2n) = -B_{2n}/2n \qquad (n > 0). \tag{2.17}$$

Proposition 2.6 then follows forthwith.

PROPOSITION 2.7. *For $n > 1$ and odd,*

$$\sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\pi k} - 1} = \frac{B_{2n}}{4n}.$$

PROOF. Set $\alpha = \beta = \pi$ in (2.16).

The latter two results were also stated by Ramanujan in his Notebooks. Proposition 2.6 is found in [54, vol. I, p. 259, no. 14] and Proposition 2.7 is found in [54, vol. II, p. 171, Cor. iv]. Proposition 2.6 is also found in [52, p. 269] or [53, p. 190], but no proof is indicated. Proposition 2.7, however, was first established by Glaisher [19] in 1889. Proofs of the more general Proposition 2.6 have been given by S. L. Malurkar [44], B. M. Rao and M. V. Aiyar [55], Hardy [24], [25, pp. 537–539], Nanjundiah [47], Lagrange [36], and Grosswald

[21]. The special case $n = 7$ of Proposition 2.7 was stated in a letter of Ramanujan to Hardy, dated January 16, 1913 [53, p. xxvi]. A proof of Proposition 2.7 has been given by Watson [67]. Sandham [58] proved the special case communicated in the letter to Hardy and later [59] proved the general result. Special cases of Proposition 2.6 were also proved by Aiyar [2].

Propositions 2.6 and 2.7 arise also in the following way. As usual, let $\sigma_\nu(n) = \sum_{d|n} d^\nu$. Then, for $y = 0$ and $n > 1$, we shall see shortly that Proposition 2.6 is equivalent to

$$\sum_{k=1}^{\infty} \sigma_{2n-1}(k)e^{-2\pi ky}$$

$$(2.18) \qquad = (-1)^n y^{-2n} \sum_{k=1}^{\infty} \sigma_{2n-1}(k)e^{-2\pi k/y} + \frac{B_{2n}}{4n}\{1 - (-1)^n y^{-2n}\}.$$

For any ν,

$$(2.19) \qquad \sum_{k=1}^{\infty} \sigma_\nu(k)e^{-2\pi ky} = \sum_{r=1}^{\infty}\sum_{d=1}^{\infty} d^\nu e^{-2\pi rdy} = \sum_{d=1}^{\infty} \frac{d^\nu}{e^{2\pi dy} - 1}.$$

If we use (2.19) in (2.18), let $y = \alpha/\pi$, and suppose that $\alpha\beta = \pi^2$ with $\alpha, \beta > 0$, we obtain Proposition 2.6 once more.

It is difficult to say who first proved (2.18). It, at least, predates Hurwitz's thesis [26] in 1881. Koshliakov [34] in 1928, the same year as Hardy's proof of Proposition 2.6 and Watson's proof of Proposition 2.7, gives (2.18). There are also proofs of (2.18) by Guinand [22] and Chandrasekharan and Narasimhan [15, pp. 15–17].

PROPOSITION 2.8. *Let $n > 1$ be an integer such that $n \not\equiv 0 (\mathrm{mod}\ 3)$. Then*

$$\sum_{k=1}^{\infty} \frac{k^{2N-1}}{(-1)^k e^{\pi k\sqrt{3}} - 1} = \frac{B_{2n}}{4n}.$$

PROOF. In (2.14) let $N = -n$, where $n > 1$. Employing (2.17), we are done.

Proposition 2.8 was first established by Rao and Aiyar [55], [56].

In fact, Propositions 2.7 and 2.8 are special cases of a considerably more general theorem.

THEOREM 2.9. *Let V be an elliptic transformation with fixed point* $\gamma \in \mathcal{H}$, *and let* $n > 1$ *be an integer. Then if* $(c\gamma + d)^{2n} \neq 1$,

$$\sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{-2\pi ik\gamma} - 1} = \frac{B_{2n}}{4n}.$$

PROOF. In Theorem 2.1 set $r_1 = r_2 = 0$ and $m = -2n$, where $n > 1$. Then $h(z, 2n, 0, 0) = 0$ by (2.5). Using (2.7) we find that (2.6) becomes

$$(cz + d)^{-2n} H(Vz, 2n) = H(z, 2n) + \{1 - (cz + d)^{-2n}\}\zeta(1 - 2n).$$

Setting $z = \gamma$ in the formula above and using (2.9) and (2.17), we deduce Theorem 2.9 at once.

COROLLARY 2.10. *Let S denote the set of all the fixed points in* $\mathcal{H}$ *of all the elliptic modular transformations. Then S does not have a limit point in* $\mathcal{H}$.

PROOF. Fix an integer $n > 1$ and consider

$$f(z) = \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{-2\pi ikz} - 1} - \frac{B_{2n}}{4n}.$$

Clearly, $f(z)$ is analytic on $\mathcal{H}$. By Theorem 2.9, $f(z)$ has a zero at each fixed point $\gamma \in \mathcal{H}$. Since the zeros of an analytic function are isolated, Corollary 2.10 follows.

Corollary 2.10 is a well-known result and can be found in [64, p. 9].

The foregoing results required that $n > 1$. We next examine the case $n = 1$.

PROPOSITION 2.11. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Then*

$$\alpha \sum_{k=1}^{\infty} \frac{k}{e^{2\alpha k} - 1} + \beta \sum_{k=1}^{\infty} \frac{k}{e^{2\beta k} - 1} = \frac{\alpha + \beta}{24} - \frac{1}{4}. \tag{2.20}$$

PROOF. In (2.12), put $N = -1$. The result now follows upon the use of (2.17) for $n = 1$.

If we set $\alpha = \beta = \pi$ in (2.20), we deduce the following proposition.

PROPOSITION 2.12. *We have*

(2.21)
$$\sum_{k=1}^{\infty} \frac{k}{e^{2\pi k}-1} = \frac{1}{24} - \frac{1}{8\pi}.$$

Formulas (2.20) and (2.21) also appear in Ramanujan's Notebooks [54, vol. I, p. 257, no. 9; vol. II, p. 170, Cor. 1]. Proposition 2.12 is also stated by Ramanujan as a problem in [50], [53, p. 326]. Ramanujan gave a proof of (2.21) in [53, p. 34], [51, p. 361] that depends upon formulae from the theory of elliptic functions. However, in fact, (2.20) and (2.21) appear to have been first proved by O. Schlömilch [61], [62, p. 157]. Proofs of (2.21) have also been given by Watson [67], Sandham [58], Lewittes [40], [41], and Grosswald [21]. Malurkar [44], Rao and Aiyar [55], Lagrange [36], and Grosswald [21] have also given proofs of the more general result (2.20). The general transformation formulae for $A(z, 2)$ were first proven by Hurwitz [26], [27], and so the latter author had also essentially established the last two propositions before Ramanujan. Another proof of the transformation formula for $A(z, 2)$ in the case $Vz = -1/z$ has been given by Guinand [22].

PROPOSITION 2.13. We have

(2.22)
$$\sum_{k=1}^{\infty} \frac{k}{(-1)^k e^{\pi k\sqrt{3}}-1} = \frac{1}{24} - \frac{1}{4\pi\sqrt{3}}.$$

PROOF. Put $N = -1$ in (2.14), and (2.22) easily follows after an elementary calculation.

Proposition 2.13 has also been proved by Rao and Aiyar [55] and Lewittes [40].

Propositions 2.12 and 2.13 are, in fact, special cases of a much more general theorem.

THEOREM 2.14. *Let V be an elliptic transformation with fixed point $\gamma \in \mathcal{H}$. Then*

$$\sum_{k=1}^{\infty} \frac{k}{e^{-2\pi i k\gamma}-1} = \frac{1}{24} + \frac{c}{4\pi i\{(c\gamma+d)-(c\gamma+d)^{-1}\}}.$$

PROOF. In Theorem 2.1, set $r_1 = r_2 = 0$ and $m = -2$. From (2.5),

$$h(z, 2, 0, 0) = -\sum_{j=1}^{c} (cz+d)^{-1} = -c/(cz+d).$$

Thus, with the aid of (2.7), (2.9), (2.17), and the above, we find that for $z = \gamma$,

$$2(c\gamma + d)^{-2} \sum_{k=1}^{\infty} \frac{k}{e^{-2\pi ik\gamma} - 1} = 2 \sum_{k=1}^{\infty} \frac{k}{e^{-2\pi ik\gamma} - 1} - \frac{\{1 - (c\gamma + d)^{-2}\}}{12} - \frac{c}{2\pi i(c\gamma + d)}.$$

The theorem now easily follows.

Next, we examine Theorem 2.2 in the case $m = 0$. This gives the transformation formulae of $\log \eta(z)$ for the two transformations $-1/z$ and $-(z + 1)/z$.

PROPOSITION 2.15. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Then*

$$\sum_{k=1}^{\infty} \frac{1}{k(e^{2\alpha k} - 1)} - \frac{1}{4} \log \alpha + \frac{\alpha}{12} = \sum_{k=1}^{\infty} \frac{1}{k(e^{2\beta k} - 1)} - \frac{1}{4} \log \beta + \frac{\beta}{12}. \tag{2.23}$$

PROOF. Using (2.5), (2.8), and (2.9) in (2.6) with $m = 0$, we get

$$\sum_{k=1}^{\infty} \frac{1}{k(e^{-2\pi ikVz} - 1)} = \sum_{k=1}^{\infty} \frac{1}{k(e^{-2\pi ikz} - 1)} - \frac{1}{2} \log z - \frac{\pi i}{12} (z + 1/z) + \frac{\pi i}{4}. \tag{2.24}$$

Setting $Vz = -1/z$ and $z = \pi i/\alpha$ in (2.24), we arrive at (2.23) after simple manipulation.

