ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 3, Summer 1976

A NOTE ON SOME LEBESGUE CONSTANTS

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Given a continuous function f on $[0, 2\pi]$ and the set of nodes

(1) $$x_j = \frac{2j+1}{2n+1}\pi, \quad j = 0, 1, 2, \cdots, 2n,$$

there exists a unique trigonometric polynomial t_n of degree at most n such that $t_n(x_j) = f(x_j)$, $j = 0, 1, 2, \cdots, 2n$. We write $L_n f = t_n$, thereby defining the interpolating projection L_n. The norm of this projection

(2) $$\lambda_n = \|L_n\| = \max\{\|L_n f\| : \|f\| \leqq 1\}$$

is called the *Lebesgue constant* of order n for trigonometric interpolation at the nodes (1). In (2) the function norms are supremum norms on $[0, 2\pi]$. It is known (cf. Morris and Cheney [3]) that

(3) $$\lambda_n = \frac{1}{2n+1}\left\{1 + 2\sum_{j=1}^{n} \sec \frac{j\pi}{2n+1}\right\}$$

$$= \frac{2}{2n+1}\sum_{j=1}^{n} \csc \frac{2j-1}{2n+1}\cdot\frac{\pi}{2} + \frac{1}{2n+1}.$$

Our purpose here is to present a detailed analysis of the asymptotic behavior of λ_n. The analysis depends upon interpreting the expression in (3) as a Riemann sum for a certain integral. We apply the same technique to the classical Lebesgue constants of the Fourier series.

The main tool in the analysis is the following lemma.

LEMMA. *For any function $f \in C^3[0, 1]$ satisfying the inequalities*

(i) $f'''(x) \geqq 0, 0 \leqq r \leqq 1,$ *and*

(ii) $3f'(0) + 2f''(0) \geqq 0,$

the Riemann sums

Recieved by the editors on June 23, 1975.

$$Q_n(f) = \frac{2}{2n+1}\sum_{j=1}^{n} f\left(\frac{2j-1}{2n+1}\right) + \frac{1}{2n+1}f(1)$$

converge monotonically downward to $\int_0^1 f(x)\,dx$.

PROOF. Three integrations by parts yield the identity

$$\text{(4)} \qquad Q_n(f) - \int_0^1 f(x)\,dx = \frac{3f'(0) + 2f''(0)}{24(n+1/2)^2}$$

$$+ \frac{1}{24}\int_0^1 \frac{4\{\frac{1}{2} + [(n+\frac{1}{2})x] - (n+\frac{1}{2})x\}^3 + 3(n+\frac{1}{2})x - [(n+\frac{1}{2})x] - \frac{1}{2}}{x^3(n+\frac{1}{2})^3}$$

$$\cdot\, x^3 f'''(1-x)\,dx.$$

The square bracket in (4) denotes the integer-part function. We put $t = x(n+1/2)$ and note that the function

$$g(t) = \frac{4(1/2 + [t] - t)^3 + 3t - [t] - 1/2}{t^3}$$

is differentiable for $t > 0$. Moreover, we assert that $g'(t) < 0$ for $t > 0$. In proving this, it suffices to consider $k < t < k+1$, in which interval

$$g'(t) = 3t^{-4}\{-4(1/2+k)(1/2+k-t)^2 + 1/2 + k - 2t\} < 0.$$

Since $x^3 f'''(1-X) \geqq 0$ on $[0, 1]$, the Lemma now follows.

We now apply the Lemma to the function

$$f(x) = \csc\frac{\pi}{2}x - \frac{2}{\pi x} = \frac{\pi}{12}x + \frac{7}{360}\frac{\pi^3}{8}x^3 + \cdots$$

which is analytic in $|x| < 2$ and whose power series has nonnegative coefficients. Thus $f'''(x) \geqq 0$ for $0 \leqq x \leqq 1$, $f'(0) = \pi/12$, and $f''(0) = 0$, verifying hypotheses (i) and (ii) of the Lemma. Therefore, we conclude that the sequence of numbers

$$\text{(5)} \quad q_n = \frac{2}{2n+1}\sum_{j=1}^{n}\left\{\csc\left(\frac{2j-1}{2n+1}\cdot\frac{\pi}{2}\right) - \frac{2(2n+1)}{\pi(2j-1)}\right\} + \frac{1-(2/\pi)}{2n+1}$$

converges monotonically downward to

$$\int_0^1 \left(\csc\frac{\pi}{2}t - \frac{2}{\pi t}\right) dt = \frac{2}{\pi}\log\frac{4}{\pi}.$$