The last result, of course, is just a consequence of the transformation formulae of $\log \eta(z)$ [**6**, pp. 167–168], and so it is difficult to attach any priority to it. Proposition 2.15 is found in Ramanujan's Notebooks [**54**, vol. I, p. 255, no. 8]. Nanjundiah [**47**] and Grosswald [**21**] have also proven Proposition 2.15.

Many interesting results will now be obtained from the differentiation of (2.6) and (2.10).

THEOREM 2.16. *For* $z \in \mathcal{H}$, *m integral, and* $Vz = -1/z$ *or* $Vz = -(z + 1)/z$,

$$mz^{m-1}(1 + (-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{-2\pi ikVz} - 1}$$

$$+\pi(1+(-1)^N)\sum_{k=1}^{\infty}\frac{k^{-2N}}{\sinh^2(\pi k)} \tag{2.28}$$

$$=(2\pi)^{2N+1}\sum_{k=0}^{N+1}(-1)^{k+1}(2k-1)\frac{B_{2k}}{(2k)!}\frac{B_{2N+2-2k}}{(2N+2-2k)!}.$$

PROPOSITION 2.19. *Let $N \neq 0$ be an integer. Then*

$$4N\rho^{2N-1}\left\{(1/2)\zeta(2N+1)+\sum_{k=1}^{\infty}\frac{k^{-2N-1}}{(-1)^k e^{\pi k\sqrt{3}}-1}\right\}$$

$$+\pi i(1-\rho^{2N-2})\sum_{k=1}^{\infty}\frac{k^{-2N}}{\sin^2(\pi k\rho)} \tag{2.29}$$

$$=(2\pi)^{2N+1}(-1)^{N+1}i\sum_{k=0}^{N+1}(2k-1)\frac{B_{2k}}{(2k)!}\frac{B_{2N+2-2k}}{(2N+2-2k)!}\rho^{2k-2}.$$

PROOF. In (2.25), let $Vz = -(z+1)/z$, and set $z=\rho$ which is fixed by V.

For $N > 0$, Proposition 2.17 is the analogue of Theorem 2.4, and Proposition 2.18 is analogous to formula (2.13). Equation (2.28), in fact, includes (2.13), for if we let N be an odd, positive integer in (2.28), we get (2.13) after some manipulation. Proposition 2.19 is, of course, analogous to Proposition 2.5 for $N > 0$ and is more general than the latter proposition which is vacuous for $N \equiv 0 \pmod 3$. The first proof of (2.28) for $N > 0$ appears to be by Grosswald [20]. Further formulae for $\zeta(2N+1)$ may be achieved by further differentiations of (2.6) or (2.10). The formulas for $\zeta(3)$ and $\zeta(5)$ that one gets after two differentiations of (2.10) were discovered by A. Terras [66] in a completely different way. A similar type of formula for $\zeta(3)$ was established by Koshliakov [35].

We now examine in detail some interesting special cases of the previous three results. First, we suppose that $N = -n$, where $n > 1$.

PROPOSITION 2.20. *For $n > 0$ and even,*

$$\sum_{k=1}^{\infty}\frac{k^{2n-1}}{e^{2\pi k}-1}-\frac{\pi}{2n}\sum_{k=1}^{\infty}\frac{k^{2n}}{\sinh^2(\pi k)}=\frac{B_{2n}}{4n}.$$

PROOF. In (2.28), set $N = -n$, where $n > 0$ is even, and then use (2.17).

$$(2.25)\quad \begin{aligned} &+ 2\pi i z^{m-2}(1+(-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m}e^{-2\pi i k Vz}}{(e^{-2\pi i k Vz}-1)^2} \\ &= 2\pi i(1+(-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m}e^{-2\pi i k z}}{(e^{-2\pi i k z}-1)^2} + g'(z,-m) \\ &- (2\pi i)^{m+1} \sum_{k=0}^{m+2} \frac{B_k(1)}{k!} \frac{B_{m+2-k}}{(m+2-k)!}(k-1)(-z)^{k-2}, \end{aligned}$$

where

$$(2.26)\quad g'(z,-m) = \begin{cases} (-1)^{m+1} m z^{m-1}\zeta(m+1), & m \neq 0, \\ -1/z, & m = 0. \end{cases}$$

PROOF. Differentiate (2.10) with respect to z. Note that for each V, $V'(z) = 1/z^2$. Use (2.7) and (2.8) to determine $g'(z,-m)$.

If m is odd, no new information is obtained from (2.25). Thus, we shall assume that m is even and put $m = 2N$.

THEOREM 2.17. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Le N be a non-zero integer. Then*

$$(2.27)\quad \begin{aligned} &4N\alpha^{-N} \left\{ (1/2)\zeta(2N+1) + \sum_{k=1}^{\infty} \frac{k^{-2N-1}}{e^{2\alpha k}-1} \right\} \\ &+ \alpha^{-N+1} \sum_{k=1}^{\infty} \frac{k^{-2N}}{\sinh^2(\alpha k)} - (-\beta)^{-N+1} \sum_{k=1}^{\infty} \frac{k^{-2N}}{\sinh^2(\beta k)} \\ &= 2^{2N+1} \sum_{k=0}^{N+1} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!}(2k-1)\alpha^{N+1-k}\beta^k. \end{aligned}$$

PROOF. Apply Theorem 2.16 with $Vz = -1/z$, and set $z = \pi i/\alpha$. The final form is gotten after multiplying both sides of the resulting equation by $(-1)^N i\alpha^{-1/2}\beta^{-N+1/2}$.

The next proposition is obtained by setting $\alpha = \beta = \pi$ in the previous theorem.

PROPOSITION 2.18. *Let $N \neq 0$ be an integer. Then*

$$4N \left\{ (1/2)\zeta(2N+1) + \sum_{k=1}^{\infty} \frac{k^{-2N-1}}{e^{2\pi k}-1} \right\}$$

Proposition 2.20 then is the natural complement of Proposition 2.7. In fact, if we set $N = -n$, where $n > 1$ is odd, in (2.28), Proposition 2.7 is achieved again. Proposition 2.20 can be found in Ramanujan's Notebooks [54, vol. II, p. 269, no. 5] and is stated in a letter from to be by Rao and Aiyar [55]. Other proofs have been given by Grosswald [21] and C. T. Preece [48].

PROPOSITION 2.21. *Let $n > 1$ be integral. Then*

$$(2.30)\quad \sum_{k=1}^{\infty} \frac{k^{2n-1}}{(-1)^k e^{\pi k\sqrt{3}} - 1} + \frac{\pi i}{4n}(\rho^{-1} - \rho^{-2n+1}) \sum_{k=1}^{\infty} \frac{k^{2n}}{\sin^2(\pi k\rho)} = \frac{B^{2n}}{4n}.$$

PROOF. In (2.29), let $N = -n$, $n > 1$, and use (2.17).

COROLLARY 2.22. *Let $n > 0$ and $n \equiv 0 \pmod 3$. Then*

$$\sum_{k=1}^{\infty} \frac{k^{2n-1}}{(-1)^k e^{\pi k\sqrt{3}} - 1} + \frac{\pi\sqrt{3}}{4n} \sum_{k=1}^{\infty} \frac{k^{2n}}{\sin^2(\pi k\rho)} = \frac{B_{2n}}{4n}.$$

Thus, Corollary 2.22 is complementary to Proposition 2.8. If $n \equiv 1 \pmod 3$, Proposition 2.21 reduces to Proposition 2.8. However, if $n \equiv 2 \pmod 3$, since $\sin^2(\pi k\rho)$ is real, we deduce the following surprising fact from (2.30).

COROLLARY 2.23. *Let $n > 1$ with $n \equiv 2 \pmod 3$. Then*

$$\sum_{k=1}^{\infty} \frac{k^{2n}}{\sin^2(\pi k\rho)} = 0.$$

In analogy with Theorem 2.9, one could easily state a general proposition that would include Propositions 2.20 and 2.21 as special cases. Merely, set $m = 2n$, $n > 1$, in (2.6), differentiate both sides of (2.6) with respect to z, and then let $z = \gamma$.

We consider now the case $N = -1$. Putting $N = -1$ in (2.28) simply gives another verification of Proposition 2.12.

PROPOSITION 2.24. *We have*

$$(2.31)\quad \sum_{k=1}^{\infty} k^2/\sin^2(\pi k\rho) = 1/6\pi^2.$$

PROOF. In Proposition 2.19 let $N = -1$. Employing (2.17), we find that

$$(2.32)\quad \sum_{k=1}^{\infty} \frac{k}{(-1)^k e^{\pi k\sqrt{3}} - 1} + \frac{\pi i}{4}(\rho^{-1} - 1) \sum_{k=1}^{\infty} \frac{k^2}{\sin^2(\pi k\rho)} = \frac{1}{24} + \frac{i}{8\pi\rho^2}.$$

If we equate imaginary parts in (2.32), we arrive at (2.31).

If we equate real parts in (2.32) and use (2.31), we deduce Proposition 2.13 again.

We return to Theorem 2.16 to discern some very striking results for $m = 0$. Rewriting (2.25) for $m = 0$, we have, by the use of (2.26),

$$\sum_{k=1}^{\infty} \frac{e^{-2\pi ikVz}}{(e^{-2\pi ikVz} - 1)^2} = z^2 \sum_{k=1}^{\infty} \frac{e^{-2\pi ikz}}{(e^{-2\pi ikz} - 1)^2} + \frac{iz}{4\pi} - \frac{z^2}{24} + \frac{1}{24}. \tag{2.33}$$

PROPOSITION 2.25. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then*

$$\alpha \sum_{k=1}^{\infty} \operatorname{csch}^2(\alpha k) + \beta \sum_{k=1}^{\infty} \operatorname{csch}^2(\beta k) = -1 + (\alpha + \beta)/6. \tag{2.34}$$

PROOF. In (2.33) let $Vz = -1/z$ and then put $z = \pi i/\alpha$. Equation (2.34) is then obtained very easily.

PROPOSITION 2.26. *We have*

$$\sum_{k=1}^{\infty} \operatorname{csch}^2(\pi k) = 1/6 - 1/2\pi.$$

PROOF. Set $\alpha = \beta = \pi$ in (2.34).

PROPOSITION 2.27. *We have*

$$\sum_{k=1}^{\infty} \csc^2(\pi k\rho) = 1/\pi\sqrt{3} - 1/6.$$

PROOF. In (2.33) let $Vz = -(z + 1)/z$, and then let $z = \rho$. Elementary manipulation then yields the desired result.

Equation (2.34) appears to have been first stated by Lagrange [36]. Proposition 2.26 was evidently first proved by Nanjundiah [47], however. It was later stated as a problem by Shafer [63]. Muckenhoupt [46], Kiyek and Schmidt [33], and Ling [42] have also given proofs. Kiyek and Schmidt [33] and Ling [42], [43] have further summed higher powers of $\operatorname{csch}(\pi k)$.

It is clear that one could produce further identities and summations of the types described above by successive differentiations of (2.6) and (2.10), but we shall not pursue the matter further.

3. In this section, several fascinating relations, some analogous and some generalizing those in § 2, will be derived from Theorem 2.1 when $r_1 = 0$ and $0 < r_2 = r < 1$. Define for k and m integral and w complex,

$$f(k, -m, w) = \begin{cases} 2i \sin(2\pi kw), & m \text{ odd}, \\ 2\cos(2\pi kw), & m \text{ even}. \end{cases} \tag{3.1}$$

Then from (2.1) and (2.2), we find that for $z \in \mathcal{H}$,

$$H(z, -m, 0, r) = \sum_{k=1}^{\infty} \frac{k^{-m-1} f(k, -m, r)}{e^{-2\pi ikz} - 1}. \tag{3.2}$$

For Im $w \geqq 0$ and $\sigma > 1$, define

$$\varphi(w, s) = \sum_{n=1}^{\infty} e^{2\pi inw} n^{-s}. \tag{3.3}$$

If Im $w > 0$, then s may be an arbitrary complex number in the definition (3.3). Then, from (2.1), (2.2) and (3.3), for $z \in \mathcal{H}$,

$$\begin{aligned} &H(z, -m, r, 0) \\ &= \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k^{-m-1} e^{2\pi ik(n+r)z} + (-1)^m \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k^{-m-1} e^{2\pi ik(n-r)z} \\ &= \sum_{k=1}^{\infty} \frac{k^{-m-1} f(k, -m, rz)}{e^{-2\pi ikz} - 1} + \varphi(rz, m+1). \end{aligned} \tag{3.4}$$

Let $\sigma < 0$. Then by Hurwitz's formula for $\zeta(s, \alpha)$ [68, p. 269],

$$\begin{aligned} &\Gamma(s)\{\zeta(s, r) + e^{\pi is}\zeta(s, -r)\} \\ &\quad = \frac{(2\pi)^s}{\sin(\pi s)} \left\{ \sum_{n=1}^{\infty} \frac{\sin(2\pi nr + \pi s/2)}{n^{1-s}} \right. \\ &\qquad \left. + e^{\pi is} \sum_{n=1}^{\infty} \frac{\sin(-2\pi nr + \pi s/2)}{n^{1-s}} \right\} \\ &\quad = (2\pi i)^s \varphi(-r, 1-s). \end{aligned} \tag{3.5}$$

This provides the analytic continuation of $\varphi(w, s)$, when w is real, into the whole complex s-plane. From (3.3) and (3.5), we deduce that $\varphi(w, s)$, for each w with Im $w \geqq 0$, is an entire function of s.

If R_1 is not an integer, we find from (2.4) and (3.5) that

$$g(z, -m, 0, r) = (-1)^{m+1}(cz + d)^m \varphi(-r, m+1). \tag{3.6}$$

Suppose that $Vz = -1/z$ or $Vz = -(z+1)/z$. In either case, $R_1 = r$, $R_2 = 0$ and $\rho = 0$. Thus, from (2.5) we find that

$$h(z, -m, 0, r) = \sum_{k=0}^{m+2} \frac{B_k(1-r)}{k!} \frac{B_{m+2-k}}{(m+2-k)!} (-z)^{k-1}. \tag{3.7}$$

THEOREM 3.1. *For $z \in \mathcal{H}$ and an arbitrary integer m, we have*

$$\begin{aligned} z^m \sum_{k=1}^{\infty} \frac{k^{-m-1} f(k, -m, r)}{e^{-2\pi i k Vz} - 1} &= \sum_{k=1}^{\infty} \frac{k^{-m-1} f(k, -m, rz)}{e^{-2\pi i k z} - 1} \\ &+ \varphi(rz, m+1) + (-1)^{m+1} z^m \varphi(-r, m+1) \\ &- (2\pi i)^{m+1} \sum_{k=0}^{m+2} \frac{B_k(r)}{k!} \frac{B_{m+2-k}}{(m+2-k)!} z^{k-1}. \end{aligned} \tag{3.8}$$

PROOF. Use (3.2), (3.4), (3.6), and (3.7) in Theorem 2.2. Then employ the property [1, p. 804] $B_k(1-r) = (-1)^k B_k(r)$.

We give together the proofs of the next two theorems. Quite remarkably, the first shows that the Fourier series of the Bernoulli polynomials are easily deduced from Theorem 3.1.

THEOREM 3.2. *Let N be an integer and let $0 < r < 1$. Then, for $N \geqq 0$,*

$$B_{2N+1}(r) = \frac{2(2N+1)!(-1)^{N+1}}{(2\pi)^{2N+1}} \sum_{n=1}^{\infty} \frac{\sin(2\pi n r)}{n^{2N+1}}; \tag{3.9}$$

for $N \geqq 1$,

$$B_{2N}(r) = \frac{2(2N)!(-1)^{N+1}}{(2\pi)^{2N}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n r)}{n^{2N}}. \tag{3.10}$$

THEOREM 3.3. *Let N be an integer, let $0 < r < 1$, and let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then if $N \geqq 0$,*

$$\begin{aligned} &\alpha^{-N} \left\{ (1/2) \sum_{k=1}^{\infty} \frac{\cos(2\pi k r)}{k^{2N+1}} + \sum_{k=1}^{\infty} \frac{k^{-2N-1}\cos(2\pi k r)}{e^{2\alpha k} - 1} \right\} \\ &= (-\beta)^{-N} \left\{ (1/2) \sum_{k=1}^{\infty} \frac{e^{-2\beta k r}}{k^{2N+1}} + \sum_{k=1}^{\infty} \frac{k^{-2N-1}\cosh(2\beta k r)}{e^{2\beta k} - 1} \right\} \\ &- 2^{2N} \sum_{k=0}^{N+1} (-1)^k \frac{B_{2k}(r)}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!} \alpha^{N+1-k} \beta^k; \end{aligned} \tag{3.11}$$

if $N \geqq 1$,

$$\alpha^{-N+1/2}\left\{(1/2)\sum_{k=1}^{\infty}\frac{\sin(2\pi kr)}{k^{2N}}+\sum_{k=1}^{\infty}\frac{k^{-2N}\sinh(2\pi kr)}{e^{2\alpha k}-1}\right\}$$

$$(3.12)\qquad =(-1)^N\beta^{-N+1/2}\left\{(1/2)\sum_{k=1}^{\infty}\frac{e^{-2\beta kr}}{k^{2N}}-\sum_{k=1}^{\infty}\frac{k^{-2N}\sinh(2\beta kr)}{e^{2\beta k}-1}\right\}$$

$$-\,2^{2N-1}\sum_{k=0}^{N}(-1)^k\frac{B_{2k+1}(r)}{(2k+1)!}\frac{B_{2N-2k}}{(2N-2k)!}\alpha^{N-k}\beta^k.$$

PROOFS. In Theorem 3.1 let $Vz=-1/z$ and put $z=\pi i/\alpha$. Then

$$(\pi i/\alpha)^m\sum_{k=1}^{\infty}\frac{k^{-m-1}f(k,-m,r)}{e^{2\alpha k}-1}=\sum_{k=1}^{\infty}\frac{k^{-m-1}f(k,-m,\pi ir/\alpha)}{e^{2\beta k}-1}$$

$$(3.13)\qquad +\varphi(\pi ir/\alpha,m+1)+(-1)^{m+1}(\pi i/\alpha)^m\varphi(-r,m+1)$$

$$-\,(2\pi i)^{m+1}\sum_{k=0}^{m+2}\frac{B_k(r)}{k!}\frac{B_{m+2-k}}{(m+2-k)!}(\pi i/\alpha)^{k-1}.$$

First, suppose that m is even and put $m=2N$. Using (3.1) and multiplying both sides of (3.13) by $(-\beta)^{-N}$, we get

$$2\alpha^{-N}\sum_{k=1}^{\infty}\frac{k^{-2N-1}\cos(2\pi kr)}{e^{2\alpha k}-1}=2(-\beta)^{-N}\sum_{k=1}^{\infty}\frac{k^{-2N-1}\cosh(2\beta kr)}{e^{2\beta k}-1}$$

$$(3.14)\qquad +(-\beta)^{-N}\sum_{k=1}^{\infty}k^{-2N-1}e^{-2\beta kr}-\alpha^{-N}\varphi(-r,2N+1)$$

$$-\,2^{2N+1}\sum_{k=0}^{2N+2}i^k\frac{B_k(r)}{k!}\frac{B_{2N+2-k}}{(2N+2-k)!}\alpha^{N+1-k/2}\beta^{k/2}.$$

Let $N\geqq 0$. Equating real parts in (3.14), we arrive at (3.11). Equating imaginary parts in (3.14), we deduce (3.9).