Now using (3) and (5) we obtain

$$\lambda_n - q_n = \frac{2}{\pi}(\log n + v_n) \tag{6}$$

where we have put

$$v_n = \sum_{j=1}^{n} \frac{2}{2j-1} + \frac{1}{2n+1} - \log n. \tag{7}$$

In order to see that the sequence defined in (7) is decreasing, first compute

$$v_{n-1} - v_n = \log\left(1 + \frac{1}{n-1}\right) - \frac{4n}{4n^2-1} \qquad (n \geqq 2),$$

and then verify that the function

$$h(x) = \log\left(1 + \frac{1}{x-1}\right) - \frac{4x}{4x^2-1}$$

is positive for $x = 2$, satisfies $h'(x) < 0$ for $x \geqq 2$, and has limit 0 as x becomes infinite. Therefore, $h(x) > 0$ for $x \geqq 2$, and $v_{n-1} > v_n$ for $n \geqq 2$. This proves that the sequence of numbers

$$\lambda_n - \frac{2}{\pi}\log n = q_n + \frac{2}{\pi}v_n \qquad n = 1, 2, 3, \cdots$$

is monotone decreasing. An easy calculation establishes that

$$\lim_{n\to\infty} v_n = \gamma + \log 4 \tag{8}$$

where $\gamma = \lim_{n\to\infty}(\sum_{\nu=1}^{n} \nu^{-1} - \log n) = .5772156649 \cdots$ (Euler's constant). Hence, we have proved the following theorem.

THEOREM 1. *The Lebesgue constants for trigonometric interpolation at equidistant nodes satisfy the relation*

$$\lambda_n = \frac{2}{\pi}\log n + \delta_n, \qquad n = 1, 2, \cdots,$$

in which δ_n decreases monotonically from 5/3 *to*

$$\frac{2}{\pi}\left(\log\frac{16}{\pi} + \gamma\right) = 1.40379 \cdots.$$

REMARK. A similar result for the Lebesgue constant associated with algebraic polynomial interpolation at the zeros of the Chebyshev polynomials is given in Rivlin [4].

The n-th Lebesgue constant for the classical Fourier series is

$$\rho_n = \frac{1}{\pi}\int_0^{2\pi} \left| \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} \right| dt.$$

Fejér [2] obtained the elegant representation

$$\rho_n = \frac{1}{2n+1} + \frac{2}{\pi}\sum_{j=1}^{n} \frac{1}{j}\tan\frac{j\pi}{2n+1}, \tag{9}$$

which in turn can be easily transformed to

$$\rho_n = \frac{1}{2n+1} + \frac{2}{2n+1}\sum_{j=1}^{n} \frac{\cot\left(\frac{2j-1}{2n+1}\cdot\frac{\pi}{2}\right)}{\frac{\pi}{2}\left(1-\frac{2j-1}{2n+1}\right)}.$$

We wish to show now that the function

$$f(x) = \frac{\cot\frac{\pi}{2}x}{\frac{\pi}{2}(1-x)} - \frac{1}{\left(\frac{\pi}{2}\right)^2 x}$$

satisfies the hypotheses of the Lemma. To this end, we introduce the function

$$g(z) = \frac{1}{z} - \cot z = \frac{z}{3} + \frac{z^3}{45} + \cdots$$

which is analytic in $|z| < \pi$ and has a power series in which only odd powers appear, and these with positive coefficients. The relation between f and g is

$$f(x) = \frac{1 - \frac{\pi}{2} g\left(\frac{\pi}{2}x\right)}{\left(\frac{\pi}{2}\right)^2 (1-x)},$$

and f is analytic in $|x| < 2$. If we write

$$g\left(\frac{\pi}{2}x\right) = \sum_{j=0}^{\infty} g_j x^j,$$

then $g_j \geqq 0$ for $j = 0, 1, 2, \cdots$. Putting $s_k = g_0 + \cdots + g_k$, we have $s_k < g(\pi/2) = 2/\pi$, and hence, in the power series

$$f(x) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \left(1 - \frac{\pi}{2} s_k \right) x^k,$$

all the coefficients are positive. Thus $f'''(x) > 0$ on $[0, 1]$, and $3f'(0) + 2f''(0) > 0$. The Lemma then implies that the sequence of numbers

$$r_n = \frac{2}{2n+1} \sum_{j=1}^{n} \left\{ \frac{\cot\left(\frac{2j-1}{2n+1} \cdot \frac{\pi}{2}\right)}{\frac{\pi}{2}\left(1 - \frac{2j-1}{2n+1}\right)} - \frac{1}{\left(\frac{\pi}{2}\right)^2 \frac{2j-1}{2n+1}} \right\} + \frac{1}{2n+1}\left(1 - \frac{4}{\pi^2}\right) \tag{10}$$

is monotone decreasing to

$$C = \frac{2}{\pi} \int_0^1 \left(\frac{\cot \frac{\pi}{2} x}{1 - x} - \frac{2}{\pi x} \right) dx. \tag{11}$$

By (9), (10), and (7) we obtain

$$\rho_n - \frac{4}{\pi^2} \log n = r_n + \frac{4}{\pi^2} v_n.$$

With (8), this proves the following result.

THEOREM 2. (Cf. Cheney and Price [1]) *The Lebesgue constants associated with Fourier series satisfy the equation*

$$\rho_n = \frac{4}{\pi^2} \log n + \epsilon_n, \qquad n = 1, 2, \cdots,$$

in which ϵ_n *decreases monotonically from* $(1/3) + (2\sqrt{3})/\pi = 1.4359 \cdots$ *to* $C + (4/\pi^2)(\gamma + \log 4) = 1.2703 \cdots$ *where* C *is given by equation* (11).

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