Secondly, suppose that m is odd and put $m=2N-1$. Using (3.1) and multiplying both sides of (3.13) by $(-1)^N\beta^{-N+1/2}$, we obtain

$$2\alpha^{-N+1/2}\sum_{k=1}^{\infty}\frac{k^{-2N}\sin(2\pi kr)}{e^{2\alpha k}-1}$$

$$(3.15)\qquad =2(-1)^{N+1}\beta^{-N+1/2}\sum_{k=1}^{\infty}\frac{k^{-2N}\sinh(2\beta kr)}{e^{2\beta k}-1}$$

$$+ (-1)^N\beta^{-N+1/2} \sum_{k=1}^{\infty} k^{-2N}e^{-2\beta kr} - i\alpha^{-N+1/2}\varphi(-r, 2N)$$

$$- 2^{2N} \sum_{k=0}^{2N+1} i^{k-1} \frac{B_k(r)}{k!} \frac{B_{2N+1-k}}{(2N+1-k)!} \alpha^{N+1/2-k/2}\beta^{k/2}.$$

If we equate real parts in (3.15), we get (3.12), and if we equate imaginary parts in (3.15), we get (3.10).

Let us examine in detail (3.11) when $N = 0$. Putting $N = 0$ in (3.11), for $\alpha\beta = \pi^2$ we arrive at

$$\sum_{k=1}^{\infty} \frac{\cos(2\pi kr)}{k(e^{2\alpha k}-1)} = 2 \sum_{k=1}^{\infty} \frac{\cosh(2\beta kr)}{k(e^{2\beta k}-1)} + \sum_{k=1}^{\infty} \frac{e^{-2\beta kr}}{k} - \sum_{k=1}^{\infty} \frac{\cos(2\pi kr)}{k} - \alpha/6 + B_2(r)\beta. \tag{3.16}$$

Now

$$\sum_{k=1}^{\infty} \frac{e^{-2\beta kr}}{k} = -\log(1 - e^{-2\beta r}) = \beta r - \log\{2\sinh(\beta r)\}$$

and

$$\sum_{k=1}^{\infty} \frac{\cos(2\pi kr)}{k} = -\log\{2\sin(\pi r)\}.$$

Since also $B_2(r) = r^2 - r + 1/6$, (3.16) becomes

$$2\sum_{k=1}^{\infty} \frac{\cos(2\pi kr)}{k(e^{2\alpha k}-1)} - 2\sum_{k=1}^{\infty} \frac{\cosh(2\beta kr)}{k(e^{2\beta k}-1)} = r^2\beta - \alpha/6 + \beta/6 + \log\left\{\frac{\sin(\pi r)}{\sinh(\beta r)}\right\}. \tag{3.17}$$

We will put (3.17) in a slightly different form. Replace α by α^2 and β by β^2, and let $r = \eta/\beta$ in the new notation for β. We then obtain the following version of (3.17) which evinces more symmetry.

PROPOSITION 3.4. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi$. Suppose that $0 < \alpha\eta < \pi$. Then*

$$2\sum_{k=1}^{\infty} \frac{\cos(2\alpha\eta k)}{k(e^{2\alpha^2 k}-1)} - 2\sum_{k=1}^{\infty} \frac{\cosh(2\beta\eta k)}{k(e^{2\beta^2 k}-1)} = \eta^2 - \alpha^2/6 + \beta^2/6 + \log\left\{\frac{\sin(\alpha\eta)}{\sinh(\beta\eta)}\right\}. \tag{3.18}$$

Formula (3.18) is found in Ramanujan's Notebooks [54, vol. I, p. 257, no. 12; vol. II, p. 169, no. 8 (ii)]. A proof of (3.18) has been given by Lagrange [36].

If we put $\alpha = \beta = \pi$ and $r = 1/2$ in either (3.16) or (3.17), we get

$$2\sum_{k=1}^{\infty}\frac{(-1)^k}{k(e^{2\pi k}-1)} = \sum_{k=1}^{\infty}\frac{\operatorname{csch}(\pi k)}{k} + \log 2 - \pi/4.$$

Theorems 3.2 and 3.3 are generalizations of Theorems 2.3 and 2.4. In fact, if we let r tend to 0 in (3.10) and (3.12), we obtain Euler's formula again in each case. Letting r tend to 0 in (3.11), we get Ramanujan's formula (2.11) once more. Formulas (3.11) and (3.12) are natural generalizations of (2.11) and may be considered as "Ramanujan formulas" for

$$\sum_{k=1}^{\infty}\frac{\cos(2\pi kr)}{k^{2N+1}} \quad\text{and}\quad \sum_{k=1}^{\infty}\frac{\sin(2\pi kr)}{k^{2N}},$$

respectively.

Putting $r = 1/4$ in (3.9), using the fact that [1, p. 806]

$$B_n(1/4) = -n4^{-n}E_{n-1} \qquad (n \geqq 1), \tag{3.19}$$

where E_j is the j-th Euler number, and using (1.1), we have

$$\begin{aligned} L(2N+1) &= \frac{(2\pi)^{2N+1}(-1)^{N+1}}{2(2N+1)!}B_{2N+1}(1/4) \\ &= \frac{(\pi/2)^{2N+1}(-1)^N}{2(2N)!}E_{2N}, \end{aligned}$$

which, of course, is well-known [1, p. 807].

Proposition 3.5. *Let $N \geqq 1$ be integral, and let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then*

$$\begin{aligned} &\alpha^{-N+1/2}\left\{(1/2)L(2N) + \sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)^{-2N}}{e^{2\alpha(2k+1)}-1}\right\} \\ &= \frac{(-1)^N\beta^{-N+1/2}}{4}\sum_{k=1}^{\infty}\frac{\operatorname{sech}(\beta k/2)}{k^{2N}} \\ &\quad + 2^{2N-3}\sum_{k=0}^{N}(-1)^k\frac{E_{2k}}{(2k)!}\frac{B_{2N-2k}}{(2N-2k)!}\alpha^{N-k}(\beta/8)^k. \end{aligned} \tag{3.20}$$

PROOF. Put $r = 1/4$ in (3.12) and use (1.1) and (3.19). After simplification we get (3.20).

Equation (3.20) is also in Ramanujan's Notebooks [54, vol. I, p. 274; vol. II, pp. 177–178, no. 21 (iii)]. The first proof of (3.20) was given by Chowla [16].

PROPOSITION 3.6. *Let $N \geqq 1$ be integral, and let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then*

$$\alpha^{-N}\left\{ (1/2)(2^{-2N} - 1)\zeta(2N+1) + \sum_{k=1}^{\infty} \frac{k^{-2N-1}(-1)^k}{e^{2\alpha k} - 1} \right\}$$

$$(3.21) \qquad = (1/2)(-\beta)^{-N} \sum_{k=1}^{\infty} \frac{\operatorname{csch}(\beta k)}{k^{2N+1}} - 2^{2N} \sum_{k=0}^{N+1} (-1)^k(2^{1-2k} - 1) \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!} \alpha^{N+1-k}\beta^k.$$

PROOF. Put $r = 1/2$ in (3.11). Recall that for $\sigma > 1$

$$\sum_{k=1}^{\infty} (-1)^k k^{-s} = (2^{1-s} - 1)\zeta(s)$$

and that [1, p. 805]

$$(3.22) \qquad B_{2k}(1/2) = (2^{1-2k} - 1)B_{2k}.$$

After some simplification, (3.21) is readily achieved.

By putting $\alpha = \beta = \pi$ in (3.14) and (3.15), we easily deduce the next proposition.

PROPOSITION 3.7. *Let N be any integer and let $0 < r < 1$. Then*

$$2 \sum_{k=1}^{\infty} \frac{k^{-2N-1}\cos(2\pi kr)}{e^{2\pi k} - 1} = 2(-1)^N \sum_{k=1}^{\infty} \frac{k^{-2N-1}\cosh(2\pi kr)}{e^{2\pi k} - 1}$$

$$(3.23) \qquad + (-1)^N \sum_{k=1}^{\infty} k^{-2N-1}e^{-2\pi kr} - \operatorname{Re}\{\varphi(-r, 2N+1)\} - (2\pi)^{2N+1} \sum_{k=0}^{N+1} (-1)^k \frac{B_{2k}(r)}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!}$$

and

$$2\sum_{k=1}^{\infty}\frac{k^{-2N}\sin(2\pi kr)}{e^{2\pi k}-1} = 2(-1)^{N+1}\sum_{k=1}^{\infty}\frac{k^{-2N}\sinh(2\pi kr)}{e^{2\pi k}-1}$$

$$(3.24)\qquad + (-1)^N\sum_{k=1}^{\infty} k^{-2N}e^{-2\pi kr} + \operatorname{Im}\{\varphi(-r,2N)\}$$

$$-(2\pi)^{2N}\sum_{k=0}^{N}(-1)^k\frac{B_{2k+1}(r)}{(2k+1)!}\frac{B_{2N-2k}}{(2N-2k)!}.$$

If $N<0$, $\operatorname{Re}\{\varphi(-r,2N+1)\}$ *may be replaced by* $\varphi(-r,2N+1)$ *and* $\operatorname{Im}\{\varphi(-r,2N)\}$ *may be replaced by* $-i\varphi(-r,2N)$ *in the above formulas.*

Interesting consequences will now be deduced from the preceding formulas when $N\leqq 0$. Some facts about Eulerian numbers will be needed. For $\lambda\neq 1$, the Eulerian numbers $H_n[\lambda]$, $n\geqq 0$, may be defined by

$$(3.25)\qquad \frac{1-\lambda}{e^x-\lambda}=\sum_{n=0}^{\infty}H_n[\lambda]\frac{x^n}{n!},$$

where $|x|$ is sufficiently small. Apostol [4, p. 164] has calculated $\varphi(-r,-n)$, where n is a non-negative integer. Rewriting his findings in terms of Eulerian numbers, we find that

$$(3.26)\qquad \varphi(-r,-n)=\frac{(-1)^nH_n[e^{-2\pi ir}]}{e^{2\pi ir}-1}.$$

PROPOSITION 3.8. *Let* $0<r<1$ *and let* n *be an integer. If* $n>1$, *then*

$$(3.27)\qquad 2\sum_{k=1}^{\infty}\frac{k^{2n-1}\cos(2\pi kr)}{e^{2\pi k}-1} - 2(-1)^n\sum_{k=1}^{\infty}\frac{k^{2n-1}\cosh(2\pi kr)}{e^{2\pi k}-1}$$

$$-(-1)^n\sum_{k=1}^{\infty}k^{2n-1}e^{-2\pi kr}=\frac{H_{2n-1}[e^{-2\pi ir}]}{e^{2\pi ir}-1};$$

if $n\geqq 1$, *then*

$$(3.28)\qquad 2\sum_{k=1}^{\infty}\frac{k^{2n}\sin(2\pi kr)}{e^{2\pi k}-1} + 2(-1)^n\sum_{k=1}^{\infty}\frac{k^{2n}\sinh(2\pi kr)}{e^{2\pi k}-1}$$

$$-(-1)^n\sum_{k=1}^{\infty}k^{2n}e^{-2\pi kr}=-i\,\frac{H_{2n}[e^{-2\pi ir}]}{e^{2\pi ir}-1}.$$

PROOF. Put $N = -n$ in (3.23) and (3.24) and use (3.26) to obtain (3.27) and (3.28), respectively.

PROPOSITION 3.9. *If $n > 1$ is integral, then*

$$2\sum_{k=1}^{\infty} \frac{(-1)^k k^{2n-1}}{e^{2\pi k}-1} - (-1)^n \sum_{k=1}^{\infty} k^{2n-1}\operatorname{csch}(\pi k) = \frac{(2^{2n}-1)B_{2n}}{2n}.$$

PROOF. Put $r = 1/2$ and $N = -n$ in (3.27) and use the fact that [14, p. 257]

$$H_{n-1}[-1] = 2(1-2^n)B_n/n \quad (n \geqq 1).$$

An equivalent formulation of Proposition 3.9 has been given by Glaisher [19].

PROPOSITION 3.10. *Let $0 < r < 1$ and $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then*

$$\alpha \sum_{k=1}^{\infty} \frac{k\cos(2\pi kr)}{e^{2\alpha k}-1} + \beta \sum_{k=1}^{\infty} \frac{k\cosh(2\beta kr)}{e^{2\beta k}-1} = -1/4 + (1/8)(\alpha\csc^2(\pi r) - \beta\operatorname{csch}^2(\beta r)). \tag{3.29}$$

PROOF. In (3.14) put $N = -1$. An elementary calculation shows that

$$\sum_{k=1}^{\infty} ke^{-2\beta kr} = (1/4)\operatorname{csch}^2(\beta r).$$

From (3.25), $H_1[\lambda] = -1/(1-\lambda)$. Hence, from (3.26),

$$\varphi(-r, -1) = -(1/4)\csc^2(\pi r).$$

With these calculations, (3.29) is now immediate.

Proposition 3.10 is due originally to Schlömilch [61], [62, p. 157]. The special case when $\alpha = \beta = \pi$ is given by Watson [67]. Formula (3.29) may be written in a slightly more symmetric form by employing the device used to produce (3.18).

COROLLARY 3.11. *We have*

$$\sum_{k=1}^{\infty} \frac{k(-1)^k}{e^{2\pi k}-1} + (1/2)\sum_{k=1}^{\infty} k\operatorname{csch}(\pi k) = 1/8 - 1/4\pi.$$

PROOF. Put $\alpha = \beta = \pi$ and $r = 1/2$ in (3.29).

PROPOSITION 3.12. *Let* $0 < r < 1$ *and* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Then*

$$\alpha^{1/2} \sum_{k=1}^{\infty} \frac{\sin(2\pi kr)}{e^{2\alpha k} - 1} + \beta^{1/2} \sum_{k=1}^{\infty} \frac{\sinh(2\beta kr)}{e^{2\beta k} - 1} \tag{3.30}$$
$$= (1/4)(\beta^{1/2} \coth(\beta r) - \alpha^{1/2} \cot(\pi r)) - (1/2) r\beta^{1/2}.$$

PROOF. Put $N = 0$ in (3.15). From (3.26),

$$\varphi(-r, 0) = \frac{1}{e^{2\pi i r} - 1} = -(1/2)i \cot(\pi r) - 1/2.$$

Of course,

$$\sum_{k=1}^{\infty} e^{-2\beta kr} = \frac{1}{e^{2\beta r} - 1}.$$

Putting the above values in (3.15), we obtain (3.30) after a little simplification.

The first proof of Proposition 3.12 is due to Schlömilch [61], [62, p. 156]. Another proof has been given by Lagrange [36].

Formula (3.30) takes a somewhat more symmetric shape if we replace α by α^2, β by β^2, and let $r = \eta/\beta$. Then for $\alpha\beta = \pi$ and $0 < \alpha\eta < \pi$, (3.30) is transformed into

$$\alpha \sum_{k=1}^{\infty} \frac{\sin(2\alpha\eta k)}{e^{2\alpha^2 k} - 1} + \beta \sum_{k=1}^{\infty} \frac{\sinh(2\beta\eta k)}{e^{2\beta^2 k} - 1} \tag{3.31}$$
$$= (1/4)\beta \coth(\beta\eta) - (1/4)\alpha \cot(\alpha\eta) - (1/2)\eta.$$

In the latter form, (3.31) is found in Ramanujan's Notebooks [54, vol. I, p. 257, 259, no. 13; vol. II, p. 169, no. 8(i)].

We have only proven identities in this section in the case $Vz = -1/z$. Theorem 3.1, of course, can be also used to generate identities involving ρ. Further identities may be achieved by differentiating (3.8). Lastly, we mention that beautiful identities may also be obtained from Theorem 2.1 when $0 < r_1, r_2 < 1$.

4. In this section we derive many interesting series relations involving characters. For one particular character, that character associated with $L(s)$, the results are largely due to Ramanujan and can be found in his Notebooks. We shall confine our results to the case $Vz = -1/z$.

First, we introduce some definitions and notation in order to state a result of [11] in the case $Vz = -1/z$. As before, let r_1 and r_2 be real.

Let χ_1 and χ_2 be primitive characters, each of modulus k. Extend the definition of χ_1 to the set of all real numbers by defining $\chi_1(r) = 0$ if r is not an integer. For $z \in \mathcal{H}$ and m integral, define

$$\begin{aligned} H(z, -m; \chi_1, \chi_2; r_1, r_2) &= A(z, -m; \chi_1, \chi_2; r_1, r_2) \\ &+ \chi_1(-1)\chi_2(-1)(-1)^m A(z, -m; \chi_1, \chi_2; -r_1, -r_2), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &A(z, -m; \chi_1\chi_2; r_1, r_2) \\ &= \sum_{\mu > -r_1} \chi_1(\mu) \sum_{\nu=1}^{\infty} \chi_2(\nu)\nu^{-m-1} e^{2\pi i\nu\{(\mu+r^1)z+r^2\}/k} \end{aligned} \tag{4.2}$$

Let $G(z, \chi)$ denote the Gaussian sum,

$$G(z, \chi) = \sum_{j=1}^{k-1} \chi(j)^{2\pi ijz/k},$$

and put $G(1, \chi) = G(\chi)$. For primitive characters, we have the fundamental result [**6**, p. 313]

$$G(\chi)G(\bar{\chi}) = \chi(-1)k. \tag{4.3}$$

The generalized Bernoulli polynomials $B_n(x, \chi)$, $n \geqq 0$, are defined by

$$B_n(x, \chi) = k^{n-1} \sum_{j=1}^{k-1} \bar{\chi}(j) B_n\left(\frac{x+j}{k}\right), \tag{4.4}$$

and the generalized Bernoulli numbers $B_n(\chi)$, $n \geqq 0$, are defined by $B_n(\chi) = B_n(0, \chi)$. In the sequel, without explicitly saying so, we shall repeatedly use the facts that for $j \geqq 0$ [**10**],

$$B_{2j+1}(\chi) = 0 \quad (\chi \text{ even})$$

and

$$B_{2j}(\chi) = 0 \quad (\chi \text{ odd}).$$

THEOREM 4.1. *Let $z \in \mathcal{H}$ and let m be an integer. Suppose that $\chi_1(r_1) = \chi_2(r_2) = 0$. Then*

$$\begin{aligned} &(-zk/2\pi i)^m G(\chi_2) H(-1/z, -m; \chi_1, \bar{\chi}_2; r_1, r_2) \\ &= \chi_1(-1)(-k/2\pi i)^m G(\chi_1) H(z, -m; \chi_2; \bar{\chi}_1; r_2, -r_1) \\ &+ \chi_1(-1)\chi_2(-1)(-1)^m 2\pi i \cdot \sum_{j=0}^{m+2} \frac{B_j(-r_2, \bar{\chi}_2)}{j!} \frac{B_{m+2-j}(-r_1, \bar{\chi}_1)}{(m+2-j)!}(-z)^{j-1}. \end{aligned} \tag{4.5}$$

PROOF. In Theorem 1(i) of [11], let $s = -m$ and $Vz = -1/z$. Using (4.4) in the calculation of $f(z, -m; r_1, r_2; j, \mu, \nu)$ in (3.2) of [11], we obtain (4.5).

THEOREM 4.2. *Let m be an integer and suppose that $\chi_1(-1)\chi_2(-1)(-1)^m = 1$. Then for $z \in \mathcal{H}$,*

$$(-zk/2\pi i)^m G(\chi_2) \sum_{\nu=1}^{\infty} \frac{\bar{\chi}_2(\nu)\nu^{-m-1}G(-\nu/z, \chi_1)}{1 - e^{-2\pi i\nu/z}}$$

$$= \chi_1(-1)(-k/2\pi i)^m G(\chi_1) \sum_{\nu=1}^{\infty} \frac{\bar{\chi}_1(\nu)\nu^{-m-1}G(\nu z, \chi_2)}{1 - e^{2\pi i\nu z}} \tag{4.6}$$

$$+ \pi i \sum_{j=1}^{m+1} \frac{B_j(\bar{\chi}_2)}{j!} \frac{B_{m+2-j}(\bar{\chi}_1)}{(m+2-j)!}(-z)^{j-1}.$$

PROOF. In (4.5), let $r_1 = r_2 = 0$. Letting $\mu = rk + \ell, 0 \leqq r < \infty$, $0 \leqq \ell \leqq k - 1$, and using the hypothesis $\chi_1(-1)\chi_2(-1)(-1)^m = 1$, we find from (4.1) and (4.2) that for $z \in \mathcal{H}$,

$$H(z, -m; \chi_1, \chi_2; 0, 0) = 2A(z, -m; \chi_1, \chi_2; 0, 0)$$

$$= 2\sum_{\nu=1}^{\infty} \chi_2(\nu)\nu^{-m-1} \sum_{\ell=1}^{k-1} \chi_1(\ell)e^{2\pi i\ell\nu z/k} \sum_{r=0}^{\infty} e^{2\pi i r\nu z}$$

$$= 2\sum_{\nu=1}^{\infty} \frac{\chi_2(\nu)\nu^{-m-1}G(\nu z, \chi_1)}{1 - e^{2\pi i\nu z}}.$$

Using the above calculation in (4.5), we conclude (4.6) upon the realization from (4.4) that $B_0(\chi) = 0$.

Let us further specialize by setting $\chi = \chi_1 = \chi_2$. Hence, m is now even, and so we write $m = 2N$. Let $z = \pi i/k\alpha$, where $\alpha > 0$, and determine $\beta > 0$ by the relation $\alpha\beta = \pi^2/k^2$. Thus, we deduce the following theorem.

THEOREM 4.3. *Let N be an integer. Let $\alpha, \beta > 0$ satisfy the relation $\alpha\beta = \pi^2/k^2$. Then*

$$\alpha^{-N}G(\chi)\sum_{\nu=1}^{\infty}\frac{\bar{\chi}(\nu)\nu^{-2N-1}G(i\alpha k\nu/\pi,\chi)}{1-e^{-2\alpha k\nu}}$$

$$(4.7)\qquad =\chi(-1)(-\beta)^{-N}G(\chi)\sum_{\nu=1}^{\infty}\frac{\bar{\chi}(\nu)\nu^{-2N-1}G(i\beta k\nu/\pi,\chi)}{1-e^{-2\beta k\nu}}$$

$$-\,2^{2N}k\sum_{j=1}^{2N+1}(-i)^{j}\frac{B_j(\bar{\chi})}{j!}\frac{B_{2N+2-j}(\bar{\chi})}{(2N+2-j)!}\alpha^{N+1-j/2}\beta^{j/2}.$$

THEOREM 4.4. *Let N be an integer such that* $\chi(-1)(-1)^N = -1$. *Then*

$$(4.8)\qquad \sum_{\nu=1}^{\infty}\frac{\bar{\chi}(\nu)\nu^{-2N-1}G(i\nu,\chi)}{1-e^{-2\pi\nu}}$$

$$= -(1/4)(2\pi/k)^{2N+1}\chi(-1)G(\bar{\chi})\sum_{j=1}^{2N+1}(-i)^{j}\frac{B_j(\bar{\chi})}{j!}\frac{B_{2N+2-j}(\bar{\chi})}{(2N+2-j)!}.$$

PROOF. Let $\alpha = \beta = \pi/k$ in Theorem 4.3 and employ (4.3).

We now examine some examples of the above results for certain characters. Throughout the remainder of the paper, χ_1, χ_2, χ_3, and χ_4 shall denote the following primitive characters. First,

$$\chi_1(n) = \begin{cases} (-1)^{(n-1)/2}, & n \text{ odd},\\ 0, & n \text{ even},\end{cases}$$

and

$$\chi_2(n) = \begin{cases} (-1)^{(n^2-1)/8}, & n \text{ odd},\\ 0, & n \text{ even}.\end{cases}$$

Next, $\chi_3(n) = (\frac{n}{3})$ and $\chi_4(n) = (\frac{n}{5})$, where $(\frac{n}{3})$ and $(\frac{n}{5})$ denote Legendre symbols. χ_1 and χ_3 are odd while χ_2 and χ_4 are even.

First, we discuss the implications of our results for χ_1. We have $G(ix,\chi_1) = e^{-2\pi x/4} - e^{-6\pi x/4} = 2e^{-\pi x}\sinh(\pi x/2)$ and $G(\chi_1) = 2i$. From (4.4), for odd j,

$$(4.9)\qquad B_j(\chi_1) = 4^{j-1}\{B_j(1/4) - B_j(3/4)\} = -jE_{j-1}/2,$$

where we have used (3.19) and the fact that $B_j(1/4) = -B_j(3/4)$ [1, p. 806].

PROPOSITION 4.5. *Let* $\alpha,\beta > 0$ *with* $\alpha\beta = \pi^2/16$. *Then if N is any integer,*

$$\alpha^{-N}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu+1)^{2N+1}\cosh\{2(2\nu+1)\alpha\}}$$

$$(4.10)\qquad +(-\beta)^{-N}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu+1)^{2N+1}\cosh\{2(2\nu+1)\beta\}}$$

$$=2^{2N-2}\pi\sum_{j=0}^{N}(-1)^{j}\frac{E_{2j}}{(2j)!}\frac{E_{2N-2j}}{(2N-2j)!}\alpha^{N-j}\beta^{j}.$$

PROOF. In (4.7), let $\chi=\chi_1$. Use (4.9) and the calculations immediately prior to (4.9).

Formula (4.10) is also found in Ramanujan's Notebooks [54, vol. I, p. 279; vol. II, p. 177, no. 21 (ii)]. The first published proof of (4.10) is due to Malurkar [44]. Nanjundiah [47] has also proven (4.10). Formula (4.10) may be considered as an analogue of (2.11).

PROPOSITION 4.6. *Let M be any integer. Then*

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\operatorname{sech}\{(2\nu+1)\pi/2\}}{(2\nu+1)^{4M+1}}$$

$$(4.11)\qquad =(1/4)(\pi/2)^{4M+1}\sum_{J=0}^{2M}(-1)^{j}\frac{E_{2j}}{(2j)!}\frac{E_{4M-2j}}{(4M-2j)!}.$$

PROOF. In (4.10), put $N=2M$, where M is an integer, and set $\alpha=\beta=\pi/4$. Formula (4.11) follows forthwith.

For $M>0$, Watson [67], Sandham [59], and Riesel [57] have also given proofs of (4.11).

PROPOSITION 4.7. *Let $\alpha,\beta>0$ with $\alpha\beta=\pi^2/16$. If n is any positive integer, then*

$$\alpha^{n}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(2\nu+1)^{2n-1}}{\cosh\{2(2\nu+1)\alpha\}}+(-\beta)^{n}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(2\nu+1)^{2n-1}}{\cosh\{2(2\nu+1)\beta\}}=0.$$

PROOF. Let $N=-n$ in (4.10).

This last result is stated explicitly in Ramanujan's Notebooks [54, vol. I, p. 276; vol. II, p. 172, no. 14].

PROPOSITION 4.8. *Let n be any positive integer. Then*

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(2\nu+1)^{4n-1}}{\cosh\{(2\nu+1)\pi/2\}}=0.$$

PROOF. Put $M=-n$ in (4.11).

Proposition 4.8 was stated as a problem by Ramanujan in [49]. It is also stated in Ramanujan's Notebooks [54, vol. II, p. 172, Cor. to no. 14]. Besides the proof of Malurkar [44], proofs have also been given by Chowla [16], Nanjundiah [47], Sandham [59], and Riesel [57].

Proposition 4.9. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2/16$. Then*

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}\operatorname{sech}\{2(2\nu+1)\alpha\}}{2\nu+1} + \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}\operatorname{sech}\{2(2\nu+1)\beta\}}{2\nu+1} = \pi/4.$$

Proof. Put $N = 0$ in (4.10).

This last result is found in Ramanujan's Notebooks [54, vol. I, p. 277].

Secondly, we examine examples involving χ_2.

Proposition 4.10. *Let M be any integer. Then*

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(\nu+1)/2}\sinh\{(2\nu+1)\pi/4\}}{(2\nu+1)^{4M+3}\cosh\{(2\nu+1)\pi/2\}} = 2^{-1/2}(\pi/4)^{4M+3} \sum_{j=1}^{2M+1} (-1)^{j+1} \frac{B_{2j}(\chi_2)}{(2j)!} \frac{B_{4M+4-2j}(\chi_2)}{(4M+4-2j)!}. \tag{4.12}$$

Proof. We have

$$G(ix, \chi_2) = e^{-\pi x/4} - e^{-3\pi x/4} - e^{-5\pi x/4} + e^{-7\pi x/4} = 2e^{-\pi x}\{\cosh(3\pi x/4) - \cosh(\pi x/4)\},$$

and $G(\chi_2) = 2^{3/2}$. Apply Theorem 4.4 and note that N must be odd. Thus, set $N = 2M + 1$. Since

$$\frac{G(ix, \chi_2)}{1 - e^{-2\pi x}} = \frac{\sinh(\pi x/4)}{\cosh(\pi x/2)},$$

formula (4.12) follows.

An equivalent formulation of Proposition 4.10 is due to Chowla [16].

Proposition 4.11. *Let n be any positive integer. Then*

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(\nu+1)/2}(2\nu+1)^{4n-3}\sinh\{(2\nu+1)\pi/4\}}{\cosh\{(2\nu+1)\pi/2\}} = 0.$$

PROOF. Put $M = -n$, where $n > 0$, in (4.12).

PROPOSITION 4.12. *We have*

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(\nu+1)/2}\sinh\{(2\nu+1)\pi/4\}}{(2\nu+1)^3\cosh\{(2\nu+1)\pi/2\}} = \pi^3/64\sqrt{2}.$$

PROOF. From (4.4), $B_2(\chi_2) = 2$. Apply (4.12) with $M = 0$.

Thirdly, we apply our theorem to χ_3.

PROPOSITION 4.13. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2/9$. *Then if* N *is any integer,*

$$(4.13) \quad \begin{aligned} &\alpha^{-N} \sum_{\nu=1}^{\infty} \left(\frac{\nu}{3}\right) \frac{\sinh(\alpha\nu)}{\nu^{2N+1}\sinh(3\alpha\nu)} \\ &+ (-\beta)^{-N} \sum_{\nu=1}^{\infty} \left(\frac{\nu}{3}\right) \frac{\sinh(\beta\nu)}{\nu^{2N+1}\sinh(3\beta\nu)} \\ &= (2^{2N}\pi/\sqrt{3}) \sum_{j=0}^{N} (-1)^j \frac{B_{2j+1}(\chi_3)}{(2j+1)!} \frac{B_{2N+1-2j}(\chi_3)}{(2N+1-2j)!} \alpha^{N-j}\beta^j. \end{aligned}$$

PROOF. Apply Theorem 4.3. Note that $G(\chi_3) = i\sqrt{3}$, and $G(ix, \chi_3) = 2e^{-\pi x}\sinh(\pi x/3)$. Formula (4.13) then readily follows.

PROPOSITION 4.14. *Let* M *be any integer. Then*

$$(4.14) \quad \begin{aligned} &\sum_{\nu=1}^{\infty} \left(\frac{\nu}{3}\right) \frac{\sinh(\pi\nu/3)}{\nu^{4M+1}\sinh(\pi\nu)} \\ &= \frac{\sqrt{3}(2\pi/3)^{4M+1}}{4} \sum_{j=0}^{2M} (-1)^j \frac{B_{2j+1}(\chi_3)}{(2j+1)!} \frac{B_{4M+1-2j}(\chi_3)}{(4M+1-2j)!}. \end{aligned}$$

PROOF. Let $N = 2M$ and $\alpha = \beta = \pi/3$ in (4.13).

PROPOSITION 4.15. *Let* n *be any positive integer. Then*

$$\begin{aligned} &\sum_{\nu=0}^{\infty} \frac{(3\nu+1)^{4n-1}\sinh\{(3\nu+1)\pi/3\}}{\sinh\{(3\nu+1)\pi\}} \\ &= \sum_{\nu=0}^{\infty} \frac{(3\nu+2)^{4n-1}\sinh\{(3\nu+2)\pi/3\}}{\sinh\{(3\nu+2)\pi\}}. \end{aligned}$$

PROOF. Let $M = -n$, $n > 0$, in (4.14).

PROPOSITION 4.16. *We have*

$$\sum_{\nu=1}^{\infty}\left(\frac{\nu}{3}\right)\frac{\sinh(\pi\nu/3)}{\nu\sinh(\pi\nu)}=\pi/18\sqrt{3}.$$

PROOF. From (4.4), $B_1(\chi_3) = -1/3$. Now let $M = 0$ in (4.14).

Lastly, we consider identities for χ_4.

PROPOSITION 4.17. *Let M be any integer. Then*

$$\sum_{\nu=1}^{\infty}\left(\frac{\nu}{5}\right)\frac{\cosh(3\pi\nu/5)-\cosh(\pi\nu/5)}{\nu^{4M+3}\sinh(\pi\nu)}$$

$$=\frac{\sqrt{5}(2\pi/5)^{4M+3}}{4}\sum_{j=1}^{2M+1}(-1)^{j+1}\frac{B_{2j}(\chi_4)}{(2j)!}\frac{B_{4M-2j+4}(\chi_4)}{(4M-2j+4)!}.$$

PROOF. Apply Theorem 4.4. Here $G(\chi_4) = \sqrt{5}$ and $G(ix,\chi_4) = 2e^{-\pi x}\{\cosh(3\pi x/5) - \cosh(\pi x/5)\}$. The result now follows easily.

In analogy with our work in § 2, we can obtain further formulae from differentiating (4.6) with respect to z. For example, one can obtain a general formula for $\chi(-1)(-1)^N = \pm 1$ that includes Theorem 4.4 when $\chi(-1)(-1)^N = -1$.

Next, series relations will be derived when $r_1 = 0$ and $0 < r_2 = r < 1$ in Theorem 4.1. Using (4.1) and (4.2) and letting $\mu = rk + \ell$, $0 \leqq r < \infty$, $0 \leqq \ell \leqq k-1$, we easily deduce that for $z \in \mathcal{H}$,

$$H(z,-m;\chi,\bar{\chi};0,r)=\sum_{\nu=1}^{\infty}\frac{\bar{\chi}(\nu)\nu^{-m-1}f(\nu,-m,r/k)G(\nu z,\chi)}{1-e^{2\pi i\nu z}}\tag{4.15}$$

and

$$H(z,-m;\chi,\bar{\chi};r,0)=\sum_{\nu=1}^{\infty}\frac{\bar{\chi}(\nu)\nu^{-m-1}f(\nu,-m,rz/k)G(\nu z,\chi)}{1-e^{2\pi i\nu z}},\tag{4.16}$$

where $f(\nu, -m, w)$ is defined by (3.1).

THEOREM 4.18. *Let m be any integer,* $\alpha,\beta > 0$ *with* $\alpha\beta = \pi^2/k^2$, *and* $0 < r < 1$. *Then*

$$\alpha^{-m/2} \sum_{\nu=1}^{\infty} \frac{\bar{\chi}(\nu)\nu^{-m-1} f(\nu, -m, r/k) G(i\alpha k\nu/\pi, \chi)}{1 - e^{2\alpha k\nu}}$$

$$(4.17) \quad = \chi(-1) e^{-\pi i m/2} \beta^{-m/2} \sum_{\nu=1}^{\infty} \frac{\bar{\chi}(\nu)\nu^{-m-1} f(\nu, -m, r\beta i/\pi) G(i\beta k\nu/\pi, \chi)}{1 - e^{-2\beta k\nu}} - 2^{m+1} G(\bar{\chi}) \sum_{j=0}^{m+2} i^j \frac{B_j(r, \bar{\chi})}{j!} \frac{B_{m+2-j}(\bar{\chi})}{(m+2-j)!} \alpha^{(m+2-j)/2} \beta^{j/2}.$$

PROOF. In Theorem 4.1, put $\chi = \chi_1 = \chi_2$, $r_1 = 0$, $r_2 = r$, and $z = \pi i/\alpha k$. From [10],

$$B_j(-r, \chi) = \chi(-1)(-1)^j B_j(r, \chi) \qquad (j \geqq 0).$$

Using the above, multiplying the equation resulting from (4.5) by $(-2\alpha^{1/2})^m / G(\chi)$, and then using (4.3), we arrive at (4.17).

If we put $\alpha = \beta = \pi/k$ in (4.17), we derive the following theorem.

THEOREM 4.19. *Let m be any integer and suppose that $0 < r < 1$. Then*

$$\sum_{\nu=1}^{\infty} \frac{\bar{\chi}(\nu)\nu^{-m-1} f(\nu, -m, r/k) G(i\nu, \chi)}{1 - e^{-2\pi\nu}}$$

$$(4.18) \quad = \chi(-1) e^{-\pi i m/2} \sum_{\nu=1}^{\infty} \frac{\bar{\chi}(\nu)\nu^{-m-1} f(\nu, -m, ri/k) G(i\nu, \chi)}{1 - e^{-2\pi\nu}} - G(\bar{\chi})(2\pi/k)^{m+1} \sum_{j=0}^{m+2} i^j \frac{B_j(r, \bar{\chi})}{j!} \frac{B_{m+2-j}(\bar{\chi})}{(m+2-j)!}.$$

If we let m be even and let r tend to 0 in (4.17) and (4.18), we obtain (4.7) and (4.8), respectively.

We now consider a couple of examples to illustrate the two previous theorems.

From (4.4),

$$(4.19) \qquad B_j(x, \chi_1) = 4^{j-1}\{B_j((x+1)/4) - B_j((x+3)/4)\},$$

Furthermore, if $E_j(x)$, $j \geqq 0$, denotes the j-th Euler polynomial, we know that [1, p. 806]

$$(4.20) \qquad E_{j-1}(x) = \frac{2^j}{j}\{B_j((x+1)/2) - B_j(x/2)\} \qquad (j \geqq 1).$$

From (4.19) and (4.20), we deduce that

$$B_j(x,\chi_1) = -2^{j-2}jE_{j-1}((x+1)/2) \qquad (j \geqq 1). \tag{4.21}$$

PROPOSITION 4.20. *Let N be an arbitrary integer and let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2/16$. Then*

$$\begin{aligned} &\alpha^{-N} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \cos\{(2\nu+1)\pi r/2\}}{(2\nu+1)^{2N+1}\cosh\{2(2\nu+1)\alpha\}} \\ &+ (-\beta)^{-N} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \cosh\{2(2\nu+1)\beta r\}}{(2\nu+1)^{2N+1}\cosh\{2(2\nu+1)\beta\}} \\ &= 2^{2N-2}\pi \sum_{j=0}^{N} (-1)^j 2^{2j} \frac{E_{2j}((r+1)/2)}{(2j)!} \frac{E_{2N-2j}}{(2N-2j)!} \alpha^{N-j}\beta^j \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} &\alpha^{-N+1/2} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \sin\{(2\nu+1)\pi r/2\}}{(2\nu+1)^{2N}\cosh\{2(2\nu+1)\alpha\}} \\ &- (-1)^N \beta^{-N+1/2} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \sinh\{2(2\nu+1)\beta r\}}{(2\nu+1)^{2N}\cosh\{2(2\nu+1)\beta\}} \\ &= -2^{2N-2} \sum_{j=1}^{N} (-1)^j 2^{2j} \frac{E_{2j-1}((r+1)/2)}{(2j-1)!} \frac{E_{2N-2j}}{(2N-2j)!} \alpha^{N-j+1/2}\beta^j. \end{aligned} \tag{4.23}$$

PROOF. In (4.17), let $\chi = \chi_1$. Use the calculations prior to Proposition 4.5. Using also (4.9) and (4.21), we find that

$$\begin{aligned} &\alpha^{-m/2} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu f(2\nu+1, -m, r/4)}{(2\nu+1)^{m+1}\cosh\{2(2\nu+1)\alpha\}} \\ &= -e^{-\pi i m/2}\beta^{-m/2} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu f(2\nu+1, -m, r\beta i/\pi)}{(2\nu+1)^{m+1}\cosh\{2(2\nu+1)\beta\}} \\ &\quad - \sum_{j=1}^{m+1} i^{j+1} 2^{m+j} \frac{E_{j-1}((r+1)/2)}{(j-1)!} \frac{E_{m+1-j}}{(m+1-j)!} \alpha^{(m+2-j)/2}\beta^{j/2}. \end{aligned} \tag{4.24}$$

Consider first the case when m is even. Put $m = 2N$ in (4.24), use (3.1), and in the finite sum on the right side of (4.24) replace j by $2j+1$. Formula (4.22) then follows forthwith. If m is odd, put $m = 2N-1$ in (4.24), use (3.1), and in the finite sum on the right side of (4.24) replace j by $2j$. We then get (4.23).

If we let r tend to 0 in (4.22), we obtain Proposition 4.5 again, since [1, p. 804]

$$E_j = 2^j E_j(1/2) \qquad (j \geqq 0). \tag{4.25}$$

The preceding formulas assume a somewhat more symmetric shape if we change variables in a manner similar to what we have done before. For example, let us examine (4.23). Replace α by $\alpha^2/4$, β by $\beta^2/4$, and, in the new notation, let $r = 2\alpha/\pi$. Thus, for $\alpha\beta = \pi$ and $\alpha < \pi/2$, we now have

$$\alpha^{-2N+1} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \sin\{(2\nu+1)\alpha\}}{(2\nu+1)^{2N}\cosh\{(2\nu+1)\alpha^2/2\}}$$

$$- (-1)^N \beta^{-2N+1} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \sinh\{(2\nu+1)\beta\}}{(2\nu+1)^{2N}\cosh\{(2\nu+1)\beta^2/2\}} \tag{4.26}$$

$$= \sum_{j=1}^{N} (-1)^{j+1} 2^{2j-2N-2} \frac{E_{2j-1}(\alpha/\pi + 1/2)}{(2j-1)!} \frac{E_{2N-2j}}{(2N-2j)!} \alpha^{2N+1-2j}\beta^{2j}.$$

If we let $N = 0$ in (4.26), we obtain

$$\alpha \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \sin\{(2\nu+1)\alpha\}}{\cosh\{(2\nu+1)\alpha^2/2\}} = \beta \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \sinh\{(2\nu+1)\beta\}}{\cosh\{(2\nu+1)\beta^2/2\}},$$

which may be found in Ramanujan's Notebooks [54, vol. I, p. 276; vol. II, p. 171, no. 12].

We shall not write out explicitly any more examples to illustrate Proposition 4.20. However, the cases of (4.22) and (4.23) when $N \leqq 0$ are especially striking. If we let $\alpha = \beta = \pi/4$ in (4.22), let r tend to 0, and use (4.25), we obtain Proposition 4.6 if N is even.

PROPOSITION 4.21. *Let N be any integer. Then*

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(\nu+1)/2}\cos\{(2\nu+1)\pi r/4\}\sinh\{(2\nu+1)\pi/4\}}{(2\nu+1)^{2N+1}\cosh\{(2\nu+1)\pi/2\}}$$

$$- (-1)^N \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(\nu+1)/2}\cosh\{(2\nu+1)\pi r/4\}\sinh\{(2\nu+1)\pi/4\}}{(2\nu+1)^{2N+1}\cosh\{(2\nu+1)\pi/2\}} \tag{4.27}$$

$$= \sqrt{2}\,(\pi/4)^{2N+1} \sum_{j=1}^{N} (-1)^{j+1} \frac{B_{2j}(r,\chi_2)}{(2j)!} \frac{B_{2N+2-2j}(\chi_2)}{(2N+2-2j)!}$$

and

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(\nu+1)/2}\sin\{(2\nu+1)\pi r/4\}\sinh\{(2\nu+1)\pi/4\}}{(2\nu+1)^{2N}\cosh\{(2\nu+1)\pi/2\}}$$

$$+ (-1)^N \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(\nu+1)/2}\sinh\{(2\nu+1)\pi r/4\}\sinh\{(2\nu+1)\pi/4\}}{(2\nu+1)^{2N}\cosh\{(2\nu+1)\pi/2\}}$$

$$= \sqrt{2}\,(\pi/4)^{2N} \sum_{\nu=0}^{N-1} (-1^{j+1}) \frac{B_{2j+1}(r,\chi_2)}{(2j+1)!} \frac{B_{2N-2j}(\chi_2)}{(2N-2j)!}\,.$$

Proof. Apply Theorem 4.19 and use the calculations from the proof of Proposition 4.10.

If we let N be odd and let r tend to 0 in (4.27), we get Proposition 4.10.

We shall conclude the paper with some examples of another class of series identities involving primitive characters. As usual, let $L(s,\chi)$ denote the Dirichlet L-function associated with χ. In [11, Theorem 4], we proved the following theorem. Special cases were first proved by Chowla [16]. See also the papers of Katayama [30], [31].

Theorem 4.22. *Let N be any integer. If χ is even, then*

$$L(2N+1,\chi) = (2/k)(-1)^N G(\chi) \sum_{\nu=1}^{\infty} \frac{G(i\nu,\bar{\chi})\nu^{-2N-1}}{1-e^{-2\pi\nu}}$$

$$- 2\sum_{\nu=1}^{\infty} \frac{\chi(\nu)\nu^{-2N-1}}{e^{2\pi\nu}-1} \tag{4.28}$$

$$+ (2/\pi) \sum_{j=0}^{N} (-1)^{j+1}\zeta(2j)L(2N+2-2j,\chi);$$

if χ is odd, then

$$L(2N,\chi) = (2i/k)(-1)^{N+1} G(\chi) \sum_{\nu=1}^{\infty} \frac{G(i\nu,\bar{\chi})\nu^{-2N}}{1-e^{-2\pi\nu}}$$

$$- 2\sum_{\nu=1}^{\infty} \frac{\chi(\nu)\nu^{-2N}}{e^{2\pi\nu}-1} \tag{4.29}$$

$$+ (2/\pi) \sum_{j=0}^{N} (-1)^{j+1}\zeta(2j)L(2N+1-2j,\chi).$$

We shall work out a few beautiful examples that can be deduced from this theorem when $N \leqq 0$.

PROPOSITION 4.23. *Let n be a positive integer. Then*

$$(4.30) \qquad E_{2n} = (-1)^n \sum_{\nu=1}^{\infty} \frac{\nu^{2n}}{\cosh(\pi\nu/2)} - 4 \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (2\nu+1)^{2n}}{e^{2\pi(2\nu+1)} - 1}.$$

PROOF. Apply (4.29) with $\chi = \chi_1$, and let $N = -n$ with $n > 0$. Use the calculations prior to the proof of Proposition 4.5. If j is a positive integer, then [10, § 4], [37]

$$(4.31) \qquad L(1-j, \bar{\chi}) = -B_j(\chi)/j,$$

and so by (4.31) and (4.9), $L(-2n, \chi_1) = E_{2n}/2$. Formula (4.30) now follows immediately.

An alternative proof of (4.30) may be obtained by putting $\alpha = \beta = \pi$ and $r = 1/4$ in (3.15) and observing that $\varphi(-1/4, -2n) = -iL(-2n, \chi_1)$. Formula (4.30) was first shown by Chowla [16].

PROPOSITION 4.24. *If n is any positive integer, then*

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(\nu+1)/2}(2\nu+1)^{2n-1}}{e^{2\pi(2\nu+1)} - 1} + \frac{(-1)^{n+1}\sqrt{2}}{4} \sum_{\nu=1}^{\infty} \frac{\sinh(\pi\nu/4)\nu^{2n-1}}{\cosh(\pi\nu/2)} = \frac{B_{2n}(\chi_2)}{4n}.$$

PROOF. Apply (4.28) with $\chi = \chi_2$, let $N = -n$ with $n > 0$, use the calculations from the proof of Proposition 4.10, and employ (4.31).

PROPOSITION 4.25. *If n is any positive integer, then*

$$\sum_{\nu=1}^{\infty} \left(\frac{\nu}{3}\right) \frac{\nu^{2n}}{e^{2\pi\nu} - 1} + \frac{(-1)^{n+1}}{\sqrt{3}} \sum_{\nu=1}^{\infty} \frac{\sinh(\pi\nu/3)\nu^{2n}}{\sinh(\pi\nu)} = \frac{B_{2n+1}(\chi_3)}{2(2n+1)}.$$

PROOF. Apply (4.29) with $\chi = \chi_3$, and use (4.31).

In [11, Theorem 5] we also derived formulae for $L(2N+1,\chi)$, where χ is even, and $L(2N,\chi)$, where χ is odd, involving $\rho = (-1+i\sqrt{3})/2$. Formulae, analogous to those of the three preceding propositions but involving ρ, may be easily deduced from this theorem. Further formulae may be derived from Theorem 3 of [11].

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UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801